Quantum discord and Maxwell's demons

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Quantum discord was proposed as an information-theoretic measure of the "quantumness" of correlations. I show that discord determines the difference between the efficiency of quantum and classical Maxwell's demons—that is, entities that can or cannot measure nonlocal observables or carry out conditional quantum operations—in extracting work from collections of correlated quantum systems.

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I. DEMONS

Maxwell's demon [1] was introduced to explore the role of information and, more generally, to investigate the place of "intelligent observers" in physics. In modern discussions of the subject [2], "intelligence" is often regarded as predicated upon or even synonymous with the information processing ability—with computing. Thus, Maxwell's demon is frequently modeled by a Turing machine—a classical computer—endowed with the ability to measure and act depending on the outcome. The role of a demon is to implement an appropriate conditional dynamics—to react to the state of the system as revealed through its correlation with the state of the apparatus.

It is now known that quantum logic—i.e., the logic employed by quantum computers—is in some applications more powerful than its classical counterpart. It is therefore intriguing to enquire whether a quantum demon—an entity that can measure nonlocal states and implement quantum conditional operations—could be more efficient than a classical one. I show that quantum demons can extract more work than classical demons from correlations between quantum systems, and that the difference is given by the *quantum discord*, a recently introduced [3–5] measure of the "quantumness" of correlations.

Maxwell's demon sets up a useful conceptual framework that provides an operational interpretation of discord. The role played by the quantum demon—carrying out conditional quantum operations on pairs of systems—could be also played by a classical device that can outright measure nonlocal quantum observables. This is especially apparent in Sec. IV where we alternate between the quantum and classical demon on one hand, and "Alice and Bob" on the other. The real point of employing demons is to draw attention to the thermodynamic (and information-theoretic) costs of various operations and—in a sense—to hold Alice and Bob accountable for their thermodynamic expenditures which are usually simply ignored.

II. DISCORD

Quantum discord [3–5] is the difference between two classically identical formulas that measure the information content of a pair of quantum systems. Several closely related variants can be obtained starting from the original definition [3] given in terms of mutual information [6]. Mutual information quantifies the strength of correlations between, say, the apparatus $\mathcal{A}$ and the system $\mathcal{S}$,

$$I(\mathcal{S}:\mathcal{A}) = H(\mathcal{S}) + H(\mathcal{A}) - H(\mathcal{S},\mathcal{A}). \tag{1}$$

It measures the difference between the missing information about the two objects when they are taken separately, $H(\mathcal{S}) + H(\mathcal{A})$, and jointly, $H(\mathcal{S},\mathcal{A})$ (see Fig. 1). In the extreme case $\mathcal{S}$ and $\mathcal{A}$ may be identical—e.g., copies of the same

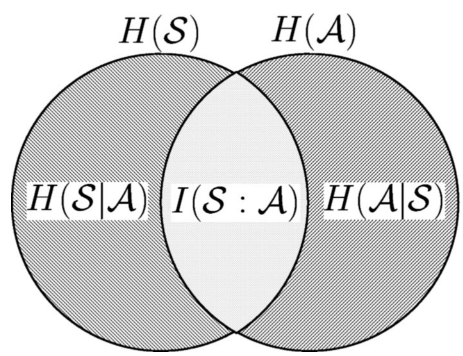

FIG. 1. Information-theoretic measures of the relationship between $\mathcal{A}$ and $\mathcal{S}$ can be illustrated by means of the *Venn diagram* shown above. Shaded areas represent various uncertainties. Joint entropy $H(\mathcal{S},\mathcal{A})$ is the measure of uncertainty about the combined state of $\mathcal{S}$ and $\mathcal{A}$. Individual circles correspond to the uncertainty about $\mathcal{S}$ and $\mathcal{A}$. When their states are correlated, the two circles overlap. Mutual information $I(\mathcal{S}:\mathcal{A})$ is the area of that overlap. Conditional entropy $H(\mathcal{S}|\mathcal{A})$ is one of the half-moons above—the one left when the lens corresponding to the mutual information $I(\mathcal{S}:\mathcal{A})$ is subtracted from $H(\mathcal{S})$. Obviously, $H(\mathcal{S},\mathcal{A}) = H(\mathcal{A}) + H(\mathcal{S}|\mathcal{A}) = H(\mathcal{S}) + H(\mathcal{A}|\mathcal{S}) = H(\mathcal{S}) + H(\mathcal{A}) - I(\mathcal{S}:\mathcal{A})$. These equalities are predicated on the classical assumption that the states of $\mathcal{S}$ and $\mathcal{A}$ exist objectively, and, thus, a measurement need not disturb them. In quantum theory this is not the case: a measurement will, in general, redefine the state of the measured object, even for an "outsider" who does not know its outcome. Indeed, for a generic quantum state of the pair $\mathcal{SA}$, a measurement of $\mathcal{A}$ alone would increase the uncertainty of the outsider, i.e., would increase the entropy he attributes to the pair. This is a consequence of the difference between the nature of joint states in classical physics (where they are represented by Cartesian products of subspaces of the constituents) and quantum physics (where they exist in a tensor product of the two Hilbert spaces). It has profound effects on the accessibility of the information and leads to a difference in the efficiency of Maxwell's demons.

