Classification of multipartite entangled states by multidimensional determinants

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We find that multidimensional determinants ''hyperdeterminants,'' related to entanglement measures (the so-called concurrence, or 3-tangle for two or three qubits, respectively), are derived from a duality between entangled states and separable states. By means of the hyperdeterminant and its singularities, the single copy of multipartite pure entangled states is classified into an onion structure of every closed subset, similar to that by the local rank in the bipartite case. This reveals how inequivalent multipartite entangled classes are partially ordered under local actions. In particular, the generic entangled class of the maximal dimension, distinguished as the nonzero hyperdeterminant, does not include the maximally entangled states in Bell's inequalities in general (e.g., in the $n \geqslant 4$ qubits), contrary to the widely known bipartite or three-qubit cases. It suggests that not only are they never locally interconvertible with the majority of multipartite entangled states, but they would have no grounds for the canonical n-partite entangled states. Our classification is also useful for the mixed states.

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I. INTRODUCTION

Entanglement is the quantum correlation exhibiting nonlocal (nonseparable) properties. It is supposed to be never strengthened, on average, by local operations and classical communication (LOCC). In particular, entanglement in multiparties is of fundamental interest in quantum many-body theory [1], and makes quantum information processing (QIP), e.g., distillation protocol, more efficient than relying on entanglement only in two parties [2]. Here, we classify and characterize the multipartite entanglement that has yet to be understood, compared with the bipartite one.

For the single copy of bipartite pure states on $\mathcal{H}(\mathbb{C}^{k+1}) \otimes \mathcal{H}(\mathbb{C}^{k+1})$, we are interested in whether a state $|\Psi\rangle$ can convert to another state $|\Phi\rangle$ by LOCC. It is convenient to consider the Schmidt decomposition,

$$|\Psi\rangle = \sum_{i_1, i_2=0}^{k} a_{i_1, i_2} |i_1\rangle \otimes |i_2\rangle = \sum_{j=0}^{k} \lambda_j |e_j\rangle \otimes |e'_j\rangle, \qquad (1)$$

where the computational basis $|i_j\rangle$ is transformed to a local biorthonormal basis $|e_j\rangle, |e'_j\rangle$, and the Schmidt coefficients λ_j can be taken as $\lambda_j \geqslant 0$. We call the number of nonzero λ_j the (Schmidt) local rank r. Then the LOCC convertibility is given by a majorization rule between the coefficients λ_j of $|\Psi\rangle$ and those of $|\Phi\rangle$ [3]. This suggests that the structure of entangled states consists of partially ordered, continuous classes labeled by a set of λ_j. In particular, $|\Psi\rangle$ and $|\Phi\rangle$ belong to the same class under the LOCC classification if and only if all continuous λ_j coincide.

Suppose we are concerned with a coarse-grained classification by the so-called stochastic LOCC (SLOCC) [4,5], where we identify $|\Psi\rangle$ and $|\Phi\rangle$ that are interconvertible back and forth with (maybe different) nonvanishing probabilities. This is because $|\Psi\rangle$ and $|\Phi\rangle$ are supposed to perform the same tasks in QIP although their probabilities of success differ. Later, we find that this SLOCC classification is still fine grained to classify the multipartite entanglement. Mathematically, two states belong to the same class under SLOCC if and only if they are converted by an *invertible* local operation having a nonzero determinant [5]. Thus the SLOCC classification is equivalent to the classification of orbits of the natural action: direct product of general linear groups $\mathrm{GL}_{k+1}(\mathbb{C}) \times \mathrm{GL}_{k+1}(\mathbb{C})$ [6]. The local rank r in Eq. (1) [7], equivalently the rank of a_{i_1,i_2}, is found to be preserved under SLOCC. A set S_j of states of the local rank $\leqslant j$ is a *closed* subvariety under SLOCC, and S_{j-1} is the singular locus of S_j. This is how the local rank leads to an ''onion'' structure (mathematically the stratification):

$$S_{k+1} \supset S_k \supset \cdots \supset S_1 \supset S_0 = \varnothing, \qquad (2)$$

and $S_j - S_{j-1} (j=1, \ldots, k+1)$ give $k+1$ classes of entangled states. Since the local rank can decrease by *noninvertible* local operations, i.e., general LOCC [4,5,8], these classes are totally ordered such that, in particular, the outermost generic set $S_{k+1} - S_k$ is the class of maximally entangled states and the innermost set $S_1 (= S_1 - S_0)$ is that of separable states.

For the single copy of multipartite pure states,

$$|\Psi\rangle = \sum_{i_1, \ldots, i_n=0}^{k_1, \ldots, k_n} a_{i_1, \ldots, i_n} |i_1\rangle \otimes \cdots \otimes |i_n\rangle, \qquad (3)$$

there are difficulties in extending the Schmidt decomposition for a multiorthonormal basis [9]. Moreover, an attempt to use the tensor rank of $a_{i_1, \ldots, i_n}$ [10] falls down since S_j, defined by it, is not always closed [11,12]. For three qubits, Dür *et al.* showed that SLOCC classifies the whole states M into *finite* classes, and in particular there exist two inequivalent, Greenberger-Horne-Zeilinger (GHZ) and W, classes of the tripartite entanglement [5]. They also pointed out that this case is exceptional since the action $\mathrm{GL}_{k_1+1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{k_n+1}(\mathbb{C})$ has *infinitely many* orbits in general (e.g., for $n \geqslant 4$).

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In this paper, we classify multipartite entanglement in a unified manner based on the hyperdeterminant. The advantages are three fold.

(i) This classification is equivalent to the SLOCC classification when SLOCC has finitely many orbits. So it naturally includes the widely known bipartite and three-qubit cases.

(ii) In the multipartite case, we need further SLOCC invariants in addition to the local ranks. For example, in the three-qubit case [5], the 3-tangle τ, the absolute value of the hyperdeterminant [see Eq. (10)], is utilized to distinguish GHZ and W classes. This work clarifies why the 3-tangle τ appears and how these SLOCC invariants are related to the hyperdeterminant in general.

(iii) Our classification is also useful to multipartite mixed states. A mixed state ρ can be decomposed as a convex combination of projectors onto pure states. Considering how ρ needs at least the outer class in the onion structure of pure states, we can also classify multipartite mixed states into totally ordered classes (for details, see Appendix B). We concentrate on the pure states here.

The rest of the paper is organized as follows. In Sec. II, a duality between separable states and entangled states is introduced. We find that the hyperdeterminant, associated to this duality, and its singularities lead to the SLOCC-invariant onionlike structure of multipartite entanglement. The characteristics of the hyperdeterminant and its singularities are explained in Sec. III. Classifications of multipartite entangled states are exemplified in Sec. IV so as to reveal how they are ordered under SLOCC. Finally, the conclusion is given in Sec. V.

II. DUALITY BETWEEN SEPARABLE STATES AND ENTANGLED STATES

In this section, we find that there is a duality between the set of separable states and that of entangled states. This duality derives the hyperdeterminant our classification is based on.

