Theory of light-induced drift. III. Models of surface and bulk light-induced drift in one dimension

Frank O. Goodman*
Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1
(Received 3 May 2002; published 30 January 2003)

Light-induced drift (LID) of a rarefied gas in a cell is studied, and exact analytical closed-form solutions to the model rate equations, which model the gas motion in one dimension, are obtained for cases of both surface LID (SLID) and bulk LID (BLID); the special case of the limit of low radiation absorption by the gas is given particular attention. Similarities and differences among the results for SLID and BLID are discussed. This is part III of a series of papers, parts I and II having studied LID, but concentrating on SLID, with flat-plate and circular-cylindrical cell geometries, respectively [F. O. Goodman, Phys. Rev. A **65**, 064309 (2002); **65**, 064310 (2002)].

DOI: 10.1103/PhysRevA.67.013410 PACS number(s): 34.50.Rk

I. INTRODUCTION

In parts I and II (Refs. [1] and [2]) of this series of papers, the phenomenon of light-induced drift (LID) was discussed, and exact treatments of models of surface LID (SLID), in three dimensions, were presented for the cases of flat-plate [1] (FP) and circular-cylindrical [2] (CC) geometries, in the limit of large cell length (and large FP cell width), and in the free-molecule limit. Because of considerations of feasibility and interpretation of possible experimental measurements, and of a desire for relative simplicity, interest is essentially confined to the limit of low radiation intensity absorption. Although general Maxwell-Boltzmann rate equations (MBREs) were presented [1,2], no calculations for cases involving bulk LID (BLID) were made.

Here (''here'' means ''in the present paper''), in the spirit of parts I and II, we consider models of both (pure) SLID and (pure) BLID in order to compare the two phenomena, and we use models in one dimension (1D) in order that the simplest possible reasonable comparison may be made. SLID (but not BLID) is strictly impossible in 1D without some assumption about the transverse molecular motion (in order that the active surfaces may be deemed to have scattered the molecules during collisions), and our assumption is explained below. Specializations analogous to those made in parts I and II are also made here. We do not want our comparisons to be jeopardized by uncertainties caused by approximations (other than the unavoidable use of approximate MBREs), and so exact treatments of both SLID and BLID are presented first, with specializations made afterward.

As was the case with part II, in order to understand the material here, it is necessary for readers to have read and understood part I, and an understanding of part II is also necessary here. References to entities in parts I and II are made here with the notation (I.5.11), (II.3.4), (I.D1), and so on. Symbols used here have the same meanings as in part I unless otherwise indicated.

Our assumption in SLID is that, superimposed on the 1D molecular motion (along the cell axis), there is a Maxwellian (in thermal equilibrium with the cell) transverse molecular motion which is not discussed in further detail, but which gives rise to molecule-surface collisions with average frequency per molecule, denoted by $\langle \nu \rangle_m$ [3,4], of the same value as those for Maxwellian gases in parts I and II, that is, $\langle \nu \rangle_m = \pi^{-1/2}$; this assumption implies that we choose characteristic lengths $\ell_c^* = \chi^*$ with $\chi^* = Z^*$ and R^*, respectively [3,5]. In 1D BLID, where θ^* is [1] the average inverse mean-free-path for molecule-buffer particle collisions, we have $\langle \nu \rangle_m = \pi^{-1/2} \theta^* \ell_c^* = \pi^{-1/2} \theta$ [3,6]; although the choice $\theta^* \ell_c^* = 1$ for ℓ_c^* (giving $\theta = 1$) may seem at first sight the most natural [because θ would not appear in the working and it gives the same value of $\langle \nu \rangle_m$ as in SLID], the choice $\ell_c^* = \chi^*$ (giving $\theta = \theta^* \chi^*$) is also natural, and we carry θ through the analysis.

It is necessary to discuss the choice of system transmission parameter κ; for SLID, it seems logical to make the same choices as made in (I.H5) and (II.1.2), respectively, whereas for BLID, in which active surfaces are not necessary (and not mentioned here), $\kappa = 1$ is a candidate. We simply use unspecified κ in the analysis here.

We recall, on the basis of the results (I.2.1), (I.2.2), and (I.3.1)–(I.3.6), that all of the important physical conclusions from the models may be drawn from the three integrals I_{1e}, I_{xs}, and I_{qd}, and, more particularly, from the ratios I_{qd}/a, I_{xs}/I_{qd}, and I_{1e}/I_{qd}; for example,

$$Xc_e/(r/\rho) = I_{1e}/I_{qd}, \tag{1.1}$$

$$\delta/\kappa = (\Delta P/P)/(\kappa r/\rho) = I_{xs}/I_{qd}, \tag{1.2}$$

and we proceed directly to their calculation. The intention here is to get as good a comparison of SLID and BLID as is possible using simple models, and not to compare with experimental results, for example, although a generalized analog of (I.5.11) is presented.

