Decay of superfluid turbulence via Kelvin-wave radiation

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The decay of superfluid turbulence in the limit of low temperature is studied by numerical simulations of vortex ring collisions. In particular we excite Kelvin waves and study the loss of vortex line length due to Kelvin-wave radiation. Although the effect is small, the decay constant is not inconsistent with recent experiments on vortex lattices in dilute Bose-Einstein condensates, and on superfluid turbulence in helium at very low temperature. We also consider the character of the decay when both vortex reconnections and Kelvin-wave radiation are present.

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One of the most challenging problems in fluid mechanics is the development of a complete hydrodynamical description of turbulence. Progress is often made by studying fluids where the theoretical description can be simplified. For example, low-temperature superfluids such as liquid helium [1] or the recently discovered dilute Bose-Einstein condensates [2] are particularly attractive due to the dramatic reduction in viscosity. In a superfluid, the turbulent state consists of a network of interacting vortices known as a ‘‘vortex tangle’’ [1]. In superfluid helium-4 the vortex tangle is strongly coupled to an interpenetrating normal fluid and the system shares many of the features of classical turbulence [3]. However, in the limit of low temperature, the normal fluid is negligible, but experiments still indicate a temperature-independent decay of the turbulence state [4]. In this case, the emission of sound either by vortex reconnections or vortex motion is the only active dissipation mechanism. Similarly, the time scale for the formation of vortex arrays in dilute Bose-Einstein condensates is also temperature independent suggesting that vortex-sound interactions may also be important in this relaxation process [6]. In an earlier work we showed that significant sound energy is released during vortex reconnections [7]. However, at the typical vortex line densities of superfluid helium-4 experiments [4,5], or in the regular structures found in a vortex lattice [6] one might expect that vortex reconnections are relatively infrequent, and that the continuous emission of sound by accelerating vortices may be the dominant dissipation mechanism. Unfortunately there is a scarcity of quantitative predictions of the significance of vortex motion as a dissipation mechanism in superfluid turbulence. Analytical results for the sound radiated by moving vortices exist only for simple cases such as a corotating pair [8]. Furthermore, conventional numerical simulations of superfluid turbulence based on vortex filaments governed by incompressible Euler dynamics (the Biot-Savart law) [9] are unable to describe sound emission.

An elegant model of quantum fluid mechanics is provided by the Gross-Pitaevskii (GP) equation. The GP model represents an extension of Euler fluid mechanics to include the quantization of circulation, vortex core structure and vortex-sound interactions. It provides a good description of the dynamics of dilute atomic Bose-Einstein condensates in situations where the dissipative effect of the thermal component does not dominate, i.e., in the limit of low temperature. Although the GP model does not accurately represent the physics of superfluid helium-4, it has been shown to provide qualitative and sometimes quantitative insight into the critical velocity for vortex nucleation [10,11], vortex line reconnections [12], vortex ring collisions [13], the decay of superfluid turbulence [14], and sound emission due to vortex reconnections [7].

To study sound emission due to vortex motion we excite Kelvin waves on a vortex ring and measure the length of the ring as a function of time. The Kelvin waves are produced by colliding two or more vortex rings. The initial state is constructed from stationary vortex ring solutions of the uniform flow equation found by Newton's method [15]. The desired configuration is obtained by multiplying the individual vortex ring states. This initial state is then evolved according to the dimensionless GP equation,