A. Preliminary: Segrè variety

To introduce our idea, we first recall the geometry of pure states. In a complex (finite) $(k+1)$-dimensional Hilbert space $\mathcal{H}(\mathbb{C}^{k+1})$, let $|\Psi\rangle$ be a (not necessarily normalized) vector given by $(k+1)$tuple of complex amplitudes $x_j(j=0,\ldots,k)\in\mathbb{C}^{k+1}-\{0\}$ in a computational basis. The physical state in $\mathcal{H}(\mathbb{C}^{k+1})$ is a ray, an equivalence class of vectors up to an overall nonzero complex number. Then the set of rays constitutes the complex projective space $\mathbb{C}P^k$ [the projectivization of $\mathcal{H}$ $(\mathbb{C}^{k+1})$], and $x := (x_0:\cdots:x_k)$, considered up to a complex scalar multiple, gives homogeneous coordinates in $\mathbb{C}P^k$.

For a composite system that consists of $\mathcal{H}(\mathbb{C}^{k_1+1})$ and $\mathcal{H}(\mathbb{C}^{k_2+1})$, the whole Hilbert space is the tensor product $\mathcal{H}(\mathbb{C}^{k_1+1})\otimes\mathcal{H}(\mathbb{C}^{k_2+1})$, and the associated projective space is $M=\mathbb{C}P^{(k_1+1)(k_2+1)-1}$. A set X of the separable states is the mere Cartesian product $\mathbb{C}P^{k_1}\times\mathbb{C}P^{k_2}$, whose dimension k_1+k_2 is much smaller than that of the whole space M, $(k_1+1)(k_2+1)-1$. This X is a closed, smooth, algebraic subvariety (Segrè variety) defined by the Segrè embedding into $\mathbb{C}P^{(k_1+1)(k_2+1)-1}$ [12,13],

$$\mathbb{C}P^{k_1}\times\mathbb{C}P^{k_2}\hookrightarrow\mathbb{C}P^{(k_1+1)(k_2+1)-1},$$
$$((x_0^{(1)}:\cdots:x_{k_1}^{(1)}),(x_0^{(2)}:\cdots:x_{k_2}^{(2)}))$$
$$\mapsto(x_0^{(1)}x_0^{(2)}:\cdots:x_0^{(1)}x_{k_2}^{(2)}:x_1^{(1)}x_0^{(2)}:\cdots:x_{k_1}^{(1)}x_{k_2}^{(2)}). \tag{4}$$

Denoting homogeneous coordinates in $\mathbb{C}P^{(k_1+1)(k_2+1)-1}$ by $b_{i_1,i_2}=x_{i_1}^{(1)}x_{i_2}^{(2)}(0\leqslant i_j\leqslant k_j)$, we find that the Segrè variety X is given by the common zero locus of $k_1(k_1+1)k_2(k_2+1)/4$ homogeneous polynomials of degree 2:

$$b_{i_1,i_2}b_{i_1',i_2'}-b_{i_1,i_2'}b_{i_1',i_2}, \tag{5}$$

where $0\leqslant i_1<i_1'\leqslant k_1$, and $0\leqslant i_2<i_2'\leqslant k_2$. Note that this condition implies that all 2×2 minors of "matrix" b_{i_1,i_2} equal 0; i.e., the rank of b_{i_1,i_2} is 1. Thus we have $X=S_1$, which agrees with the SLOCC classification by the local rank in the bipartite case.

Now consider the multipartite Cartesian product $X=\mathbb{C}P^{k_1}\times\cdots\times\mathbb{C}P^{k_n}$ in the Segrè embedding into $M=\mathbb{C}P^{(k_1+1)\cdots(k_n+1)-1}$. Because this Segrè variety X is the projectivization of a variety of the matrices $b_{i_1,\ldots,i_n}=x_{i_1}^{(1)}\cdots x_{i_n}^{(n)}$, it gives a set of completely separable states in $\mathcal{H}(\mathbb{C}^{k_1+1})\otimes\cdots\otimes\mathcal{H}(\mathbb{C}^{k_n+1})$. By another Segrè embedding, say $X'=\mathbb{C}P^{(k_1+1)(k_2+1)-1}\times\mathbb{C}P^{k_3}\times\cdots\times\mathbb{C}P^{k_n}$, we also distinguish a set of separable states where only first and second parties can be entangled; i.e., when we regard first and second parties as one party, an element of this set is completely separable for "$n-1$" parties. This is how, also in the multipartite case, we can classify all kinds of *separable* states, typically lower-dimensional sets. Note that, in the multipartite case, this check for the separability is more strict than the check by local ranks [14].

B. Main idea: Duality

We rather want to classify *entangled* states, typically higher-dimensional complementary sets of separable states. Our strategy is based on the duality in algebraic geometry; a hyperplane in $\mathbb{C}P$ forms the point of a dual projective space $\mathbb{C}P^*$, and conversely every point p of $\mathbb{C}P$ is tied to a hyperplane $p^\vee$ in $\mathbb{C}P^*$ as the set of all hyperplanes in $\mathbb{C}P$ passing through p. Remarkably, the projective duality between projective subspaces, like the above example, can be extended to an involutive correspondence between irreducible algebraic subvarieties in $\mathbb{C}P$ and $\mathbb{C}P^*$. So we define a projectively dual (irreducible) variety $X^\vee\subset\mathbb{C}P^*$ as the closure of the set of all hyperplanes tangent to the Segrè variety X.

Let us observe (and see the reason later) that, in the bipartite case seen in Sec. I, the variety S_k of the *degenerate* $(k+1)\times(k+1)$ matrices $A=a_{i_1,i_2}$ is projectively dual to the variety $S_1=X$ of the matrices $B=b_{i_1,i_2}=x_{i_1}^{(1)}x_{i_2}^{(2)}$. That is, S_k is the dual variety $X^\vee$. Following an analogy with a two-dimensional (bipartite) case, an n-dimensional matrix $A=a_{i_1,\ldots,i_n}$ is called *degenerate* if and only if it (precisely, its projectivization) lies in the projectively dual variety $X^\vee$ of the Segrè variety X. In other words, identifying the space of n-dimensional matrices with its dual by means of the pairing,

$$F(A,B)=\sum_{i_1,\ldots,i_n=0}^{k_1,\ldots,k_n} a_{i_1,\ldots,i_n} b_{i_1,\ldots,i_n}, \tag{6}$$

we see that A is degenerate if and only if its orthogonal hyperplane $F(A,B)=0$ is tangent to X at some nonzero point $x=(x^{(1)},\ldots,x^{(n)})$. Analytically, a set of equations,

$$F(A,x)=\sum_{i_1,\ldots,i_n=0}^{k_1,\ldots,k_n} a_{i_1,\ldots,i_n} x_{i_1}^{(1)}\cdots x_{i_n}^{(n)}=0,$$
$$\frac{\partial}{\partial x_{i_j}^{(j)}} F(A,x)=0 \quad \text{for all } j,i_j \tag{7}$$

($j=1,\ldots,n$ and $0\leqslant i_j\leqslant k_j$), has at least one nontrivial solution $x=(x^{(1)},\ldots,x^{(n)})$ of every $x^{(j)}\neq 0$, and then x is called a critical point. The above condition is also equivalent to saying that the kernel ker$F$ of $F(A,x)$ is not empty, where ker$F$ is the set of points $x=(x^{(1)},\ldots,x^{(n)})\in X$ such that, in every $j_0=1,\ldots,n$,

$$F(A,(x^{(1)},\ldots,x^{(j_0-1)},z^{(j_0)},x^{(j_0+1)},\ldots,x^{(n)}))=0 \tag{8}$$

for the *arbitrary* $z^{(j_0)}$.