book, or a state of the apparatus pointer $\mathcal{A}$ after a perfect but as yet unread measurement of $\mathcal{S}$. Then the joint entropy $H(\mathcal{S},\mathcal{A})$ is equal to $H(\mathcal{A})=H(\mathcal{S})$, so $I(\mathcal{S}:\mathcal{A})=H(\mathcal{A})$. By contrast, when the two objects are not correlated, $H(\mathcal{S},\mathcal{A})=H(\mathcal{S})+H(\mathcal{A})$, and $I(\mathcal{S}:\mathcal{A})=0$.

The other formula for mutual information employs classical identity for joint entropy [6],

$$H(\mathcal{S},\mathcal{A})=H(\mathcal{A})+H(\mathcal{S}|\mathcal{A})=H(\mathcal{S})+H(\mathcal{A}|\mathcal{S}). \tag{2}$$

Above, $H(\mathcal{S}|\mathcal{A})$ is the conditional entropy—e.g., the measure of the lack of information about the state of $\mathcal{S}$, given the state of $\mathcal{A}$. Substituting this in Eq. (1) leads to an asymmetric looking formula for mutual information,

$$J_{\mathcal{A}}(\mathcal{S}:\mathcal{A})=H(\mathcal{S})+H(\mathcal{A})-[H(\mathcal{A})+H(\mathcal{S}|\mathcal{A})]. \tag{3}$$

We have refrained from carrying out the obvious cancellation above that would have yielded $J_{\mathcal{A}}(\mathcal{S}:\mathcal{A})=H(\mathcal{S})-H(\mathcal{S}|\mathcal{A})$ for a reason that will be soon apparent.

Discord is defined as

$$\begin{aligned}\delta(\mathcal{S}|\mathcal{A})&=I(\mathcal{S}:\mathcal{A})-J_{\mathcal{A}}(\mathcal{S}:\mathcal{A})\\&=[H(\mathcal{A})+H(\mathcal{S}|\mathcal{A})]-H(\mathcal{S},\mathcal{A}).\end{aligned} \tag{4}$$

Classically, discord disappears as a consequence of Eq. (2)—information about a collection of classical objects can be acquired one object at a time. In quantum theory, however, measurements can modify the quantum state [7]. Thus, in order to properly define conditional entropy, one must specify how the apparatus is ''interrogated'' about $\mathcal{S}$. After a measurement of the observable with eigenstates $\{|A_k\rangle\}$, observer's own description of the pair is the conditional density matrix,

$$\rho_{\mathcal{S}\mathcal{A}|A_k\rangle}=\rho_{\mathcal{S}|A_k\rangle}\otimes|A_k\rangle\langle A_k|. \tag{5}$$

Given an outcome $|A_k\rangle$, he will attribute $\rho_{\mathcal{S}|A_k\rangle}=\mathrm{Tr}_{\mathcal{A}}\langle A_k|\rho_{\mathcal{S}\mathcal{A}}|A_k\rangle/p_{\mathcal{A}}(k)$ to $\mathcal{S}$ with the probability $p_{\mathcal{A}}(k)=\mathrm{Tr}\langle A_k|\rho_{\mathcal{S}\mathcal{A}}|A_k\rangle$. Even for an outsider (who has not yet found out the outcome), postmeasurement density matrix $\rho'_{\mathcal{S}\mathcal{A}}$ usually differs from the premeasurement $\rho_{\mathcal{S}\mathcal{A}}$. This outsider's state of knowledge should be contrasted with the viewpoint of the insider who made the measurement. Insider knows that the apparatus is in the state $|A_k\rangle$. Outsider does not, so he obtains his postmeasurement $\rho'_{\mathcal{S}\mathcal{A}}$ by averaging over the outcomes,

$$\rho'_{\mathcal{S}\mathcal{A}}=\sum_k p_{\mathcal{A}}(k)\rho_{\mathcal{S}|A_k\rangle}\otimes|A_k\rangle\langle A_k|. \tag{6}$$

Outsider's description of the pair is unaffected by the insiders measurements—$\rho'_{\mathcal{S}\mathcal{A}}=\rho_{\mathcal{S}\mathcal{A}}$—only when the measured observable commutes with $\rho_{\mathcal{S}\mathcal{A}}$. We shall find outsider's viewpoint useful because it represents a statistical ensemble of all possible outcomes.