$$i\partial_t\psi = -\tfrac{1}{2}\nabla^2\psi + (|\psi|^2-1)\psi, \tag{1}$$

using a semi-implicit Crank-Nicholson algorithm. In dimensionless units, distance and velocity are measured in terms of the healing length ξ and the sound speed c, respectively. In addition, the asymptotic number density n_0 is rescaled to unity. The computation box with volume $V=(50)^3$ is divided into 10^6 grid points with a spacing of 0.5. A grid spacing of 0.25 was also used to test the accuracy of the numerical methods. The time step is 0.02 and a typical simulation is run for 1.5×10^5 steps. Simulations have been performed with two, three, and four vortex rings with radii ranging from 2.86 to 18.1 healing lengths. To convert the dimensionless units into values applicable to superfluid helium-4, we take the number density as $n_0=2.18\times10^{28}$ m^{-3}, the quantum of circulation as $\kappa=h/m=9.92\times10^{-8}$ m^{2}s^{-1}, and the healing length as $\xi/\sqrt{2}=0.128$ nm [16]. This gives a time unit $2\pi\xi^2/\kappa=2$ ps, and therefore, for superfluid helium-4 our simulations would correspond to a real time of 6 ns. Whereas for a sodium vapor condensate with $\xi\sim0.2$ μm [6], the time unit is $\sim$15 ns giving a simulation time of $\approx$45 μs.

In the first example we consider a collision between a large and a small vortex rings. Such collisions are important in vortex tangle dynamics because small vortex rings are often produced in collisions between larger structures [13]. A

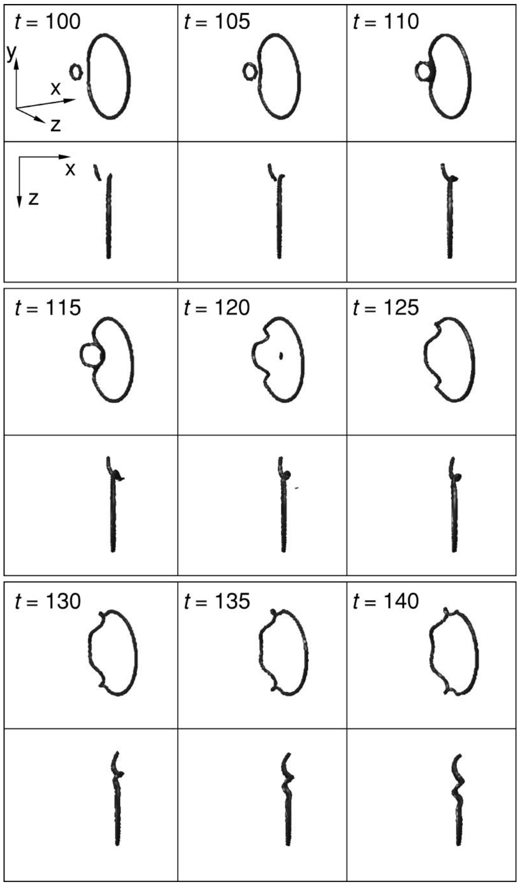

FIG. 1. Sequence of density isosurfaces ($|\psi|^2=0.35$) illustrating a vortex ring collision with ring radii $R_1=18.1$ and $R_2=2.86$. A top view is shown below each frame. The rings collide at $t=110$. The reconnection produces a sound pulse which appears as a small dot in the $t=120$ frame.

sequence of density isosurface plots illustrating the collision are shown in Fig. 1. Both vortex rings propagate in the positive x direction with their propagation axes offset by the radius of the larger ring, $R_1=18.1$. The collision occurs at $t=110$, and leads to the destruction of the small vortex ring. The vortex energy is converted into a sound pulse and Kelvin waves. The sound pulse propagates away from the large vortex ring and appears as a density minimum at $t=120$ in Fig. 1. The Kelvin-waves appear as two kinks which propagate around the vortex ring in opposite directions.

In Fig. 2 we plot the vortex line length ℓ as a function of time. The vortex line length is evaluated numerically by searching for phase singularities, estimating the length of vortex line within a grid cube and summing over all cubes. We have checked the accuracy of the method by performing the same calculation with a grid spacing of 0.25 and 0.5. The two calculations agree within a fraction of the healing length. During the reconnection at $t=110$, the vortex line length is stretched from its initial value of $2\pi(R_1+R_2)\sim130$, and then decreases dramatically due to the conversion of energy into sound [7]. After the collision the small vortex ring disappears completely while the mean radius of the large vortex ring is slightly larger, $R_1'=18.8$.