In the case of $n=2$, the condition for Eq. (7) coincides with the usual notion of degeneracy, and means that A does not have the full rank. It shows that $X^\vee$ is nothing but S_k. In particular, $X^\vee$ (defined by this condition) is of codimension 1 and is given by the ordinary determinant $\det A=0$, if and only if A is a square ($k_1=k_2=k$) matrix. In the n-dimensional case, if $X^\vee$ is a hypersurface (of codimension 1), it is given by the zero locus of a unique (up to sign) irreducible homogeneous polynomial over $\mathbb{Z}$ of $a_{i_1,\ldots,i_n}$. This polynomial is the hyperdeterminant introduced by Cayley and is denoted by Det$A$. As usual, if $X^\vee$ is not a hypersurface, we set Det$A$ to be 1.

Remember that, in the *bipartite* case, we classify the states $\in S_{k+1}-S_k=M-X^\vee$ as the generic entangled states, the states $\in S_k-S_{k-1}=X^\vee-X^\vee_{\text{sing}}$ as the next generic entangled states, and so on. Likewise, we aim to classify the *multipartite* entangled states into the onion structure by the dual variety $X^\vee$ (Det$A=0$), its singular locus $X^\vee_{\text{sing}}$, and so on (i.e., by every closed subvariety), instead of the tensor rank [10].

III. HYPERDETERMINANT AND ITS SINGULARITIES

In order to classify multipartite entanglement into the SLOCC-invariant onion structure, we explore the dual variety $X^\vee$ (zero hyperdeterminant) and its singular locus in this section.

A. Hyperdeterminant

We utilize the hyperdeterminant, the generalized determinant for higher-dimensional matrices, by Gelfand *et al.* [15,16]. Its absolute value is also known as an entanglement measure, the concurrence C [17], or 3-tangle τ [18], for the two-, three-qubit pure case, respectively.

$$C=2|\mathrm{Det}A_2|=2|\det A|=2|a_{00}a_{11}-a_{01}a_{10}|, \tag{9}$$

$$\begin{aligned}\tau=4|\mathrm{Det}A_3|=4|&a_{000}^2a_{111}^2+a_{001}^2a_{110}^2+a_{010}^2a_{101}^2+a_{100}^2a_{011}^2\\&-2(a_{000}a_{001}a_{110}a_{111}+a_{000}a_{010}a_{101}a_{111}\\&+a_{000}a_{100}a_{011}a_{111}+a_{001}a_{010}a_{101}a_{110}\\&+a_{001}a_{100}a_{011}a_{110}+a_{010}a_{100}a_{011}a_{101})\\&+4(a_{000}a_{011}a_{101}a_{110}+a_{001}a_{010}a_{100}a_{111})|.\end{aligned} \tag{10}$$

The following useful facts are found in Ref. [16]. Without loss of generality, we assume that $k_1\geqslant k_2\geqslant\cdots\geqslant k_n\geqslant 1$. The n-dimensional hyperdeterminant Det$A$ of format $(k_1+1)\times\cdots\times(k_n+1)$ exists, i.e., $X^\vee$ is a hypersurface if and only if a "polygon inequality" $k_1\leqslant k_2+\cdots+k_n$ is satisfied. For $n=2$, this condition is reduced to $k_1=k_2$ as desired, and Det$A$ coincides with det$A$. The matrix format is called boundary if $k_1=k_2+\cdots+k_n$ and interior if $k_1<k_2+\cdots+k_n$. Note that (i) the boundary format includes the "bipartite cut" between the first party and the others so that it is mathematically tractable; (ii) the interior format includes the $n\geqslant 3$qubit case. We treat hereafter the format where the polygon inequality holds and $X^\vee$ is the largest closed subvariety, defined by the hypersurface Det$A=0$.

Det$A$ is relatively invariant (invariant up to constant) under the action of $\mathrm{GL}_{k_1+1}(\mathbb{C})\times\cdots\times\mathrm{GL}_{k_n+1}(\mathbb{C})$. In particular, interchanging two parallel slices (submatrices with some fixed directions) leaves Det$A$ invariant up to sign, and Det$A$ is a homogeneous polynomial in the entries of each slice. Since it is ensured that $X^\vee$, $X^\vee_{\text{sing}}$, and further singularities are invariant under SLOCC, our classification is equivalent to or coarser than the SLOCC classification. Later, we see that the former and the latter correspond to the case where SLOCC gives finitely and infinitely many classes, respectively.

B. Schläfli's construction

It would not be easy to calculate Det$A$ directly by its definition that Eq. (7) has at least one solution. Still, the Schläfli's method enables us to construct DetA_n of format 2^n (n qubits) by induction on n [15,16,19].

For $n=2$, by definition $\mathrm{Det}A_2=\det A=a_{00}a_{11}-a_{01}a_{10}$. Suppose Det$A_n$, whose degree of homogeneity is l, is given. Associating an $(n+1)$-dimensional matrix $a_{i_0,i_1,\ldots,i_n}$ ($i_j=0,1$) to a family of n-dimensional matrices $\tilde{A}(x)=\sum_{i_0}a_{i_0,i_1,\ldots,i_n}x_{i_0}$ linearly depending on the auxiliary variable x_{i_0}, we have $\mathrm{Det}\tilde{A}(x)_n$. Due to Theorems 4.1 and 4.2 of Ref. [16], the discriminant Δ of $\mathrm{Det}\tilde{A}(x)_n$ gives $\mathrm{Det}A_{n+1}$ with an extra factor R_n. The Sylvester formula of the discriminant Δ for binary forms enables us to write $\mathrm{Det}A_{n+1}$ in terms of the determinant of order $2l-1$;

$$\mathrm{Det}A_{n+1}=\Delta(\mathrm{Det}\widetilde{A}(x)_n)/R_n=\frac{1}{R_n c_l}\begin{vmatrix} c_0 & c_1 & \cdots & c_{l-2} & c_{l-1} & c_l & \cdots & 0 \\ 0 & c_0 & \cdots & \cdots & c_{l-2} & c_{l-1} & \cdots & 0 \\ \vdots & & \ddots & & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & c_0 & c_1 & \cdots & \cdots & c_l \\ 1c_1 & 2c_2 & \cdots & \cdots & lc_l & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1c_1 & 2c_2 & \cdots & lc_l \end{vmatrix}, \tag{11}$$

where each c_j is the coefficient of $x_0^{l-j}x_1^j$ in $\mathrm{Det}\widetilde{A}(x)_n$, i.e.,

$$c_j=\frac{1}{(l-j)!\,j!}\,\frac{\partial^l}{\partial x_0^{l-j}\partial_1^j}\,\mathrm{Det}\widetilde{A}(x)_n .$$

Note that because for $n=2$ or 3, the extra factor R_n is just a nonzero constant, accordingly $\mathrm{Det}A_{3,4}$ for the three or four qubits is readily calculated, respectively. It would be instructive to check that $\mathrm{Det}A_3$ in Eq. (10) is obtained in this way. On the other hand, for $n\geqslant 4$, R_n is the Chow form (related resultant) of irreducible components of the singular locus $X^\vee_{\mathrm{sing}}$. These are due to the fact that $X^\vee_{\mathrm{sing}}$ has codimension 2 in M for any format of the dimension $n\geqslant 3$ except for the format 2^3 (three-qubit case), that was conjectured in Ref. [15] and was proved in Ref. [20]. So we have to explore $X^\vee_{\mathrm{sing}}$ not only to classify entangled states in the n qubits, but to calculate $\mathrm{Det}A_{n+1}$ inductively. Although $\mathrm{Det}A_{n\geqslant 5}$ has yet to be written explicitly, only its degree l of homogeneity is known (in Corollary 2.10 of Ref. [16]) to grow very fast as 2, 4, 24, 128, 880, 6816, 60 032, 589 312, 6 384 384 for $n=2,3,\ldots,10$.