II. 1D SLID

A. Exact analysis

With our assumption, explained in Sec. I, about the transverse molecular motion, we build our MBREs as follows. In the language of part I, we put $\theta v = 0$ with $|v_z|$ and $\langle |v_z| \rangle_j$

*Also with Department of Physics, University of Waterloo, and the Guelph-Waterloo Physics Institute, Ontario, Canada.

equal to $\langle|v_z|\rangle_m=\pi^{-1/2}$, implying that $\beta_{zj}=1$, into (I.2.10); in the language of part II, we again put $\theta v=0$, now with V and $\langle V\rangle_j$ equal to $\langle V\rangle_m=\pi^{1/2}/2$, implying that $B_{Vj}=1$, into the analog of (I.2.10), which replaces [2] $|v_z|$ by $2V/\pi$ therein. Then we make the substitution

$$f_j(v_x:t)=F_j(v_x:t)m(v_x), \tag{2.1}$$

where m is the 1D version of (I.B1),

$$m(v_x)=\pi^{-1/2}e^{-v_x^2}, \tag{2.2}$$

and get, in both cases,

$$\pi^{1/2}\partial F_g/\partial t=c_g\alpha_g+c_e\alpha_e z-\alpha_g F_g+\pi^{1/2}\gamma F_e - \pi^{1/2}q(F_g-F_e), \tag{2.3a}$$

$$\pi^{1/2}\partial F_e/\partial t=c_e\alpha_e(1-z)-\alpha_e F_e-\pi^{1/2}\gamma F_e + \pi^{1/2}q(F_g-F_e), \tag{2.3b}$$

where α_j and z stand for α_{sj} and z_s, respectively [1,2]. Here we pass directly to the steady-state ($\partial F_j/\partial t=0$) solution of Eq. (2.3), that is,

$$F_j=\frac{A_j+B\pi^{1/2}q+E_j\pi^{1/2}\gamma}{C+D\pi^{1/2}q+F\pi^{1/2}\gamma}, \tag{2.4}$$

where the eight constants A_j,B,C,D,E_j,F are given by (I.5.2) with β_j $(=\beta_{zj})=1$.

The definitions here of a,g are different from those in part I:

$$a=\pi^{1/2}q_0\Sigma\alpha/\alpha_g\alpha_e, \tag{2.5a}$$

$$g=\pi^{1/2}\gamma/\alpha_e. \tag{2.5b}$$

We write (I.2.4) as $q(v_x)=q_0\Delta h$, with Δh defined by

$$\Delta h=h(v_x-v_a)-h(v_x-v_b) \tag{2.6}$$

to get

$$F_j=S_j+\frac{2aX_j+2Z_j}{1+a\Delta h+g}, \tag{2.7}$$

where S_j is defined by

$$S_j=A_j/C, \tag{2.8}$$

Z_j by

$$Z_j=gY_j, \tag{2.9}$$

and X_j,Y_j by (I.E1).

The v_x interval of laser excitation is given by ($v_a\leqslant v_x\leqslant v_b$) with $v_a<v_b$, and it clarifies the analysis if $F_j^{(\mathrm{in})},F_j^{(\mathrm{out})}$ are defined as the values of F_j (inside, outside) of this interval:

$$F_j^{(\mathrm{in})}=S_j+\frac{2W_j}{1+w}, \tag{2.10a}$$

$$F_j^{(\mathrm{out})}=S_j+\frac{2Z_j}{1+g}, \tag{2.10b}$$

with W_j defined by (II.A2), and not by (I.E7), and where the definition (I.E6) of w is recalled.

The required integrals here are I_{1j},I_{qj}, and I_{xj}, I_{zj} being absent because of our 1D assumption, and our analog of the results (I.E9) is as follows:

$$I_{1j}=S_j+\frac{Z_j[2-d(\mathrm{erf})]}{1+g}+\frac{W_j d(\mathrm{erf})}{1+w}, \tag{2.11a}$$

$$I_{qj}=\left(\frac{S_j}{2}+\frac{W_j}{1+w}\right)q_0\,d(\mathrm{erf}), \tag{2.11b}$$

$$I_{xj}=\left(\frac{W_j}{1+w}+\frac{Z_j}{1+g}\right)\frac{d(\exp)}{\pi^{1/2}}. \tag{2.11c}$$

The (exact) solution is now obtained by solving the system of (three) equations

$$c_j=I_{1j}, \tag{2.12a}$$

$$c_g+c_e=1, \tag{2.12b}$$

for the (two) quantities c_j: Eq. (2.12b) must be used with one or both of Eqs. (2.12a), and having three equations for two quantities gives a good check on the working. Exact results in closed form for the six integrals I_{1e}, I_{qk} $(k=e,d)$, and I_{xk} $(k=g,e,s)$ are presented in Appendix A (I_{1g},I_{qg} are then obtained from $I_{1g}+I_{1e}=1$, $I_{qg}-I_{qe}=I_{qd}$).