The vortex line length oscillates due to the motion of the

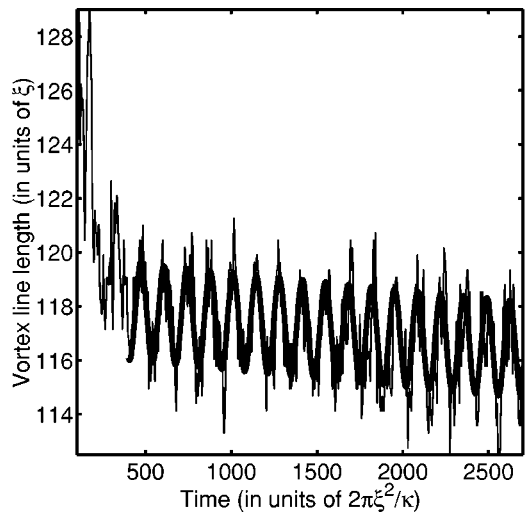

FIG. 2. The vortex line length ℓ as a function of time for the collision shown in Fig. 1. During the reconnection there is a sudden loss of line length due to the emission of a sound pulse. Subsequently the length decays due to Kelvin wave radiation. The bold line is a fit to a function of the form $\ell=\ell_0[1+a\sin(2\pi t/T+\phi)]/(1+\chi_2\ell_0 t/V)$, where the period of the oscillations, $T=134\pm1$, corresponds to the time for the Kelvin wave to go halfway around the vortex ring. A linear fit to $1/L$ against t between $t=500$ and $t=2900$ determines the coefficient in Eq. (4), $\chi_2\sim0.006\pm0.001$. These data are for a grid spacing of 0.25 healing lengths.

Kelvin-waves around the vortex ring. The minima correspond to the time when the two Kelvin wave packets overlap. By fitting the oscillations to a sine wave (shown bold in Fig. 2) we find that the period is 134 ± 1. This corresponds to a Kelvin-wave velocity of 0.44 times the speed of sound. The dispersion relationship for Kelvin waves is [17]

$$\omega=\frac{\kappa}{4\pi}k^2\ln\left(\frac{2}{ka}\right), \tag{2}$$

where in dimensionless units, $\kappa=2\pi$ and the core size $a=1/\sqrt{2}$, therefore, the group velocity of a Kelvin wave with center wavelength λ is

$$v_g=\frac{\pi}{\lambda}\left[2\ln\left(\frac{\sqrt{2}\lambda}{\pi}\right)-1\right]. \tag{3}$$

A velocity of 0.44 corresponds to a Kelvin wavelength of 5.3 healing lengths, which agrees with a measurement made from Fig. 1. In this intermediate wavelength region, the Kelvin-wave group velocity (3) is only weakly dependent on wavelength ($0.4\leqslant v_g\leqslant0.63$ for $5\leqslant\lambda\leqslant35$). Consequently, there is very little dispersion of the Kelvin-wave packet.

During the Kelvin-wave oscillations the mean vortex line length gradually decreases. It is interesting to compare the observed length decrease with that expected for a Kelvin-wave cascade, where energy is transfered to a much shorter wavelengths with a cutoff below a critical wavelength. In this case the vortex line density $L=\ell/V$ can be described by an equation of the form [18]