C. Singularities of the hyperdeterminant

We describe the singular locus of the dual variety $X^\vee$. The technical details are given in Ref. [20]. It is known that, for the boundary format, the next largest closed subvariety $X^\vee_{\mathrm{sing}}$ is always an irreducible hypersurface in $X^\vee$; in contrast, for the interior one, $X^\vee_{\mathrm{sing}}$ has generally two closed irreducible components of codimension 1 in $X^\vee$, node-type ($X^\vee_{\mathrm{node}}$) and cusp-type ($X^\vee_{\mathrm{cusp}}$) singularities. The rest of this section can be skipped for the first reading. It is also illustrated for the three-qubit case in Appendix A.

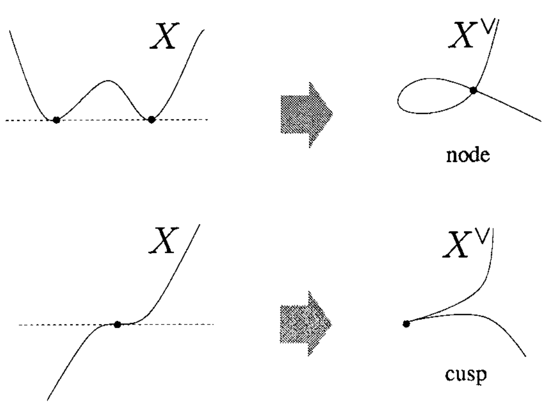

FIG. 1. Two types of singularities of $X^\vee$. $X^\vee_{\mathrm{node}}$ corresponds to the bitangent of X, where both tangencies are of the first order. $X^\vee_{\mathrm{cusp}}$ corresponds to the tangent at an inflection point of X, where its tangency is of the second order.

First, $X^\vee_{\mathrm{node}}$ is the closure of the set of hyperplanes tangent to the Segrè variety X at more than one point (cf. Fig. 1). $X^\vee_{\mathrm{node}}$ can be composed of closed irreducible subvarieties $X^\vee_{\mathrm{node}}(J)$ labeled by the subset $J\subset\{1,\ldots,n\}$, including $\varnothing$. Indicating that two solutions $x=(x^{(1)},\ldots,x^{(j)},\ldots,x^{(n)})$ of Eq. (7) coincide for $j\in J$, the label J distinguishes the pattern in these solutions. In order to rewrite $X^\vee_{\mathrm{node}}(J)$, let us pick up a point $x^0(J)$ such that its homogeneous coordinates $x^{(j)}_{i_j}=\delta_{i_j,0}$ for $j\in J$ and δ_{i_j,k_j} for $j\notin J$. It is convenient to label the positions of 1 in each $x^{(j)}$ by a multiindex $[i_1,\ldots,i_n]$. For example, $x^0(1)$ is labeled by $[0,k_2,\ldots,k_n]$ and $x^0(1,\ldots,n)$ is just written by x^0. According to Eq. (7), $X^\vee|_{x^0(J)}$, tangent to X at $x^0(J)$, consists of the matrices A of all $a_{i'_1,\ldots,i'_n}=0$ such that $[i'_1,\ldots,i'_n]$ differs from $[i_1,\ldots,i_n]$ of $x^0(J)$ in at most one index. Then we can define $X^\vee_{\mathrm{node}}(J)$ as

$$X^\vee_{\mathrm{node}}(J)=\overline{(X^\vee|_{x^0}\cap X^\vee|_{x^0(J)})\cdot G}, \tag{12}$$

where $G=\mathrm{GL}_{k_1+1}\times\cdots\times\mathrm{GL}_{k_n+1}$ acts on M from the right and the bar stands for the closure.

Second, $X^\vee_{\mathrm{cusp}}$ is the set of hyperplanes having a critical point that is not a simple quadratic singularity (cf. Fig. 1). Precisely, the quadric part of $F(A,x)$ at x^0 is a matrix $y_{(j,i_j),(j',i_{j'})}=(\partial^2/\partial x^{(j)}_{i_j}\partial x^{(j')}_{i_{j'}})F(A,x^0)$, where the pairs (j,i_j) and $(j',i_{j'})(1\leqslant i_j\leqslant k_j,1\leqslant i_{j'}\leqslant k_{j'})$ are the row and the column index, respectively. Denoting by $X^\vee_{\mathrm{cusp}}|_{x^0}$ the variety of the Hessian $\det y=0$ in $X^\vee|_{x^0}$, we can define $X^\vee_{\mathrm{cusp}}$ as

$$X^\vee_{\mathrm{cusp}}=X^\vee_{\mathrm{cusp}}|_{x^0}\cdot G. \tag{13}$$

This $X^\vee_{\mathrm{cusp}}$ is already closed without taking the closure.

IV. CLASSIFICATION OF MULTIPARTITE ENTANGLEMENT

According to Secs. II and III, we illustrate the classification of multipartite pure entangled states for typical cases.

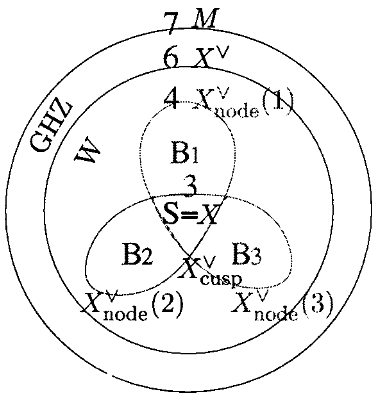

FIG. 2. The onionlike classification of SLOCC orbits in the three-qubit case. We utilize a duality between the smallest closed subvariety X and the largest closed subvariety $X^\vee$. The dual variety $X^\vee$ (zero hyperdeterminant) and its singularities constitute SLOCC-invariant closed subvarieties, so that they classify the multipartite entangled states (SLOCC orbits).

A. Three-qubit (format 2^3) case

The classification of three qubits under SLOCC has been already done in Refs. [5,6]. Surprisingly, Gelfand *et al.* considered the same mathematical problem by $\mathrm{Det}A_3$ in example 4.5 of Ref. [16]. Our idea is inspired by this example. We complement the Gelfand *et al.*'s result, analyzing additionally the singularities of $X^\vee$ in detail. The dimensions, representatives, names, and varieties of the orbits are summarized as follows. The basis vector $|i_1\rangle\otimes|i_2\rangle\otimes|i_3\rangle$ is abbreviated to $|i_1i_2i_3\rangle$.

dim 7: $|000\rangle+|111\rangle$, GHZ $\in M(=\mathbb{C}P^7)-X^\vee$;

dim 6: $|001\rangle+|010\rangle+|100\rangle$, $W\in X^\vee - X^\vee_{\mathrm{sing}} = X^\vee - X^\vee_{\mathrm{cusp}}$;

dim 4: $|001\rangle+|010\rangle, |001\rangle+|100\rangle, |010\rangle+|100\rangle$, biseparable B_j $\in X^\vee_{\mathrm{node}}(j)-X$ for $j=1,2,3$, where $X^\vee_{\mathrm{node}}(j)=\mathbb{C}P^1_{j\mathrm{th}}\times\mathbb{C}P^3$ are three closed irreducible components of $X^\vee_{\mathrm{sing}}=X^\vee_{\mathrm{cusp}}$;

dim 3: $|000\rangle$, completely separable S $\in X=\cap_{j=1,2,3}X^\vee_{\mathrm{node}}(j)=\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1$.