Attempts were made to present experiment-friendly results, in the limit of small a, for the case $g=0$, $z>0$ in (I.5.11) and (I.3.4), and for the case $z=0$, $g>0$ in (I.5.15), the last case being trivial in part II. With hindsight, it would perhaps have been better to first present the analogs of these results for the general case, with no restrictions on the parameters, with specializations made later. This could be important because, with the occasional necessary interpretation, the results are independent of cell geometry and of whether SLID or BLID is being considered, as follows:

$$\frac{\Delta P^*/\mathrm{mPa}}{r^*\mathrm{mm}^2/\mathrm{mW}}\approx 511.3\kappa\frac{I_{xs}}{I_{qd}}\left(\frac{\mu^*}{30\ \mathrm{amu}}\right)^{1/2}\left(\frac{T^*}{300\ \mathrm{K}}\right)^{1/2} \times\left(\frac{\lambda^*}{10\ \mu\mathrm{m}}\right), \tag{2.13}$$

$$\frac{1000c_e}{r^*\mathrm{mm}^2/\mathrm{mW}} \approx 5.113 \frac{I_{1e}}{I_{qd}} \left(\frac{\mathrm{Pa}}{P^*}\right)\left(\frac{100\chi^*}{X^*}\right) \times \left(\frac{\mu^*}{30\ \mathrm{amu}}\right)^{1/2}\left(\frac{T^*}{300\ \mathrm{K}}\right)^{1/2}\left(\frac{\lambda^*}{10\ \mu\mathrm{m}}\right), \tag{2.14}$$

$$\frac{100a}{r^*\mathrm{mm}^2/\mathrm{mW}} \approx \frac{0.5113}{(I_{qd}/a)} \left(\frac{\mathrm{Pa}}{P^*}\right)\left(\frac{100\chi^*}{X^*}\right) \times \left(\frac{\mu^*}{30\ \mathrm{amu}}\right)^{1/2}\left(\frac{T^*}{300\ \mathrm{K}}\right)^{1/2}\left(\frac{\lambda^*}{10\ \mu\mathrm{m}}\right). \tag{2.15}$$

The quantities of importance are clear from Eqs. (2.13)–(2.15), that is, I_{qd}/a, I_{xs}/I_{qd}, and I_{1e}/I_{qd}, and they must be calculated for each particular case. In Eq. (2.13), κ must be interpreted as an appropriate choice, $\kappa=\kappa_s$, of transmission parameter. Curious readers may ask why ''5113'' appears regularly in these results, the answer being that

$$\frac{(10\mu\mathrm{m\ mW})[(300\ \mathrm{K})(30\ \mathrm{amu})b^*)]^{1/2}}{(2^{1/2}\hbar^*\omega^*\lambda^*)(\mathrm{mm}^2\mathrm{mPa})} \approx 511.32. \tag{2.16}$$

B. Physically motivated special cases

As in part II, there is no problem here with the two limits $a\to 0$, $g\to 0$, because they commute. Because our definition (2.5a) of a, in terms of the important physical quantity q_0, for SLID is different from that (3.3a) below for BLID, we use q_0 instead of a where necessary for clarity from now on. Where

$$dv = v_b - v_a, \tag{2.17}$$

the most useful results, in view of Eqs. (2.13)–(2.15), are as follows:

$$\frac{I_{qd}}{q_0 dv} \underset{0}{\overset{q_0}{\to}} \frac{d(\mathrm{erf})}{2dv} \underset{0}{\overset{dv}{\to}} \frac{e^{-v_L^2}}{\pi^{1/2}}, \tag{2.18}$$

$$\frac{I_{xs}}{I_{qd}} = \frac{\Delta\alpha d(\exp)}{(1+g)\alpha_g\alpha_e d(\mathrm{erf})} \tag{2.19a}$$

$$\underset{0}{\overset{dv}{\to}} \frac{\pi^{1/2}v_L\Delta\alpha}{(1+g)\alpha_g\alpha_e}, \tag{2.19b}$$

$$\frac{I_{1e}}{I_{qd}} = \frac{\pi^{1/2}}{(g+z)\alpha_e}. \tag{2.20}$$

The cases $g=0$, $z>0$ and $z=0$, $g>0$ are trivially obtained from these results.

III. 1D BLID

A. Exact analysis

In the language of parts I and II, we set $|v_z|=0$ and $V=0$, respectively, for simplicity but bearing in mind the reservation concerning this (incorrect) procedure expressed in part I; with Eqs. (2.1) and (2.2), the analog of Eq. (2.3) is

$$\partial F_g/\partial t = (c_g\alpha_g\beta_g + c_e\alpha_e z\beta_e)\theta v - \alpha_g\theta v F_g + \gamma F_e - q(F_g - F_e), \tag{3.1a}$$

$$\partial F_e/\partial t = c_e\alpha_e(1-z)\beta_e\theta v - \alpha_e\theta v F_e - \gamma F_e + q(F_g - F_e), \tag{3.1b}$$

where α_j, β_j, and z stand for α_{bj}, β_{vj}, and z_b, respectively, and $v=|v_x|$.

The steady-state solution of Eq. (3.1) may be written

$$F_j = \frac{A_j v + Bq + E_j\gamma}{Cv + Dq + F\gamma}, \tag{3.2}$$

where the five constants B, D, E_j, F are given by (I.5.2c, I.5.2e, I.5.2f, I.5.2g, I.5.2h); the three constants A_j, C are given by (I.5.2a, I.5.2b, I.5.2d) except that they now carry a factor θ, for example $C=\alpha_g\alpha_e\theta$. These differences result in the following different definitions of a, g:

$$a = q_0\Sigma\alpha/\alpha_g\alpha_e\theta, \tag{3.3a}$$

$$g = \gamma/\alpha_e\theta. \tag{3.3b}$$

With the definitions (2.8) and (2.9), the analog of Eq. (2.7) is

$$F_j = S_j + \frac{2aX_j + 2Z_j}{v + a\Delta h + g}, \tag{3.4}$$

and of Eq. (2.10) is

$$F_j^{(\mathrm{in})} = S_j + \frac{2W_j}{v+w}, \tag{3.5a}$$

$$F_j^{(\mathrm{out})} = S_j + \frac{2Z_j}{v+g}, \tag{3.5b}$$

again with W_j defined by (II.A2) and with the definition (I.E6) of w recalled.