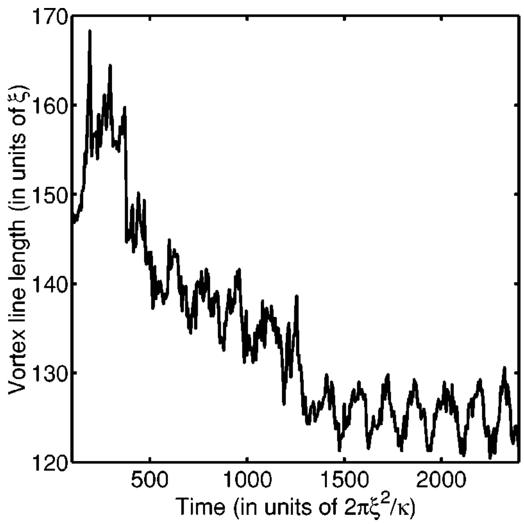

FIG. 3. The vortex line length as a function of time for a collision between vortex rings with radii $R_1 = 18.1$ and $R_2 = 5.74$. The initial increase in the vortex line length is due to stretching during the first collision. Subsequently there are a large number of reconnections which result in the production of smaller rings. After $t = 1100$ a sequence of reconnections leaves only one vortex ring which then decays by Kelvin-wave radiation. The decay between $t = 1500$ and $t = 2500$ is consistent with the coefficient found previously, $\chi_2 \sim 0.006$.

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{\kappa}{2\pi}\chi_2 L^2, \tag{4}$$

where $\kappa = 2\pi$ in dimensionless units, and χ_2 is a dimensionless coefficient. A linear fit to $1/L$ using the data from Fig. 2 for times between $t = 500$ and $t = 2900$ gives a coefficient, $\chi_2 \sim 0.006 \pm 0.001$. A much larger value, $\chi_2 \sim 0.3$, is suggested by experiments on superfluid helium [18] and by vortex dynamics simulations, where an energy cutoff is assumed [9]. However, the lower value is not inconsistent with experiments on superfluid helium and atomic condensates in the limit of low temperature. From Eq. (4) the time at which L is reduced to one half is $T = 2\pi/(\kappa\chi_2 L_0)$, where L_0 is the line density at $t = 0$. Substituting $L_0 = 10^{10}\ \mathrm{m}^{-2}$, corresponding to the low-temperature superfluid helium experiments of Davies *et al.* [5], one obtains $T \sim 1$ s, which is the same order of magnitude as the experiment. Similarly, in the case of sodium vapor condensates substituting $L_0 = 1.5 \times 10^{10}\ \mathrm{m}^{-2}$ (estimated from the parameters given in Ref. [6]) and $\kappa = 1.7 \times 10^{-8}\ \mathrm{m}^2\mathrm{s}^{-1}$ yields $T \sim 4.1$ s, compared to an experimental value $T = 8.5$ s observed at the lowest temperature.

To test the generality of this result we repeated the calculation for different initial conditions. First, a collision involving initial vortex rings with radii $R_1 = 18.1$ and $R_2 = 5.74$ produces the decay shown in Fig. 3. In this case, the small vortex ring is not completely destroyed in the initial collision. Instead after making repeated reconnections with the main vortex ring between $t = 400$ and $t = 1250$, it annihilates at $t \sim 1250$ leaving only the large ring. The large vortex ring then decays by Kelvin-wave radiation at a rate consistent

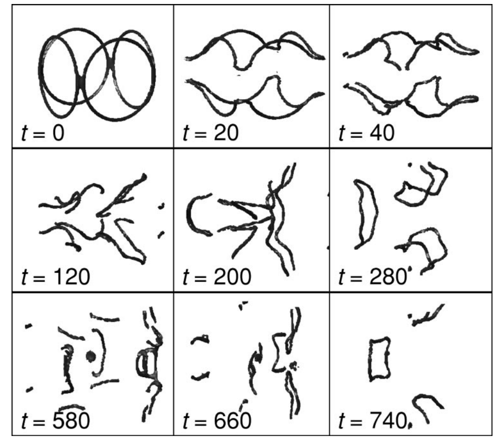

FIG. 4. Sequence of density isosurfaces ($|\psi|^2 = 0.35$) illustrating a collision involving four converging vortex rings each with radius $R = 18.1$. A large number of reconnections result in the generation of small structures which annihilate into sound energy producing only two vortex rings by $t = 740$. Note that because of the periodic boundary conditions the loop structures connect on opposite faces of the numerical box.

with the results in Fig. 2. Second, collisions involving three vortex rings reproduce the same characteristic decay with large drops in the vortex line length during reconnection events interspersed with gently sloping regions of Kelvin-wave radiation. All the simulations show that the Kelvin-wave decay coefficient is independent of the initial condition. This can be explained by the fact that the radiation spectrum is dominated by short wavelength Kelvin waves [19].