In quantum physics, one definition denoted by $J_{\mathcal{A}}(\mathcal{S}:\mathcal{A}_{\{|A_k\rangle\}})$ is the *locally accessible mutual information*. It uses Eq. (3) with the joint entropy given by

$$H_{\mathcal{A}}(\mathcal{S},\mathcal{A}_{\{|A_k\rangle\}})=[H(\mathcal{A})+H(\mathcal{S}|\mathcal{A})]_{\{|A_k\rangle\}}. \tag{7}$$

where $\{|A_k\rangle\}$ is the eigenbasis of the to-be-measured observable of the apparatus. Another acceptable and completely quantum definition of $I(\mathcal{S}:\mathcal{A})$ relies on the the von Neumann entropy of the density matrix $\rho_{\mathcal{S}\mathcal{A}}$ describing the joint state. Then, in bits

$$H(\mathcal{S},\mathcal{A})=-\mathrm{Tr}\,\rho_{\mathcal{S}\mathcal{A}}\lg\rho_{\mathcal{S}\mathcal{A}}=-\sum_l p_{\mathcal{S}\mathcal{A}}(l)\lg p_{\mathcal{S}\mathcal{A}}(l), \tag{8}$$

where $\lg=\log_2$, and the probabilities $p_{\mathcal{S}\mathcal{A}}(l)$ are the eigenvalues of $\rho_{\mathcal{S}\mathcal{A}}$ that describes the correlated pair. These eigenvalues always exist, but, in general, correspond to entangled quantum states $|\psi_{\mathcal{S}\mathcal{A}}(l)\rangle$ in the joint Hilbert space of $\mathcal{S}$ and $\mathcal{A}$. Such states cannot be found out through sequences of local measurements starting with just one subsystem of the pair—say, $\mathcal{A}$. This fundamental difference between the quantum and the classical realm (where such ''piecewise'' investigation is always possible and need not disturb the state of the pair) is responsible for nonzero discord.

A simple example of this situation is a perfectly entangled Bell state,

$$|\psi_{\mathcal{S}\mathcal{A}}\rangle=(|0_{\mathcal{S}}0_{\mathcal{A}}\rangle+|1_{\mathcal{S}}1_{\mathcal{A}}\rangle)/\sqrt{2}. \tag{9a}$$

Clearly, $\rho_{\mathcal{S}\mathcal{A}}=|\psi_{\mathcal{S}\mathcal{A}}\rangle\langle\psi_{\mathcal{S}\mathcal{A}}|$ is pure—the pair is in the state $|\psi_{\mathcal{S}\mathcal{A}}\rangle$. Hence, in accord with Eq. (8), $H(\mathcal{S},\mathcal{A})=0$. On the other hand, $\rho_{\mathcal{A}(\mathcal{S})}=Tr_{\mathcal{S}(\mathcal{A})}\rho_{\mathcal{S}\mathcal{A}}=\mathbf{1}_{\mathcal{A}(\mathcal{S})}/2$, where $\mathbf{1}$ is the unit matrix in the appropriate Hilbert space, so that $H(\mathcal{A})=H(\mathcal{S})=1$. Consequently, $I(\mathcal{S}:\mathcal{A})=2$, but the asymmetric mutual information is $J_{\mathcal{A}}(\mathcal{S}:\mathcal{A})=1$. This is because the joint information $H_{\mathcal{A}}(\mathcal{S},\mathcal{A}_{\{|A_k\rangle\}})$ defined with reference to any measurement on a $\mathcal{A}$, Eq. (5), is a sum of $H(\mathcal{A})=1$ and $H(\mathcal{S}|\mathcal{A})=0$. In our example, both of these quantities are independent of the basis because of the symmetry of Bell states.

Readers are invited to verify that a classical correlation in

$$\rho_{\mathcal{S}\mathcal{A}}=(|0_{\mathcal{S}}0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}0_{\mathcal{S}}|+|1_{\mathcal{S}}1_{\mathcal{A}}\rangle\langle 1_{\mathcal{A}}1_{\mathcal{S}}|)/2 \tag{9b}$$

results in zero discord, but only when the preferred basis $\{|A_k\rangle\}=\{|0\rangle,|1\rangle\}$ is employed. The entangled state of Eq. (9a) could be converted into the mixture of Eq. (9b) through einselection of the preferred (pointer) basis [4,8–11] or—and this is why decoherence can be regarded as monitoring by the environment—through a measurement with an undisclosed outcome carried out in the same pointer basis $\{|A_k\rangle\}=\{|0\rangle,|1\rangle\}$.