Now we need the integrals I_{1j}, I_{qj}, I_{vj}, and I_{xj}, and the analysis is more complicated than that for SLID in Sec. II [because v appears in Eq. (3.5) but not in Eq. (2.10)]. We need a generalization of the function $G(u)$ defined by (I.D1), that is, a function $G(s,u)$ of two variables defined by

$$G(s,u) = \frac{2u}{\pi^{1/2}}\int_0^s dx \frac{e^{-x^2}}{|x|+u} \tag{3.6}$$

with $G(s,0)=0$, and we stretch somewhat the meaning of ''closed-form functions'' by including $G(s,u)$ among them. We see that $G(s,u)$ is an odd function of s, and is related to $G(u)$ by

$$G(u)=G(\infty,u). \tag{3.7}$$

The following extension of our definition (I.4.10) of $d(\mathrm{erf})$ and additional definitions of quantities dG_ξ, DG_ξ are useful in presenting results of the analysis [7,8]:

$$D(\mathrm{erf})=\mathrm{erf}|v_b|-\mathrm{erf}|v_a|, \tag{3.8}$$

$$dG_\xi=G(v_b,\xi)-G(v_a,\xi), \tag{3.9a}$$

$$DG_\xi=G(|v_b|,\xi)-G(|v_a|,\xi). \tag{3.9b}$$

For example, the analog of Eq. (2.10) may now be written as follows:

$$I_{1j}=S_j+Y_j(2G_g-dG_g)+W_j dG_w/w, \tag{3.10a}$$

$$I_{qj}=[S_j d(\mathrm{erf})/2+W_j dG_w/w]q_0, \tag{3.10b}$$

$$I_{vj}=\pi^{-1/2}S_j+Z_j(2H_g+dG_g)-W_j dG_w + aX_j d(\mathrm{erf}), \tag{3.10c}$$

$$I_{xj}=Z_j DG_g-W_j DG_w+aX_j D(\mathrm{erf}). \tag{3.10d}$$

The (exact) solution is now obtained by solving the system of (five) equations

$$c_j=I_{1j}, \tag{3.11a}$$

$$c_j\beta_j=\pi^{1/2}I_{vj}, \tag{3.11b}$$

$$c_g+c_e=1 \tag{3.11c}$$

for the (four) quantities c_j,β_j: Eq. (3.11c) must be used with three or all of Eqs. (3.11a) and (3.11b) and having five equations for four quantities gives a good check. Exact results in closed form, that is, the analogs of those in Appendix A, are presented in Appendix B, with the definition (I.F2) of y recalled, and the basic results (2.13)–(2.15) apply here also. In Eq. (2.13), κ must now be interpreted as another appropriate choice, $\kappa=\kappa_b$, of transmission parameter.

B. Physically motivated special cases

There is a problem here, analogous to that found in part I with I_{1e} and I_{xs}, with the two limits $a\to 0$ and $g\to 0$ of I_{1e}. If $v_x=0$ is in the excitation interval $[v_a,v_b]$, that is, if $v_a\leqslant 0\leqslant v_b$ [9], then there is a logarithmic divergence, of the type occurring in part I, in the result for I_{1e}. Essentially, the problem is that the limit of G_w/w as $a\to 0$, $g\to 0$ or vice versa may be needed. Of course, $\lim_{g\to 0}(G_w/w)=G_a/a$ and $\lim_{a\to 0}(G_w/w)=G_g/g$, and taking the next limit in either case gives the problem.

We let $(dG_\xi/\xi)_0$ stand for $(dG_a/a)_0$ and $(dG_g/g)_0$ [7], which in turn stand for the leading terms, up to but excluding those of $O(a)$ and $O(g)$, respectively, in the expansions of dG_a/a and dG_g/g. Where an exponential integral $E(z)$ is defined by [10]

$$E(z)=\int_z^\infty dx e^{-x}/x, \tag{3.12}$$

it follows from Eq. (3.6) that the required results are as follows:

$$v_a v_b>0:(dG_\xi/\xi)_0=\pi^{-1/2}[E(v_a^2)-E(v_b^2)]\mathrm{sgn}(v_a), \tag{3.13a}$$

$$v_a v_b=0:(dG_\xi/\xi)_0=-\pi^{-1/2}[2\ln\xi+\sigma+E(v_a^2+v_b^2)], \tag{3.13b}$$

$$v_a v_b<0:(dG_\xi/\xi)_0=-\pi^{-1/2}[4\ln\xi+2\sigma+E(v_a^2)+E(v_b^2)], \tag{3.13c}$$

where it is understood that the next term in each of the three expansions is $O(\xi)$.