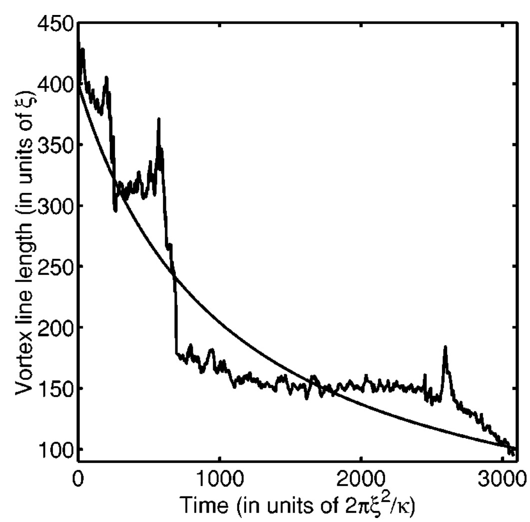

FIG. 5. The decay of vortex line length (thin line) for the collision involving four converging vortex rings shown in Fig. 4. For $t < 800$, there are a large number of reconnections resulting in a dramatic decrease in length. After $t \sim 800$, only two rings remain and there are no more reconnections until $t = 2400$. In this region the loss of vortex line length follows the prediction for Kelvin waves with a decay coefficient consistent with the data from Figs. 2 and 3. A decay characterized by $\chi_2 = 0.3$ is plotted (thick line) to illustrate the difference between a reconnection dominated decay and that due to Kelvin-wave radiation only.

Finally, it is interesting to compare the effect of Kelvin-wave radiation with the behavior at higher vortex line densities where reconnection events are more frequent and energy loss by direct emission of sound bursts is important [7]. To create a large number of reconnections we consider four initial vortex rings, propagating inwards, as in the first frame of Fig. 4. The propagation axes are offset by one healing length to break the symmetry (the symmetric case completely annihilates in the first few hundred time units). A similar system, although with a very different length scale, has been studied using a classical vortex filament calculation [20]. The decay of the vortex line length is shown in Fig. 5. The initial vortex line density is much larger than in the previous examples, therefore, there are a larger number of reconnections with a broad distribution of reconnection angles. A sequence of reconnections before $t \sim 800$ results in the formation of two large vortex rings which avoid each other until $t \sim 2600$. The decay coefficient for Kelvin-wave radiation is consistent with Fig. 2. The collision at $t \sim 2600$ sets up another phase of repeated reconnections producing smaller vortex rings which annihilate in collisions with larger structures. This scenario of vortex tangle decay involving the production of small vortex rings is also observed in the classical vortex filament calculations of Tsubota *et al.* [9].

It is clear from Fig. 5 that one may distinguish between two types of superfluid turbulence decay, a fast sound-burst dominated regime and a much slower Kelvin-wave dominated regime. In the experiments on atomic condensates [6], the different time scales for formation and decay of the vortex lattice may be associated with these two regimes. During formation, one has a disordered high-density vortex tangle corresponding to the fast reconnection dominated decay, whereas once the lattice is formed vortex reconnections become infrequent and the slow Kelvin-wave decay is observed.

In summary, we have studied the basic processes involved in the decay of quantized vorticity by numerical simulation of vortex ring collisions. Our results gave a numerical value for the decay constant that is consistent with preliminary experimental measurements on superfluid helium [5] and atomic condensates [6] in the limit of low temperature. In addition, we show that under conditions where vortex reconnections are important a much faster sound-burst dominated decay may be observed. This faster time scale may account for the shorter relaxation-times associated with the formation of vortex lattices.

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[19] The power radiated is proportional to a high power of the Kelvin-wave frequency [3]. From Eq. (2) one finds that the power spectrum is sharply peaked around a wavelength of order 5 healing lengths. As the spectrum of Kelvin-waves produced in a typical vortex ring collision is also peaked around 5 healing lengths, see Fig. 1, the decay is dominated by this wavelength region.

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