In general, the ignorance of the outsider cannot decrease (but may increase) as a result of a measurement of a known observable (by the insider), as the outsider does not know the outcome [12]. Hence,

$$\delta(\mathcal{S}|\mathcal{A}_{\{|A_k\rangle\}})=H_{\mathcal{A}}(\mathcal{S},\mathcal{A}_{\{|A_k\rangle\}})-H(\mathcal{S},\mathcal{A})\geqslant 0. \tag{10}$$

Discord disappears only when $\rho_{\mathcal{S}\mathcal{A}}$ remains unaffected by a partial measurement of $\{|A_k\rangle\}$ on the $\mathcal{A}$ end of the pair—when the information is locally accessible.

III. DEMONS AND DISCORD

The relevance of the discord for the performance of Maxwell's demon can be now appreciated. Demons are insiders. They use the acquired information to extract work from their surroundings. The traditional scenario starts with an interaction establishing initial correlation between the system and the apparatus. The demon then reads off the state of $\mathcal{A}$, and uses so acquired information about $\mathcal{S}$ to extract work by letting $\mathcal{S}$ expand throughout the available phase (or Hilbert) space of volume (dimension) $d_{\mathcal{S}}$ while in contact with the thermal reservoir at temperature T [1,2,13–19]. This yields

$$W^{+}=k_{B_2}T[\lg d_{\mathcal{S}}-H(\mathcal{S}|\mathcal{A})] \tag{11}$$

of work obtained at a price,

$$W^{-}=k_{B_2}TH(\mathcal{A}). \tag{12}$$

Above, k_{B_2} is the Boltzmann constant adapted to deal with the entropy expressed in bits and T is the temperature of the heat bath. The net gain is then

$$W=k_{B_2}T\{\lg d_{\mathcal{S}}-[H(\mathcal{A})+H(\mathcal{S}|\mathcal{A})]\}. \tag{13a}$$

The price W^{-} is the cost of restoring the apparatus to the initial ready-to-measure state. The significance of this "cost of erasure" for the second law was pointed out in the seminal paper of Szilard [13]. Its relevance in the context of information processing was elucidated and codified by Landauer [14].

It is now accepted that, because of the cost of erasure, neither classical [15–17] nor quantum [18–21] demons can violate the second law. However, a demon with a supply of empty memory (used to store measurement outcomes) can extract, on the average, W^{+} of work per step from a thermal reservoir. This strategy works, because, in effect, the demon is using its empty memory as a zero entropy (and, hence, $T=0$) reservoir. A memory block of size $d_{\mathcal{A}}$ is used up with each new measurement. This is expensive (and wasteful) and only fraudulent accounting (uncovered by Szilard and Landauer) that ignores thermodynamic cost of empty memory can create appearance of a violation of the second law.

To optimize performance, demon should use memory of $\mathcal{A}$ more efficiently. The obvious strategy here is to compress bits of the data after a sequence of measurements, freeing an unused block of $\Delta\mu$ bits. Demon can compress data A_k to the size given by $K(A_k)$, their algorithmic complexity [22]. With compression $\Delta\mu=\lg d_{\mathcal{A}}-K(A_k)$ memory bits per cycle are saved. Moreover, one can show that for long sequences of data the approximate equality $\langle K(A_k)\rangle \simeq H(\mathcal{A})$ becomes exact, so that the saved up memory is on the average $\Delta\mu=\lg d_{\mathcal{A}}-H(\mathcal{A})$. By being frugal, classical Maxwell's demon can gain, per step, net work of [17,21]

$$W=k_{B_2}T\{\lg d_{\mathcal{S}}d_{\mathcal{A}}-[H(\mathcal{A})+H(\mathcal{S}|\mathcal{A})]\}. \tag{13b}$$

When $\mathcal{S}$ and $\mathcal{A}$ are classically correlated so that Eq. (2) applies, this can be written as

$$W=k_{B_2}T\{\lg d_{\mathcal{S}}d_{\mathcal{A}}-H(\mathcal{S},\mathcal{A})\}. \tag{13c}$$

We note that the efficiency is ultimately determined by the information about $\mathcal{S}$ and $\mathcal{A}$ *accessible* to the demon, and that the same equation would have followed if we simply regarded the $\mathcal{SA}$ pair as a composite system, and the demon used it all up as a fuel.