$G=\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2$ has the onion structure of six orbits on M (see Fig. 2), by excluding the orbit $\varnothing[=X^\vee_{\mathrm{node}}(\varnothing)]$. The dual variety $X^\vee$ is given by $\mathrm{Det}A_3=0$ [cf. Eq. (10)]. Its dimension is $7-1=6$. The outside of $X^\vee$ is generic tripartite entangled class of the maximal dimension, whose representative is GHZ. This suggests that almost any state in the three qubits can be locally transformed into GHZ with a finite probability and vice versa. Next, we can identify $X^\vee_{\mathrm{sing}}$ as $X^\vee_{\mathrm{cusp}}$, which is the union of three closed irreducible subvarieties $X^\vee_{\mathrm{node}}(j)$ for $j=1,2,3$ [20] (also see Appendix A). For example, $X^\vee_{\mathrm{node}}(1)$ means by definition that, in addition to the condition of $X^\vee$ in Sec. II B, there exists some nonzero $x^{(1)}$ such that $F(A,x)=0$ for any $x^{(2)},x^{(3)}$; i.e., a set of linear equations $[y_{i_2,i_3}(x^{(1)})=(\partial^2/\partial x^{(2)}_{i_2}\partial x^{(3)}_{i_3})F(A,x)=0$ for $i_j=0,1]$ has a nontrivial solution $x^{(1)}$. This indicates that the "bipartite" matrix

$$\begin{pmatrix} a_{000} & a_{001} & a_{010} & a_{011} \\ a_{100} & a_{101} & a_{110} & a_{111} \end{pmatrix} \qquad (14)$$

never has the full rank [i.e., six 2×2 minors in Eq. (14) are zero]. We can identify $X^\vee_{\mathrm{node}}(1)$ as the set $\mathbb{C}P^1_{1\mathrm{st}}\times\mathbb{C}P^3$, seen in Sec. II A, of biseparable states between the first party and the rest of the parties. Its dimension is $1+3=4$. Likewise, $X^\vee_{\mathrm{node}}(j)$ for $j=2,3$ gives the biseparable class for the second or third party, respectively. So, the class of $X^\vee - X^\vee_{\mathrm{sing}}$ is found to be tripartite entangled states, whose representative is W. We can intuitively see that, among genuine tripartite entangled states, W is rare, compared to GHZ [5]. Finally, the intersection of $X^\vee_{\mathrm{node}}(j)$ is the completely separable class S, given by the Segrè variety X of dimension 3. Another intuitive explanation about this procedure is seen in Appendix A.

Now we clarify the relationship of six classes by *noninvertible* local operations. Because noninvertible local operations cause the decrease in local ranks [21], the partially ordered structure of entangled states in the three qubits, included in Fig. 4, appears. Two inequivalent tripartite entangled classes, GHZ and W, have the same local ranks $(2,2,2)$ for each party, so that they are not interconvertible by the noninvertible local operations (i.e., general LOCC). Two classes hold different physical properties [5]; the GHZ representative state has the maximal amount of generic tripartite entanglement measured by the 3-tangle $\tau\propto|\mathrm{Det}A_3|$, while the W representative state has the maximal amount of (average) two-partite entanglement distributed over three parties (see also Ref. [22]). Under LOCC, a state in these two classes can be transformed into any state in one of the three biseparable classes $B_j(j=1,2,3)$, where the jth local rank is 1 and the others are 2. Three classes B_j never convert into each other. Likewise, a state in B_j can be locally transformed into any state in the completely separable class S of local ranks $(1,1,1)$.

This is how the onionlike classification of SLOCC orbits reveals that multipartite entangled classes constitute the partially ordered structure. It indicates significant differences from the totally ordered one in the bipartite case. (i) In the three-qubit case, all SLOCC invariants we need to classify is the hyperdeterminant $\mathrm{Det}A_3$ in addition to local ranks. (ii) Although noninvertible local operations generally mean the transformation further inside the onion structure, an outer class cannot necessarily be transformed into the *neighboring* inner class. A good example is given by GHZ and W, as we have just seen.

B. Format $3\times2\times2$ case

Before proceeding to the $n\geqslant4$ qubit case, we drop in the format $3\times2\times2$, which would give an insight into the structure of multipartite entangled states when each party has a system consisting of more than two levels. This case is interesting since on one hand (contrary to the three-qubit case), it is typical that GHZ and W are included in $X^\vee_{\mathrm{sing}}$; on the other hand (similar to the bipartite or three-qubit cases), SLOCC has still finite classes so that it becomes another good test for the equivalence to the SLOCC classification. Besides, it is a boundary format, so that several subvarieties can be explicitly calculated, and enables us to analyze entanglement in the qubits system using an auxiliary level, like ion traps.

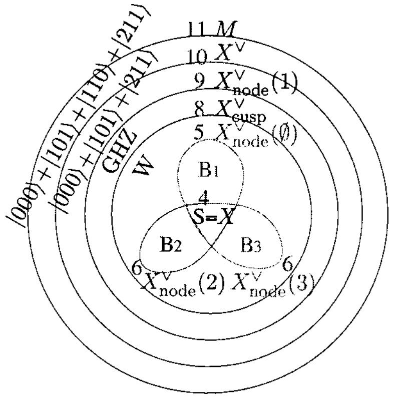

FIG. 3. The onionlike classification of SLOCC orbits in the $3\times2\times2$ format. Although this resembles Fig. 2 in the order of SLOCC orbits (two orbits are added outside), it is worthwhile to note that singularities of $X^\vee$, which classify the SLOCC orbits, have a different order.

dim 11: $|000\rangle+|101\rangle+|110\rangle+|211\rangle \quad \in M(=\mathbb{C}P^{11})-X^\vee$;
dim 10: $|000\rangle+|101\rangle+|211\rangle \quad \in X^\vee-X^\vee_{\text{sing}}=X^\vee-X^\vee_{\text{node}}(1)$;
dim 9: $|000\rangle+|111\rangle$, GHZ $\in X^\vee_{\text{sing}}[=X^\vee_{\text{node}}(1)]-X^\vee_{\text{cusp}}$;
dim 8: $|001\rangle+|010\rangle+|100\rangle$, $W\in X^\vee_{\text{cusp}}-\cup_{j=\emptyset,2,3}X^\vee_{\text{node}}(j)$;
dim 6: $|001\rangle+|100\rangle, |010\rangle+|100\rangle$, biseparable B_2, B_3 $\in X^\vee_{\text{node}}(2)-X, X^\vee_{\text{node}}(3)-X$;
dim 5: $|001\rangle+|010\rangle$, biseparable $B_1 \in X^\vee_{\text{node}}(\emptyset)-X$;
dim 4: $|000\rangle$, completely separable S $\in X=\mathbb{C}P^2\times\mathbb{C}P^1\times\mathbb{C}P^1$.