In order to write analogs of Eq. (2.19b), that is, results for the case $dv\to 0$, we need to make the following interpretations:

$$D(\mathrm{erf})/d(\mathrm{erf})\to\mathrm{sgn}(v_L), \tag{3.14a}$$

$$\mathrm{sgn}(v_L)DG_\xi/d(\mathrm{erf})\to dG_\xi/d(\mathrm{erf}) \tag{3.14b}$$

$$\to\xi/(|v_L|+\xi), \tag{3.14c}$$

$$(dG_\xi/\xi)_0/d(\mathrm{erf})\to|v_L|^{-1}, \tag{3.14d}$$

which follow from (I.4.10), (3.8), (3.9) and (3.13a). We may now write the analogs of Eqs. (2.18)–(2.20) as follows:

$$\frac{I_{qd}}{q_0 dv}\underset{0}{\overset{q_0}{\to}}\frac{d(\mathrm{erf})}{2dv}\underset{0}{\overset{dv}{\to}}\frac{e^{-v_L^2}}{\pi^{1/2}}, \tag{3.15}$$

$$\frac{I_{xs}}{I_{qd}}\underset{0}{\overset{q_0}{\to}}\frac{(D(\mathrm{erf})-DG_g)\Delta\alpha}{\alpha_g\alpha_e\theta\, d(\mathrm{erf})} \tag{3.16a}$$

$$\underset{0}{\overset{dv}{\to}}\frac{v_L\Delta\alpha}{(|v_L|+g)\alpha_g\alpha_e\theta}, \tag{3.16b}$$

$$\frac{I_{1e}}{I_{qd}}\underset{0}{\overset{q_0}{\to}}\frac{dG_g/g+\psi H_g\, d(\mathrm{erf})}{(1+g\psi H_g)\alpha_e\theta\, d(\mathrm{erf})} \tag{3.17a}$$

$$\underset{0}{\overset{dv}{\to}}\frac{(|v_L|+g)^{-1}+\psi H_g}{(1+g\psi H_g)\alpha_e\theta}, \tag{3.17b}$$

where ψ is defined by

$$\psi=y/z=\pi^{1/2}(1-z)/z, \tag{3.18}$$

and where we note that Eq. (3.15) has the same form as Eq. (2.18).

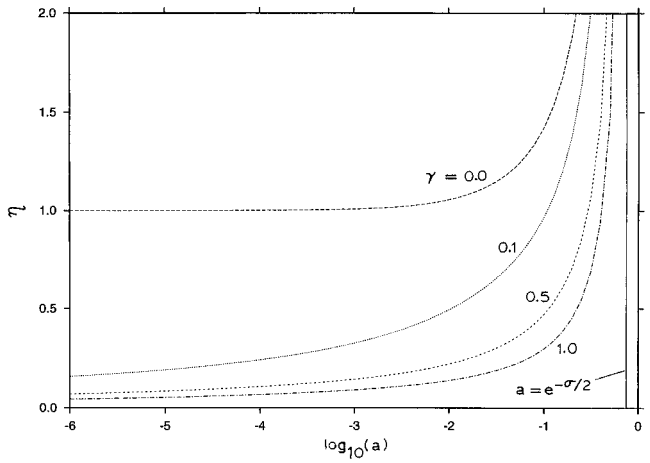

FIG. 1. Exact results from BLID on the dependence of η, defined by Eq. (3.21), on γ and a. The parameters are $v_a=0$, $v_b=\infty$, $z_b=1$, and $\alpha_{be}=0.999$; the results are independent of α_{bg} and θ. The vertical solid line at $\log_{10}a\approx-0.13$ shows where $a=e^{-\sigma/2}$, which is where the curve for every value of γ has a vertical asymptote. The meanings of the symbols are as follows: η is defined by Eq. (3.22), γ is the spontaneous decay rate parameter, $[v_a,v_b]$ is the interval of v_x in which the laser excitation function $q(v_x)$ equals q_0, a is defined by Eq. (3.3a), z_b is the diffuse-scattering quenching fraction, $\alpha_{bg,e}$ are the ground- and excited-state accommodation coefficients, θ is the average inverse mean free path for molecule-buffer particle collisions, and σ (≈0.58) is Euler's constant.

Use of Eq. (3.13) would be necessary if, for example, the small-g behavior of Eq. (3.17a) were studied. For the case $g=0$, $z>0$, applied in Eq. (3.16) and (3.17) before the limits are taken, the results are

$$\frac{I_{xs}}{I_{qd}} \underset{0}{\overset{q_0}{\rightarrow}} \frac{D(\mathrm{erf})\,\Delta\alpha}{\alpha_g\alpha_e\theta\,d(\mathrm{erf})} \tag{3.19a}$$

$$\underset{0}{\overset{dv}{\rightarrow}} \frac{\mathrm{sgn}(v_L)\Delta\alpha}{\alpha_g\alpha_e\theta}, \tag{3.19b}$$

$$\frac{I_{1e}}{I_{qd}} \underset{0}{\overset{q_0}{\rightarrow}} \frac{(dG_a/a)_0+\psi\,d(\mathrm{erf})}{\alpha_e\theta\,d(\mathrm{erf})} \tag{3.20a}$$

$$\underset{0}{\overset{dv}{\rightarrow}} \frac{|v_L|^{-1}+\psi}{\alpha_e\theta}. \tag{3.20b}$$