The efficiency of demons is then determined by the accessible information about the pair $\mathcal{SA}$—the relevant joint entropy—and we have already seen in Eqs. (7) and (8) that in quantum physics it depends on how the information about the pair is acquired. A classical demon is local—it operates on the correlated quantum pair $\mathcal{SA}$ one system at a time. In this case the above sketch of the "standard operating procedure" applies with one obvious *caveat*: It needs to be completed by the specification of the basis demon measures in $\mathcal{A}$. The cost of erasure is still given by Eq. (12), also for classical demons extracting work from quantum systems [11–13], although the relevant $H(\mathcal{A})$ may increase as a result of decoherence that converts quantum entanglement into classical data [23]. Thus classical demons operating on pairs of quantum systems gain net work of

$$W^{C}/k_{B_2}T=\lg d_{\mathcal{SA}}-[H(\mathcal{A})+H(\mathcal{S}|\mathcal{A})]_{\{|A_k\rangle\}}. \tag{14}$$

The only difference between the classical Eq. (13a) and the quantum Eq. (14) is the obvious dependence on the basis $\{|A_k\rangle\}$ demon selects to measure. The expression in square brackets is the measure of the remaining (conditional) ignorance and of the cost of erasure. We shall be interested in the $\{|A_k\rangle\}$ that maximize W^{C}.

A quantum demon can typically extract more work—get away with lower costs of erasure—because its measurement can be carried out in a global basis in the combined Hilbert space of $\mathcal{SA}$ corresponding to observables that commute with the initial $\rho_{\mathcal{SA}}$ and avoid increase of entropy associated with decoherence [4,8–11,23]. The work that can be extracted after the apparatus gets reset to its ready-to-measure state is

$$W^{Q}/k_{B_2}T=\lg d_{\mathcal{SA}}-H(\mathcal{S},\mathcal{A}). \tag{15}$$

The other way to arrive at Eq. (15) is to use quantum demon in its capacity of a universal quantum computer, which, by definition, can transform any state in the Hilbert space into any other state (see Fig. 2 for an example of a model demon that operates on pairs of qubits). This allows the quantum demon to reversibly evolve entangled eigenstates of an arbitrary known $\rho_{\mathcal{SA}}$ into product states of some $\tilde{\rho}_{\mathcal{SA}}$ with the same eigenvalues, and, hence, same entropy. This $\rho_{\mathcal{SA}}$ can be then manipulated in a local basis that does not perturb its eigenstates, and, hence, as viewed by the outsider, it will not suffer any additional increase of entropy. The work extracted by the optimal quantum demon is limited simply by the basis-independent joint von Neumann entropy of the initial $\rho_{\mathcal{SA}}$, Eq. (8).

The difference between the efficiency of the quantum and classical demons can be now immediately characterized,

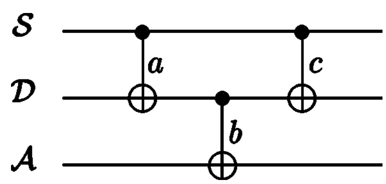

FIG. 2. A simple model of information processing built of the controlled-NOT (c-NOT) logical gates illustrates the origin of the difference between the efficiencies of the classical and quantum versions of Maxwell's demon. The aim of this demon (whose memory is marked by $\mathcal{D}$) is to use the correlation between $\mathcal{S}$ and $\mathcal{A}$ to purify their individual states, so that they can be used individually as fuel. To this end, the demon (a) finds out the state of the apparatus $\mathcal{A}$; (b) uses this information about $\mathcal{A}$ and the "promise" of the correlation—prior knowledge about the form of $\rho_{\mathcal{SA}}$ embodied in the logical circuit above—to decrease entropy of $\mathcal{S}$, so that $W^+ = k_{B_2}T[1-H(\mathcal{S}|\mathcal{A})]$ work can be extracted from $\mathcal{S}$; (c) resets his own memory to the initial ready-to-measure state, so that the same sequence of actions can be carried out cyclically on a whole ensemble of identical $\mathcal{SA}$ pairs. Thus, a quantum demon operating on an initially entangled state will result in an evolution,