The onion structure consists of eight orbits on M under SLOCC (see Fig. 3). Generic entangled states of the outermost class are given by nonzero Det$A$ that can be calculated in the *boundary* format as the determinant associated with the Cayley-Koszul complex. Although this is one of the recent successes of Gelfand *et al.* for generalized discriminants, we avoid its detailed explanation here. According to Theorem 3.3 of Ref. [16], we have

$$\text{Det}A=m_1m_4-m_2m_3 \tag{15}$$

of degree 6, where $m_j (j=1,2,3,4)$ is the 3×3 minor of

$$\begin{pmatrix} a_{000} & a_{001} & a_{010} & a_{011} \\ a_{100} & a_{101} & a_{110} & a_{111} \\ a_{200} & a_{201} & a_{210} & a_{211} \end{pmatrix} \tag{16}$$

without the jth column, respectively. Next, it is characteristic that $X^\vee_{\text{sing}}$ is $X^\vee_{\text{node}}(1)$ [20]. Similar to the three-qubit case in Sec. IV A, $X^\vee_{\text{node}}(1)$ means that the "bipartite" matrix in Eq. (16) does not have the full rank, i.e., all four 3×3 minors m_j in Eq. (16) are zero. The SLOCC orbits that appear inside $X^\vee_{\text{sing}}$ are essentially the same as the three-qubit case.

Thus we obtain the partially ordered structure of multipartite entangled states as in Fig. 4. The tripartite entanglement consists of four classes. Because the classes of $M-X^\vee$ (whose representative is $|000\rangle+|101\rangle+|110\rangle+|211\rangle$) and that of $X^\vee-X^\vee_{\text{sing}}$ (whose representative is $|000\rangle+|101\rangle+|211\rangle$) have the same local ranks $(3,2,2)$, they do not convert each other in the same reason as GHZ and W do not.

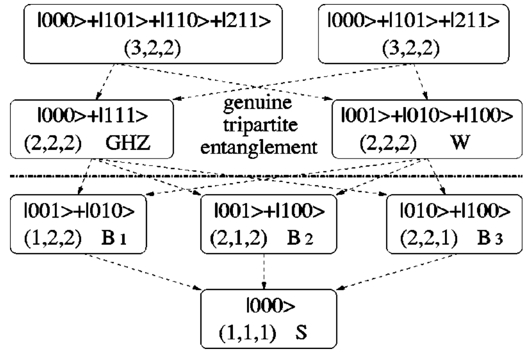

FIG. 4. The partially ordered structure of multipartite pure entangled states in the $3\times2\times2$ format, including the three-qubit case. Each class, corresponding to the SLOCC orbit, is labeled by the representative, local ranks, and the name. Noninvertible local operations, indicated by dashed arrows, degrade higher entangled classes into lower entangled ones.

However, the former two classes of the local ranks $(3,2,2)$ can convert to the latter two classes of $(2,2,2)$ by noninvertible local operations (i.e., LOCC). And we can "degrade" these tripartite entangled classes into the biseparable or completely separable classes by LOCC in a fashion similar to the three qubits.

We notice that three grades in the three-qubit case changed to four grades in the $3\times2\times2$ (one-qutrit and two-qubit) case. In general, the partially ordered structure becomes "higher" as the system of each party becomes the higher-dimensional one. We also see how the tensor rank [10] is inadequate for the onionlike classification of SLOCC orbits.

C. $n\geqslant4$ Qubit (format 2^n) case

Further in the $n\geqslant4$ qubit case, our classification works. The outermost class $M(=\mathbb{C}P^{2^n-1})-X^\vee$ of generic n-partite entangled states is given by Det$A_n\neq0$. In $n=4$, DetA_4 of degree 24 is explicitly calculated by the Schläfli's construction in Sec. III B. It would be suggestive to transform any generic four-partite state (Det$A_4\neq0$) to the "representative" of the outermost class by invertible local operations,

$$\alpha(|0000\rangle+|1111\rangle)+\beta(|0011\rangle+|1100\rangle)+\gamma(|0101\rangle+|1010\rangle)+\delta(|0110\rangle+|1001\rangle), \tag{17}$$

where the continuous complex coefficients $\alpha,\beta,\gamma,\delta$ should satisfy

$$\text{Det}A_4=\alpha^2\beta^2\gamma^2\delta^2(\alpha+\beta+\gamma+\delta)^2(\alpha+\beta+\gamma-\delta)^2(\alpha+\beta-\gamma+\delta)^2(\alpha-\beta+\gamma+\delta)^2(-\alpha+\beta+\gamma+\delta)^2(\alpha+\beta-\gamma-\delta)^2(\alpha-\beta+\gamma-\delta)^2(\alpha-\beta-\gamma+\delta)^2\neq0. \tag{18}$$

Thus three complex parameters remain in the outermost class (since we consider rays rather than normalized state vectors). This means that there are infinitely many same-dimensional SLOCC orbits in the four qubits, and the SLOCC orbits never locally convert to each other when their sets of the parameters are distinct. It is also the case for the $n>4$ qubits. Note that, in $n=4$, this outermost class $M-X^{\vee}$ corresponds to the family of generic states in Verstraete *et al.* classification of the four qubits by a different approach (generalizing the singular-value decomposition in matrix analysis to complex orthogonal equivalence classes), and $X^{\vee}$ contains other special families [23].

The next outermost class is $X^{\vee}-X^{\vee}_{\rm sing}$. In the four qubits, $X^{\vee}_{\rm sing}$ is shown to consist of eight closed irreducible components of codimension 1 in $X^{\vee}$; $X^{\vee}_{\rm cusp}, X^{\vee}_{\rm node}(\emptyset)$, and six $X^{\vee}_{\rm node}(j_1,j_2)$ for $1 \leqslant j_1 < j_2 \leqslant 4$ [20]. They neither contain nor are contained by each other. Their intersections also give (finitely) many lower-dimensional genuine four-partite entangled classes. Since the four-partite entangled classes necessarily have the same local ranks (2,2,2,2), these classes are not interconvertible by noninvertible local operations (i.e., any LOCC). As typical examples, GHZ (the maximally entangled state in Bell's inequalities [24]),

$$|{\rm GHZ}\rangle = |0000\rangle + |1111\rangle \tag{19}$$

(i.e., $a_{0000}=a_{1111}\neq 0$ and the others are 0) is included in the intersection of $X^{\vee}_{\rm node}(\emptyset)$ and six $X^{\vee}_{\rm node}(j_1,j_2)$, but is excluded from $X^{\vee}_{\rm cusp}$. In contrast, W,

$$|W\rangle = |0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle, \tag{20}$$

(i.e., $a_{0001}=a_{0010}=a_{0100}=a_{1000}\neq 0$, and the others are 0) is included in the intersection of $X^{\vee}_{\rm cusp}$ and six $X^{\vee}_{\rm node}(j_1,j_2)$ but is excluded from $X^{\vee}_{\rm node}(\emptyset)$.

In the $n>4$ qubits, $X^{\vee}_{\rm sing}$ is shown to consist of just two closed irreducible components $X^{\vee}_{\rm cusp}$ and $X^{\vee}_{\rm node}(\emptyset)$ [20]. We find that GHZ and W are contained not only in $X^{\vee}({\rm Det}A_n=0)$ but in $X^{\vee}_{\rm sing}$; i.e., they have nontrivial solutions in Eq. (7), satisfying the singular conditions. They correspond to different intersections of further singularities, similar to the four qubits. In other words, they are peculiar, living in the border dimensions between entangled states and separable ones.