Mainly in order to check the peculiar-looking result (3.20a) combined with (3.13b) with $\xi=a$, which is for $g=0$, that is, $\gamma=0$ from Eq. (3.3b), we present in Fig. 1 exact results, for several values of γ, for the ratio I_{1e}/I_{qd} with $v_a=0$, $v_b=\infty$, $z=1$, for which Eq. (3.20a) with Eq. (3.13b) reduces to

$$\eta \underset{0}{\overset{a}{\rightarrow}} 1, \tag{3.21}$$

where η is defined by

$$\eta=\frac{-\pi^{1/2}\alpha_e\theta}{(2\ln a+\sigma)}\frac{I_{1e}}{I_{qd}}. \tag{3.22}$$

For the $\gamma=0$ curve in Fig. 1, we see that η is essentially indistinguishable from unity for a less than about 0.001, in nice agreement with Eq. (3.21).

IV. COMPARISON OF SLID AND BLID

From Eqs. (2.18)–(2.20) and (3.15)–(3.17), with subscripts $(s,b)\equiv$ (SLID, BLID) attached to variables in an obvious notation, we get the following results for the important quantities, with $\theta\equiv\theta_b$:

$$\frac{(I_{qd}/q_0)_b}{(I_{qd}/q_0)_s} \underset{0}{\overset{q_0}{\rightarrow}} 1, \tag{4.1}$$

$$\frac{(I_{xs}/I_{qd})_b}{(I_{xs}/I_{qd})_s} \underset{0}{\overset{q_0}{\rightarrow}} \frac{(1+g_s)[D(\mathrm{erf})-DG_{gb}]}{\theta\,d(\exp)}\frac{\alpha_{sg}\alpha_{se}}{\alpha_{bg}\alpha_{be}}\frac{\Delta\alpha_b}{\Delta\alpha_s} \tag{4.2a}$$

$$\underset{0}{\overset{dv}{\rightarrow}} \frac{\pi^{-1/2}}{\theta}\frac{(1+g_s)}{(|v_L|+g_b)}\frac{\alpha_{sg}\alpha_{se}}{\alpha_{bg}\alpha_{be}}\frac{\Delta\alpha_b}{\Delta\alpha_s}, \tag{4.2b}$$

$$\frac{(I_{1e}/I_{qd})_b}{(I_{1e}/I_{qd})_s} \underset{0}{\overset{q_0}{\rightarrow}} \frac{(z_s+g_s)[dG_{gb}/g_b+\psi_bH_{gb}\,d(\mathrm{erf})]}{\pi^{1/2}(1+g_b\psi_bH_{gb})\,\theta\,d(\mathrm{erf})}\frac{\alpha_{se}}{\alpha_{be}} \tag{4.3a}$$

$$\underset{0}{\overset{dv}{\rightarrow}} \frac{(z_s+g_s)[(|v_L|+g_b)^{-1}+\psi_bH_{gb}]}{\pi^{1/2}(1+g_b\psi_bH_{gb})\,\theta}\frac{\alpha_{se}}{\alpha_{be}}, \tag{4.3b}$$

where use of Eq. (3.13) would again be made to study the small-g_b behavior of Eq. (4.3a). For the case $g_s=g_b=0$, $z_sz_b>0$, the analog of Eq. (4.2) is trivially obtained ($DG_{gb}=0$), while that of Eq. (4.3) is

$$\frac{(I_{1e}/I_{qd})_b}{(I_{1e}/I_{qd})_s} \underset{0}{\overset{q_0}{\rightarrow}} \frac{[z_b(dG_{ab}/a_b)_0+y_b\,d(\mathrm{erf})]}{\pi^{1/2}\theta\,d(\mathrm{erf})}\frac{z_s\alpha_{se}}{z_b\alpha_{be}} \tag{4.4a}$$

$$\underset{0}{\overset{dv}{\rightarrow}} \frac{(z_b|v_L|^{-1}+y_b)}{\pi^{1/2}\theta}\frac{z_s\alpha_{se}}{z_b\alpha_{be}}. \tag{4.4b}$$

If, as may be expected, θ is of order unity, the results for SLID and BLID are very similar for comparable values of the parameters, as is clear to a large extent from Eqs. (4.1)–(4.4); in fact, some of the results are more similar than they may appear to be at first sight. For example, let us specialize to the case $(v_a,v_b)=(0,\infty)$ which has been commonly used before [11,1,2], and consider the extreme values of the parameters z_s,z_b with $g_sg_b>0$. The results (4.1) and (4.2a) are independent of z_s,z_b, while (4.2b) and (4.3b) do not apply ($dv=\infty$); for $z_s=z_b=0$, Eq. (4.3a) becomes simply