$$\begin{aligned}(\alpha|0_{\mathcal{S}}0_{\mathcal{A}}\rangle+\beta|1_{\mathcal{S}}1_{\mathcal{A}}\rangle)|0_{\mathcal{D}}\rangle &= \alpha|0_{\mathcal{S}}0_{\mathcal{D}}0_{\mathcal{A}}\rangle+\beta|1_{\mathcal{S}}0_{\mathcal{D}}1_{\mathcal{A}}\rangle\\ &\Rightarrow^a \alpha|0_{\mathcal{S}}0_{\mathcal{D}}0_{\mathcal{A}}\rangle+\beta|1_{\mathcal{S}}1_{\mathcal{D}}1_{\mathcal{A}}\rangle\\ &\Rightarrow^b \alpha|0_{\mathcal{S}}0_{\mathcal{D}}0_{\mathcal{A}}\rangle+\beta|0_{\mathcal{S}}1_{\mathcal{D}}1_{\mathcal{A}}\rangle\\ &\Rightarrow^c \alpha|0_{\mathcal{S}}0_{\mathcal{D}}0_{\mathcal{A}}\rangle+\beta|0_{\mathcal{S}}0_{\mathcal{D}}1_{\mathcal{A}}\rangle\\ &= |0_{\mathcal{S}}\rangle|0_{\mathcal{D}}\rangle(\alpha|0_{\mathcal{A}}\rangle+\beta|1_{\mathcal{A}}\rangle),\end{aligned}$$

disentangling $\mathcal{S}$ from $\mathcal{A}$. However, a demon whose memory decoheres—e.g., entangles with the environment—will not be able to take advantage of the quantum correlations in the state of $\mathcal{SA}$ pair. Decoherence leading to the einselection of the basis $\{|0_{\mathcal{D}}\rangle,|1_{\mathcal{D}}\rangle\}$ in the memory of the demon can be represented by another CNOT that acts between $\mathcal{D}$ influencing the state of the environment $\mathcal{E}$ (not shown in the figure). As a consequence, following the CNOT (a) interaction with the environment leads to $(\alpha|0_{\mathcal{S}}0_{\mathcal{D}}0_{\mathcal{A}}\rangle+\beta|0_{\mathcal{S}}1_{\mathcal{D}}1_{\mathcal{A}}\rangle)|\varepsilon_0\rangle \Rightarrow \alpha|0_{\mathcal{S}}0_{\mathcal{D}}0_{\mathcal{A}}\rangle|\varepsilon_0\rangle + \beta|0_{\mathcal{S}}1_{\mathcal{D}}1_{\mathcal{A}}\rangle|\varepsilon_1\rangle$. Thus, when all the other C-NOT's are carried out, $|0_{\mathcal{S}}\rangle|0_{\mathcal{D}}\rangle(\alpha|0_{\mathcal{A}}\rangle|\varepsilon_0\rangle+\beta|1_{\mathcal{A}}\rangle|\varepsilon_1\rangle)$ obtains, leading to the same pure states of the system and the demon, but (in effect) a mixed state of the apparatus, $\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{E}}(\alpha|0_{\mathcal{A}}\rangle|\varepsilon_0\rangle+\beta|1_{\mathcal{A}}\rangle|\varepsilon_1\rangle)(\alpha^*\langle 0_{\mathcal{A}}|\langle\varepsilon_0| + \beta^*\langle 1_{\mathcal{A}}|\langle\varepsilon_1|) = |\alpha|^2|0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}|+|\beta|^2|1_{\mathcal{A}}\rangle\langle 1_{\mathcal{A}}|$, providing that $\langle\varepsilon_0|\varepsilon_1\rangle = 0$. In this case, decoherence that turns demon's performance from quantum to classical makes it impossible to extract all of the thermodynamic benefit from quantum correlations. We leave it as an exercise for the reader to show that the classical correlation between $\mathcal{S}$ and $\mathcal{A}$ [Eq. (9b)] leads to the same final state, thus proving that a classical demon can extract all of the work present in a classical correlation.

$$\Delta = \Delta W/k_{B_2}T = [H(\mathcal{A})+H(\mathcal{S}|\mathcal{A})]_{\{|A_k\rangle\}} - H(\mathcal{S},\mathcal{A}) \quad (16)$$

or

$$\Delta W = k_{B_2}T\,\delta(\mathcal{S}|\mathcal{A}_{\{|A_k\rangle\}}). \quad (17)$$

Equation (17) relating the extra work $\Delta W = W^Q - W^C$ to quantum discord—to the difference of the accessible joint entropy of classical (local) and quantum (global) demons—is the principal result of our paper. It answers an interesting "demonic" question while simultaneously providing an operational interpretation of discord.