In brief, the dual variety $X^{\vee}$ and its singularities lead to the *coarse* onionlike classification of SLOCC orbits, when SLOCC gives infinitely many orbits. The partially ordered structure of multipartite pure entangled states becomes "wider" as the number n of parties increases. Although many inequivalent n-partite entangled classes appear in the n qubits, they never locally convert to each other, as observed in Ref. [5]. In particular, the majority of the n-partite entangled states never convert to GHZ (or W) by LOCC, and the opposite conversion is also not possible. This is a significant difference from the bipartite or three-qubit case, where almost any entangled state and the maximally entangled state (GHZ) can convert to each other by LOCC with nonvanishing probabilities.

V. CONCLUSION

We have presented the onionlike classification of multipartite entanglement (SLOCC orbits) by the dual variety $X^{\vee}$, i.e., the hyperdeterminant Det$A$. It leads to the partially ordered structure, such as Fig. 4, of inequivalent multipartite entangled classes of pure states, which is significantly different from the totally ordered one in the bipartite case. Local ranks are not enough to distinguish these classes, and we need to calculate SLOCC invariants associated with Det$A$. In other words, the generic entangled class of the maximal dimension (the outermost class) is given by the outside of $X^{\vee}$ (Det$A\neq 0$), and other multipartite entangled classes appear as $X^{\vee}$ or its different singularities. Analytically, the classification of multipartite entanglement corresponds to that of the number and pattern of the solutions in Eq. (7).

This work reveals that the situation of the widely known bipartite or three-qubit cases, where the maximally entangled states in Bell's inequalities belong to the generic class, is exceptional. Lying far inside the onion structure, the maximally entangled states (GHZ) are included in the lower-dimensional peculiar class in general, e.g., for the $n\geqslant 4$ qubits. It suggests two points. The majority of multipartite entangled states cannot convert to GHZ by LOCC, and vice versa. So, we have given an alternative explanation to this observation, first made in Ref. [5], by comparing the number of local parameters accessible in SLOCC with the dimension of the whole Hilbert space. Moreover, there seems no *a priori* reason why we choose GHZ states as the *canonical* n-partite entangled states, that, for example, constitute a minimal reversible entanglement generating set (MREGS) in asymptotically reversible LOCC [4,25]. Since the onionlike classification is given by every closed subset, not only our work enables us to see intuitively why, say in the three qubits, the W class is rare compared to the GHZ class, but it can be also extended to the classification of multipartite mixed states (see Appendix B).

The onionlike classification seems to be reasonable in the sense that it coincides with the SLOCC classification when SLOCC gives finitely many orbits, such as the bipartite or three-qubit cases. So two states belonging to the same class can convert each other by invertible local operations with nonzero probabilities. On the other hand, when SLOCC gives infinitely many orbits, this classification is still SLOCC invariant, but may contain in one class infinitely many *same*-dimensional SLOCC orbits that cannot locally convert to each other even probabilistically. For example, in the four-qubit case, the generic entangled class in Eq. (17) has three nonlocal continuous parameters. Note that it can be possible to make the onionlike classification finer by characterizing the nonlocal continuous parameters in each class.

Then, we may ask, what is the physical interpretation of the onionlike classification in the case of infinitely many SLOCC orbits? Although a simple answer has yet to be found, we discuss two points.

(i) Let us consider *global* unitary operations that create the multipartite entanglement. On the one hand, states in distinct classes would have different complexity of the global operations, since they have the distinct number and pattern of nonlocal parameters. On the other hand, states in one class are supposed to have the equivalent complexity, since they just correspond to different "angles" of the global unitary operations.

(ii) We can consider the case where *more than one* state are shared, including the asymptotic case. Even in two shared states, there can exist a local conversion that is impossible if they are operated separately, such as the catalysis effect [26]. So we can expect that we do it more efficiently in this situation, and the coarse classification may have some physical significance. This problem remains unsettled even in the bipartite case.

Finally, two related topics are discussed. (i) The absolute value $|\mathrm{Det}A_n|$ of the hyperdeterminant, representing the amount of generic entanglement, is an entanglement monotone by Vidal [27]. This never conflicts with the property that the maximally entangled states in Bell's inequalities (GHZ) generally have a zero $\mathrm{Det}A_n$. A single-entanglement monotone is insufficient to judge the LOCC convertibility, and generic entangled states of the nonzero $\mathrm{Det}A_n$ cannot convert to GHZ in spite of decreasing $|\mathrm{Det}A_n|$. (ii) The 3-tangle $\tau = 4|\mathrm{Det}A_3|$ first appeared in the context of so-called entanglement sharing [18]; i.e., in the three qubits there is a constraint (trade-off) between the amount of two-partite entanglement and that of three-partite entanglement. By using the entanglement measure (concurrence C) for the two-qubit *mixed* entangled states, this is written as $C_{1(23)}^2 \geqslant C_{12}^2 + C_{13}^2$, and τ is defined by $\tau = C_{1(23)}^2 - C_{12}^2 - C_{13}^2$ for the three-qubit *pure* entangled states. We expect that, in turn, the hyperdeterminant $\mathrm{Det}A_n$ gives a clue to find the entanglement measure of more than two-qubit *mixed* states.

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APPENDIX A: REPRESENTATIVES OF THE THREE-QUBIT ENTANGLED CLASSES

In Sec. IV, we have classified entangled classes (SLOCC orbits), utilizing SLOCC-invariant closed subvarieties such as the dual variety $X^\vee$ and its singularities. In this appendix, we give an intuitive explanation about our technique. We obtain entangled classes *by their representatives*. The three-qubit case is exemplified, and the notation and terminology of Sec. III C is followed.

As the representative of the outermost generic entangled class, almost any state (indeed, satisfying $\mathrm{Det}A_3 \neq 0$) in the whole space $M = \mathbb{C}P^7$ is qualified. The GHZ state $|000\rangle + |111\rangle$ is chosen among them, since it can be seen as the multidimensional analog of the identity matrix.

We look for the representative of the dual variety $X^\vee$, which is qualified as that of the next outermost entangled class. When $X^\vee$ is the hyperplane tangent to the Segrè variety X at x^0 such that $x_{i_j}^{(j)} = \delta_{i_j,0}$ $(j=1,2,3)$, the "x^0 section" of $X^\vee$ is given as

$$X^\vee|_{x^0} = \{a_{000} = a_{001} = a_{010} = a_{100} = 0\}, \tag{A1}$$

in order that Eq. (7) has the nontrivial solution x^0. This suggests that the representative of $X^\vee$ is the W state $|001\rangle + |010\rangle + |100\rangle$, since the states given by Eq. (A1) and W convert to each other under some invertible local operations $G \in \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ as

$$\begin{aligned} &a_{011}|011\rangle + a_{101}|101\rangle + a_{110}|110\rangle + a_{111}|111\rangle \\ &\overset{G}{\sim} |011\rangle + |101\rangle + |110\rangle + |111\rangle \\ &\overset{G}{\sim} |001\rangle + |010\rangle + |100\rangle. \end{aligned} \tag{A2}$$

The candidates for the next outer entangled class are two, node-type $X_{\mathrm{node}}^\vee$ and cusp-type $X_{\mathrm{cusp}}^\vee$, singularities of $X^\vee$. We first consider the representative of $X_{\mathrm{node}}^\vee(1)$. According to Eq. (12), the x^0 section of $X_{\mathrm{node}}^\vee(1)$ is given as

$$\begin{aligned} X_{\mathrm{node}}^\vee(1)|_{x^0} &= X^\vee|_{x^0} \cap X^\vee|_{x^0(1)} \\ &= \{a_{000} = a_{001} = a_{010} = a_{100} = a_{011} = a_{111} = 0\}. \end{aligned} \tag{A3}$$

We find that the representative of $X_{\mathrm{node}}^\vee(1)$ is the biseparable state $|001\rangle + |010\rangle$ in B_1, checking that

$$a_{101}|101\rangle + a_{110}|110\rangle \overset{G}{\sim} |001\rangle + |010\rangle. \tag{A4}$$

In the same manner, $X_{\mathrm{node}}^\vee(2)$ and $X_{\mathrm{node}}^\vee(3)$ represent the biseparable classes B_2 and B_3, respectively.