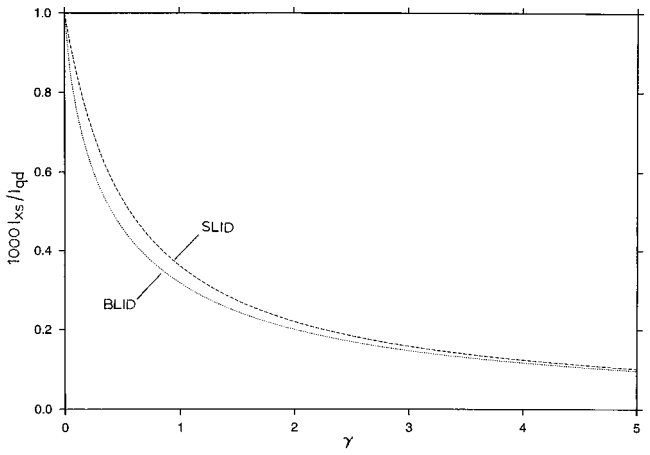

FIG. 2. Comparison of exact results from SLID and BLID on the dependence of the ratio I_{xs}/I_{qd} on γ. The parameters are $v_a=0$, $v_b=\infty$, $q_0=0.001$, $\alpha_{sg}=\alpha_{bg}=1.0$, $\alpha_{se}=\alpha_{be}=0.999$, $\gamma_s=\gamma_b=\gamma$, $z_s=z_b=0$, and $\theta=1$. The meanings of the symbols are as in Fig. 1, with z_s and $\alpha_{sg,e}$ the analogs for SLID of z_b and $\alpha_{bg,e}$, and $\gamma_{s,b}$ the values of γ used for SLID,BLID.

$$\frac{(I_{1e}/I_{qd})_b}{(I_{1e}/I_{qd})_s} \underset{0}{\overset{q_0}{\rightarrow}} \frac{\gamma_s}{\gamma_b}, \tag{4.5}$$

and, for $z_s=z_b=1$,

$$\frac{(I_{1e}/I_{qd})_b}{(I_{1e}/I_{qd})_s} \underset{0}{\overset{q_0}{\rightarrow}} \left(\frac{\alpha_{se}}{\pi^{1/2}}+\gamma_s\right)\frac{G_{gb}}{\gamma_b}, \tag{4.6}$$

in both of which we have used the definitions (2.5b) and (3.7b).

These similarities are illustrated in Figs. 2 and 3: Fig. 2 shows exact results for I_{xs}/I_{qd} for both SLID and BLID as functions of γ with $\gamma_s=\gamma_b=\gamma$ and $z_s=z_b=0$; Fig. 3 shows analogous results for I_{1e}/I_{qd}, but with $z_s=z_b=1$.

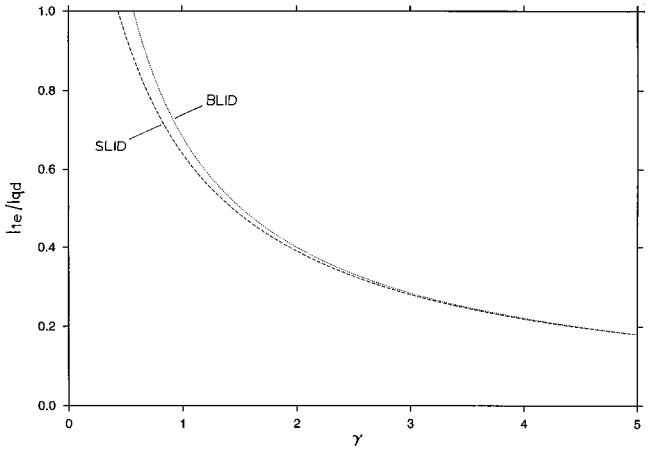

FIG. 3. As in Fig. 2, except that the ratio I_{1e}/I_{qd} is shown, and $z_s=z_b=1$.

V. CONCLUSIONS

Given the model equations (2.10) [1] for LID in one dimension, with our transverse-molecular-motion assumption for SLID, we have shown how exact solutions may be obtained for both SLID and BLID; exact solutions for SLID (in three dimensions) were presented in parts I and II.

Numerical results may be obtained by iteration of the analytical steady-state equations, and, independently, by integration of the MBREs with respect to time, using procedures analogous to those described in Secs. IV A and IV B, respectively, of part I and, implicitly, in Sec. II of part II. These numerical calculations are important, if only because they give rigorous checks on the working; however, they are also enjoyable, particularly the study, via the integration procedure, of the relaxation to steady state. Overall, if a numerical procedure is desired, then iteration is probably the best because it is easy to program, is very fast, and produces results to arbitrary accuracy. It is (hopefully) clear from part I how to implement these procedures, which is the reason why we have not discussed them in detail here.

The similarities of the results for SLID and BLID, brought out, discussed, and illustrated in Sec. IV and Figs. 2 and 3, are interesting, but, with hindsight, perhaps not unexpected. Contact of the work here with experiments and other work is best made using the basic results (2.13)–(2.15) together with appropriate values of the important quantities I_{qd}/a, I_{xs}/I_{qd}, and I_{1e}/I_{qd}. Questions of the effects of model dimensionality on the results, particularly for BLID, are obviously important, but must wait to be answered in future papers.

ACKNOWLEDGMENTS

The author is indebted to Professor C. B. Collins for discussions on the analytical properties of the function $G(s,u)$. The work was supported by the Natural Sciences and Engineering Council of Canada.