To gain further insight into implications of the above, let us first note that discord is, in general, basis dependent. Discord disappears iff the density matrix has the "post-decoherence" (or "postmeasurement") form, Eq. (6), *already before the measurement*. Given the ability of classical demons to match quantum performance standard in this case, basis $\{|A_k\rangle\}$ that allows for the disappearance of discord in the presence of nontrivial correlation can be deemed classical [3–5]. We note that $\rho_{\mathcal{SA}}$ in the locally diagonal form, Eq. (6), may emerge as a consequence of the coupling of $\mathcal{A}$ with the environment [8–11]. The preferred *pointer basis* is a result of einselection.

A typical $\rho_{\mathcal{SA}}$ does not have the form of Eq. (6), however. In that case, discord does not completely disappear for any basis. The least discord,

$$\hat{\delta}(\mathcal{S}|\mathcal{A}) = \min_{\{|A_k\rangle\}}[H(\mathcal{A})+H(\mathcal{S}|\mathcal{A})]_{\{|A_k\rangle\}} - H(\mathcal{S},\mathcal{A}), \quad (18)$$

corresponds to maximum efficiency of a classical demon. Note that to get the right answer we had to minimize the sum of the two terms contributing to the joint entropy, rather than each of them separately. The alternative $\hat{\partial}(\mathcal{S}|\mathcal{A}) = H(\mathcal{A}) + \min_{\{|A_k\rangle\}} H(\mathcal{S}|\mathcal{A})_{\{|A_k\rangle\}} - H(\mathcal{SA})$ would have followed if the cancellation in Eq. (3) was carried out. The difference between them is obvious, and $\hat{\delta}(\mathcal{S}|\mathcal{A}) \geqslant \hat{\partial}(\mathcal{S}|\mathcal{A})$.

IV. LOCALLY ACCESSIBLE INFORMATION AND DISCORD

Discord is not symmetric between the two ends of the correlation. In general, $\hat{\delta}(\mathcal{S}|\mathcal{A}) \neq \hat{\delta}(\mathcal{A}|\mathcal{S})$. In particular, for density matrices that emerge following einselection in $\mathcal{A}$ $\hat{\delta}(\mathcal{S}|\mathcal{A})$ will vanish but $\hat{\delta}(\mathcal{A}|\mathcal{S})$ may remain finite. Such locally accessible correlations are *one-way classical*. They are characterized by a preferred direction—from $\mathcal{A}$ to $\mathcal{S}$—in which more information about the joint state can be accessed. Thus, when a local demon can choose between the two "ends" of the $\mathcal{SA}$ pair, it may be more efficient than a one-way demon. Indeed, one could define polarization,

$$\varpi(\mathcal{S}|\mathcal{A}) = \hat{\delta}(\mathcal{S}|\mathcal{A}) - \hat{\delta}(\mathcal{A}|\mathcal{S}) \quad (19)$$

to quantify this directionality.

One can generalize discord to collections of several correlated quantum systems. By analogy with the case of a single pair we define it as a difference between the joint entropy accessible through a particular sequence of (possibly conditional) measurements—that is, the obvious generalization of Eq. (7)—and the joint von Neumann entropy of the unmeasured density matrix. The least discord of such a collection of systems is a minimum over all possible sequences

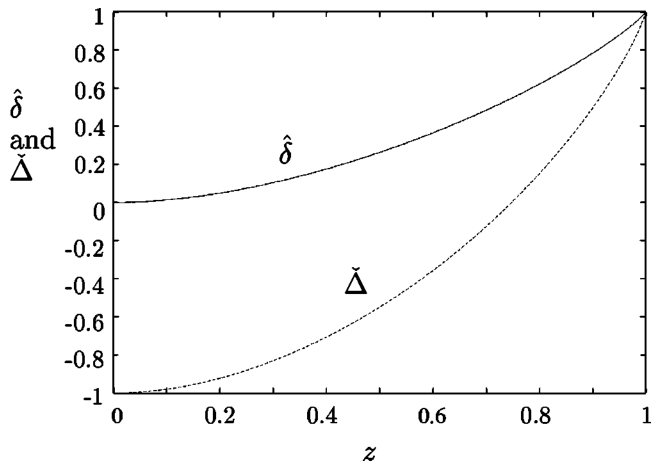

FIG. 3. Discord $\hat{\delta}(z)$, Eq. (17), and the lower bound on the work deficit $\check{\Delta}(z)$, both in bits, for Werner states, $\rho_{\mathcal{SA}}=(1-z)/4\,\mathbf{1}+z|\psi_{\mathcal{SA}}\rangle\langle\psi_{\mathcal{SA}}|$, where $|\psi_{\mathcal{SA}}\rangle=(|0_{\mathcal{S}}0_{\mathcal{A}}\rangle+|1_{\mathcal{S}}1_{\mathcal{A}}\rangle)/\sqrt{2}$. In this simple case both discord (which is equal to the work deficit) and the lower bound on the work deficit derived in Ref. [24] are independent of the basis and the same for both "ends" of the correlated pair. As argued here, there are cases where discord will actually play a role of an upper bound on the work deficit, as it is derived under the assumption of one-way classical communication.

of all possible measurements. This corresponds to the demon having a choice of the end of the pair it can measure first.