Let us then analyze the representative of $X_{\mathrm{cusp}}^\vee$. In terms of the quadric part y of $F(A,x)$ at x^0,

$$y = \begin{pmatrix} 0 & a_{110} & a_{101} \\ a_{110} & 0 & a_{011} \\ a_{101} & a_{011} & 0 \end{pmatrix}, \tag{A5}$$

the x^0 section of $X_{\mathrm{cusp}}^\vee$ is given as

$$\begin{aligned} X_{\mathrm{cusp}}^\vee|_{x^0} = \{&a_{000} = a_{001} = a_{010} = a_{100} = 0, \\ &\det y = 2a_{011}a_{101}a_{110} = 0\}. \end{aligned} \tag{A6}$$

We have three possibilities for $\det y = 0$. In the case of $a_{011} = 0$, this component of $X_{\mathrm{cusp}}^\vee$ represents the biseparable class B_1, since

$$a_{101}|101\rangle + a_{110}|110\rangle + a_{111}|111\rangle \overset{G}{\sim} |001\rangle + |010\rangle. \tag{A7}$$

Likewise, in the case of $a_{101} = 0$ or $a_{110} = 0$, each component of $X_{\mathrm{cusp}}^\vee$ corresponds to the biseparable class B_2 or B_3, respectively. Remembering that each B_j is characterized by $X_{\mathrm{node}}^\vee(j)$ for $j = 1,2,3$, we have shown that $X_{\mathrm{cusp}}^\vee$ has three irreducible components $X_{\mathrm{node}}^\vee(j)$. Thus, the next outer entangled classes are three biseparable classes B_j that never contain nor are contained by each other.

In general, remaining entangled classes are given by further singularities of $X^\vee$ such as combinations of the above $X^\vee_{\rm node}$ and $X^\vee_{\rm cusp}$, or genuine higher singularities. In the three-qubit case, since we see that $X^\vee_{\rm node}(j)$ representing the biseparable class B_j is just characterized as $\mathbb{C}P^1_{j\rm th}\times\mathbb{C}P^3$, there remains just one smaller closed irreducible subvariety $\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1=X$ as their intersection $\cap_{j=1,2,3}X^\vee_{\rm node}(j)$. This Segrè variety X represents the completely separable class S, whose representative is $|000\rangle$.

In the text, we have carried out the above procedure in the "x^0-free" manner (x^0 should be taken as *any* state on X), and have obtained entangled classes as (difference) subsets. It enables us to decide readily which entangled class a given state $|\Psi\rangle$ belongs to. After the classification of entangled classes, we can clarify their partially ordered structure under noninvertible local operations in the same manner as in the text.

APPENDIX B: CLASSIFICATION OF MULTIPARTITE MIXED STATES

The onion structure is also useful for the SLOCC-invariant classification of mixed entangled states. A mixed state ρ can be written as a convex combination of projectors onto pure states (extremal points),

$$\rho=\sum_\mu p_\mu|\Psi_\mu(\mathcal{O}_\lambda)\rangle\langle\Psi_\mu(\mathcal{O}_\lambda)|,\quad p_\mu>0, \tag{B1}$$

where $|\Psi_\mu(\mathcal{O}_\lambda)\rangle$ is the pure state belonging to the SLOCC orbit $\mathcal{O}_\lambda$ of an index λ. λ is labeled by the closed subvariety $\overline{\mathcal{O}}$ (i.e., the closure of $\mathcal{O}_\lambda$) such as $X^\vee$, $X^\vee_{\rm node}(J)$, $X^\vee_{\rm cusp}$, and X. Note that, in the multipartite case, there can be many closed subvarieties $\overline{\mathcal{O}}_\lambda$ that never contain nor are contained by each other; for example, $X^\vee_{\rm node}(1)$, $X^\vee_{\rm node}(2)$, and $X^\vee_{\rm node}(3)$ in Fig. 2. So, by taking the union of these "competitive" closed subvarieties $\overline{\mathcal{O}}_\lambda$ (it will form their convex hull in the space of ρ), we pick up only *totally ordered* ones [e.g., M, $X^\vee$, $X^\vee_{\rm cusp}=\cup_{j=1,2,3}X^\vee_{\rm node}(j)$, and X in Fig. 2] for convenience later. Now, we are concerned with at most how various classes of pure entangled states the mixed state ρ consists of. We take the maximal closure of $\overline{\mathcal{O}}_\lambda$ appeared in Eq. (B1) and denote it by $\overline{\mathcal{O}}_{\rm max}$. However, since ρ can be decomposed into the form of Eq. (B1) in infinitely many ways, we should take the minimal closure of $\overline{\mathcal{O}}_{\rm max}$ over all possible decompositions, and write it as $\min\overline{\mathcal{O}}_{\rm max}$. Every convex subset $\mathcal{S}_\lambda$ of $\lambda=\min\overline{\mathcal{O}}_{\rm max}$ is closed such that $\mathcal{S}_\lambda$ of the smaller λ is contained by that of the larger one. In other words, $\mathcal{S}_\lambda$ of the larger λ consists of more classes (SLOCC orbits) of pure entangled states. That is how the mixed state ρ is classified into the *closed* convex subsets $\mathcal{S}_\lambda$ under SLOCC.

In the bipartite case, $\min\overline{\mathcal{O}}_{\rm max}$ is called the Schmidt number [28] (since $\overline{\mathcal{O}}_\lambda$ is just labeled by the Schmidt local rank, as seen in Sec. I). Also in the three-qubit case, this kind of classification has been done in Ref. [11], and four classes appear, following the above observation: (i) GHZ class $\mathcal{S}_M-\mathcal{S}_{X^\vee}$ (consisting of all pure states); (ii) W class $\mathcal{S}_{X^\vee}-\mathcal{S}_{X^\vee_{\rm cusp}}$ (consisting of the pure W, biseparable, or separable states); (iii) biseparable class $\mathcal{S}_{X^\vee_{\rm cusp}}-\mathcal{S}_X$ (consisting of the pure biseparable or separable states); and (iv) separable class $\mathcal{S}_X$ (consisting of only the pure separable states). Needless to say, the trouble is considering all possible decompositions in Eq. (B1). So, it is very difficult to give the criterion to distinguish each closed convex subset $\mathcal{S}_\lambda$, even to distinguish the separable subset $\mathcal{S}_X$. Still, it would be interesting to observe that a witness operator $\mathcal{W}$, which forms the tangent hyperplane ${\rm tr}(\rho\mathcal{W})=0$ detecting $\mathcal{S}_\lambda$ a given ρ belongs to, shares the same idea as our dual variety.

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