APPENDIX A: EXACT RESULTS FOR THE CASE OF 1D SLID

The results may be written with the notation (I.F1) as follows:

$$I_{1e}^{(\mathrm{num})}=(1+g)a\alpha_g\, d(\mathrm{erf}), \tag{A1}$$

$$I_{qe}^{(\mathrm{num})}=[2(g+z)+(1-z)\, d(\mathrm{erf})] \times a^2\alpha_g^2\alpha_e\, d(\mathrm{erf}), \tag{A2}$$

$$I_{qd}^{(\mathrm{num})}=(1+g)(z+g)a\alpha_g\alpha_e\, d(\mathrm{erf}), \tag{A3}$$

$$I_{xg,e,s}^{(\mathrm{num})}=(z+g)a(-\alpha_e,\alpha_g,\Delta\alpha)\, d(\mathrm{exp}), \tag{A4}$$

$$I_{1e}^{(\mathrm{den})}=2(1+w)(g+z)\Sigma\alpha + (2\alpha_g+g\Delta\alpha-z\Sigma\alpha)a\, d(\mathrm{erf}), \tag{A5}$$

$$I_{qe}^{(\mathrm{den})}/(2\Sigma\alpha)=I_{qd}^{(\mathrm{den})}=I_{xk}^{(\mathrm{den})}=\pi^{1/2}I_{1e}^{(\mathrm{den})}. \tag{A6}$$

APPENDIX B: EXACT RESULTS FOT THE CASE OF 1D BLID

With the definitions

$$T = w\, d(\mathrm{erf}) - a\, dG_w, \tag{B1}$$

$$U = 2H_g + dG_g, \tag{B2}$$

and, again with the notation (I.F1), we may write the results as follows:

$$I_{1e}^{(\mathrm{num})}/(a\alpha_g) = 2z dG_w - y[aU dG_w - (wU - g dG_w) d(\mathrm{erf})], \tag{B3}$$

$$I_{qe}^{(\mathrm{num})}/(a^2\alpha_g^2\alpha_e\theta) = 2z dG_w + y\{g dG_w[U - d(\mathrm{erf})] - w[dG_w - d(\mathrm{erf})]d(\mathrm{erf})\}, \tag{B4}$$

$$I_{qd}^{(\mathrm{num})}/(a\alpha_g\alpha_e\theta) = yg[UT - g\, dG_w\, d(\mathrm{erf})] + 2zT, \tag{B5}$$

$$I_{xg,e,s}^{(\mathrm{num})}/(aw) = \{2z[D(\mathrm{erf}) - DG_w] + yg[DG_g\, dG_w - UDG_w + (U - dG_w)D(\mathrm{erf}) - (DG_g - DG_w)d(\mathrm{erf})]\} \times (-\alpha_e, \alpha_g, \Delta\alpha), \tag{B6}$$

$$I_{1e}^{(\mathrm{den})} = I_{qe}^{(\mathrm{den})}/\Sigma\alpha = I_{qd}^{(\mathrm{den})} = I_{xk}^{(\mathrm{den})} = y(2g[w(U - 2dG_w) + g dG_w]\Sigma\alpha - a^2 dG_w[4\alpha_g + (U-2)\Delta\alpha] + \{4w\alpha_g + [w(U-2) - g dG_w]\Delta\alpha\} a\, d(\mathrm{erf})) + 2z(2w\Sigma\alpha + a dG_w\Delta\alpha). \tag{B7}$$

[1] F.O. Goodman, Phys. Rev. A **65**, 063409 (2002).

[2] F.O. Goodman, Phys. Rev. A **65**, 063410 (2002).

[3] Remember that (unstarred, starred) variables are (dimensionless, dimensionful).

[4] Remember the notation $\langle\xi\rangle_m$ introduced in (I.B2a).

[5] In 3D SLID with a cell of uniform, but otherwise arbitrary, cross section having area A and perimeter P, the average molecule-surface collision frequency per molecule of a Maxwellian gas is given from $2\pi^{1/2}A\langle\nu\rangle_m = P$. With $\ell_c^* = Z^*$ in FP geometry, $A = Y$, $P = 2(Y+1) \approx 2Y$, and hence $\langle\nu\rangle_m \approx \pi^{-1/2}$; with $\ell_c^* = R^*$ in CC geometry [2], $A = \pi$, $P = 2\pi$, and hence $\langle\nu\rangle_m = \pi^{-1/2}$.

[6] In nD BLID, the average molecule-buffer particle collision frequency per molecule of a Maxwellian gas, with average inverse mean free path $\theta^{(n)}$, is given from $\Gamma(n/2)\langle\nu^{(n)}\rangle_m = \Gamma((n+1)/2)\theta^{(n)}$. In 1D BLID, then, $\pi^{1/2}\langle\nu\rangle_m = \theta$.

[7] Remember that ξ is a dummy variable.

[8] Note that, if $v_a \geqslant 0$ [9], then $D(\mathrm{erf}) = d(\mathrm{erf})$ and $DG_\xi = dG_\xi$.

[9] Remember that $v_a < v_b$.

[10] Our $E(z)$ is denoted by $E_1(z)$ in *Handbook of Mathematical Functions*, edited by M. Abramowitz and I.A. Stegun (Dover, New York, 1972), formula 5.1.1.

[11] M.A. Vaksman and I. Podgorski, Can. J. Phys. **74**, 25 (1996).