This last situation allows one to address questions raised in a recent paper on the work that can be extracted by local and global observers from correlated pairs of quantum systems [24]. The authors show that a global observer will be able to extract more work from a pair of quantum systems than "Alice and Bob," who can carry out local operations and communicate classically ("LOCC") with each other, and the difference Δ (in units of $k_{B_2}T$) is bounded from below,

$$\Delta \geqslant \check{\Delta} = \max_{\mathcal{A},\mathcal{S}}[H(\mathcal{A}),H(\mathcal{S})]-H(\mathcal{A},S). \tag{20}$$

This lower bound is illustrated in Fig. 3 along with our result given by $\hat{\delta}$ for Werner states. In this case, discord yields the work deficit, $\Delta=\hat{\delta}$. Indeed, our arguments throughout the paper show that $\hat{\delta}$ gives the difference in efficiencies when only one-way classical communication is allowed. Obviously, allowing for a two-way communication between Alice and Bob can only help, so $\hat{\delta}\geqslant\Delta$ is an upper bound on the difference of the efficiencies, and there are cases (e.g., Werner states of Fig. 3) where this upper bound is saturated.

This leads to an interesting question. When does two-way communication provide a significant advantage? Does a single round of two-way communication always suffice, or is it possible that many iterations may help even more? The nested density matrix of the form

$$\rho_{\mathcal{A},\mathcal{B},\mathcal{C},\ldots,\mathcal{S}}=\sum_i p_{\mathcal{A}}(i)\rho_{\mathcal{B},\mathcal{C},\ldots,\mathcal{S}|A_i\rangle}\otimes|A_i\rangle\langle A_i|, \tag{21a}$$

$$\rho_{\mathcal{B},\mathcal{C},\ldots,\mathcal{S}|A_i\rangle}=\sum_j p_{\mathcal{B}}^{(i)}(j)\rho_{\mathcal{C},\ldots,\mathcal{S}|A_i,B_j^{(i)}\rangle}\otimes|B_j^{(i)}\rangle\langle B_j^{(i)}|,\ \ldots,\ \text{etc.}, \tag{21b}$$

where every second of the subsystems (i.e., $\mathcal{A},\mathcal{C},\mathcal{E},\ldots$ are on Alice's side, while the complements $\mathcal{B},\mathcal{D},\ldots$ are on Bob's side) settles these questions. When the conditional density matrices $\rho_{\mathcal{B},\mathcal{C},\ldots,\mathcal{S}|A_i\rangle}$, $\rho_{\mathcal{C},\ldots,\mathcal{S}|A_i,B_j\rangle}$, etc., are not co-diagonal in the relevant Hilbert spaces describing $\mathcal{B},\mathcal{C},\mathcal{D},\ldots$ Alice and Bob will have to exchange data after each measurement to decide what to measure next if they are to extract all of the potentially accessible information. This is an example of a situation where a number of back-and-forth exchanges equal to the number of "nestings" is necessary to extract all of the work—to access all of the locally accessible information. It is tempting to suggest that such nested density matrices could be used to restrict access to information by hiding it in some sufficiently deep layer (e.g., $\mathcal{S}$), accessible only if two (or many) parties cooperate in its retrieval.

V. SUMMARY

First hints of the quantum underpinnings of the universe emerged over a century ago in a thermodynamic setting involving black body radiation. We have studied here implications of quantum physics—and, in particular, of the quantum aspects of correlations—for classical and quantum Maxwell's demons. We have seen that discord is a measure of the advantage afforded by the quantum conditional dynamics, and shown how this advantage is eliminated by decoherence and the ensuing einselection. Our discussion sheds light on the problem of transition between quantum and classical: It leads to an operational measure of the quantum aspect of correlations. As was already pointed out [3–5], the aspect of quantumness captured by discord is not the entanglement. Rather, it is related to the degree to which quantum superpositions are implicated in a state of a pair (or of a collection) of quantum systems. We expect it to be relevant in questions involving quantum theory and thermodynamics, but discord may be also of use in characterizing multiply correlated states that find applications in quantum computation.

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