Relativistic entanglement and Bell's inequality

Doyeol Ahn,[1,2,*] Hyuk-jae Lee,[1,†] Young Hoon Moon,[1,2] and Sung Woo Hwang[1,3,‡]
[1]*Institute of Quantum Information Processing and Systems, University of Seoul, Seoul 130-743, Korea*
[2]*Department of Electrical and Computer Engineering, University of Seoul, Seoul 130-743, Korea*
[3]*Department of Electronic Engineering, Korea University, Seoul 136-701, Korea*
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In this paper, the Lorentz transformation of entangled Bell states seen by a moving observer is studied. The calculated Bell observable for four joint measurements turns out to give a universal value, $\langle\hat{a}\otimes\hat{b}\rangle+\langle\hat{a}\otimes\hat{b}'\rangle+\langle\hat{a}'\otimes\hat{b}\rangle-\langle\hat{a}'\otimes\hat{b}'\rangle=(2/\sqrt{2-\beta^2})(1+\sqrt{1-\beta^2})$, where $\hat{a},\hat{b}$ are the relativistic spin observables derived from the Pauli-Lubanski pseudovector and $\beta=(v/c)$. We found that the degree of violation of the Bell's inequality is decreasing with increasing velocity of the observer and Bell's inequality is satisfied in the ultrarelativistic limit where the boost speed reaches the speed of light.

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I. INTRODUCTION

Relativistic transformation properties of quantum states, especially, the entangled states are of considerable interest partially because many novel features of the quantum information processing rely on the entanglement and the nonlocality associated with it [1–7]. One would take the teleportation [8,9] as a typical example. The problem of the entanglement and the nonlocality traces back to the famous 1935 paper by Einstein-Podolsky-Rosen (EPR) [10], almost 70 years ago, now known as the EPR paradox, and subsequent studies, most noticeably by Bell [11], which showed that the nature indeed seems to be nonlocal as far as nonrelativistic quantum mechanics is concerned. This subtle question still remains to be answered especially in the relativistic arena. Recently, Czachor [1] and Terashima and Ueda [4] suggested that the degree of violation of the Bell inequality depends on the velocity of the pair of spin-$\frac{1}{2}$ particles or the observer with respect to the laboratory.

The goal of this paper is to give a partial answer to the EPR paradox and the nonlocality. In the previous work [6], we studied the case of an entangled state shared by Alice and Bob in different frames and showed that the entangled pair satisfies Bell's inequality when the boost speed approaches the speed of light somewhat surprisingly. We also showed that the Bell state in the rest frame appears as a superposition of the Bell bases to an observer in the moving frame due to the Wigner rotation of the spin states.

In this paper, we calculate the Bell observables for entangled states in the rest frame seen by the observer moving in the x direction and show that the entangled states satisfy the Bell's inequality when the boost speed approaches the speed of light. The calculated average of the Bell observable for the Lorentz transformed entangled states turns out to be

$$c(\vec{a},\vec{a}',\vec{b},\vec{b}')=\langle\hat{a}\otimes\hat{b}\rangle+\langle\hat{a}\otimes\hat{b}'\rangle+\langle\hat{a}'\otimes\hat{b}\rangle-\langle\hat{a}'\otimes\hat{b}'\rangle$$
$$=\frac{2}{\sqrt{2-\beta^2}}(1+\sqrt{1-\beta^2}), \quad (1)$$

where $\hat{a},\hat{b}$ are the relativistic spin observables for Alice and Bob, respectively, related to the Pauli-Lubanski pseudovector [1,12] which is known to be a relativistically invariant operator corresponding to spin and $\beta=(v/c)$, the ratio of the boost speed and the speed of light.

In the nonrelativistic limit $\beta=0$, and the right-hand-side (rhs) of Eq. (1) gives the value of $2\sqrt{2}$, indicating the maximum violation of Bell's inequality. On the other hand, in the ultrarelativistic limit $\beta=1$, the rhs of Eq. (1) gives the value 2, suggesting the Bell inequality is satisfied. Moreover, Eq. (1) shows that the average Bell observable or the degree of violation of the Bell's inequality is decreasing with increasing velocity of the observer.

In the following section, we give the Lorentz transformation of the quantum states and the Wigner representation of the Lorentz group from a heuristic point of view.

II. RELATIVISTIC TRANSFORMATION OF QUANTUM STATES AND WIGNER REPRESENTATION OF THE LORENTZ GROUP

One of the conceptual barriers for the relativistic treatment of quantum information processing is the difference of the role played by the wave fields and the states vector in relativistic quantum theory. In nonrelativistic quantum mechanics both the wave function and the states vector in the Hilbert space give the probability amplitude which can be used to define conserved positive densities or density matrix. Since an attempt to unify quantum mechanics and special relativity was made by Dirac toward the end of the 1920s and the famous Dirac equation for an electron was discovered, it was found that the waves obeying the relativistic wave equations do not represent the probability amplitude by themselves. For example, the probability wave function for a photon is neither the electric nor the magnetic field which satisfies Maxwell's equations. In a way, the state vector

*Email address: dahn@uoscc.uos.ac.kr
†Email address: lhjae@iquips.uos.ac.kr
‡Email address: swhwang@korea.ac.kr

of a photon is related indirectly to Maxwell's equations. In this sense, the relativistic wave equations must be regarded only as indirect representation for the description of one-particle probability waves, and the forms of equation themselves have a direct connection to the quantum-field theory.

On the other hand, the idea that the quantum states of relativistic particles can be formulated directly without the use of wave equations, was proposed by Wigner in 1939 [13]. He showed that the states of a free particle are given by unitary irreducible representations of the Poincaré group, i.e., the group formed by translations and Lorentz transformations in the Minkowski space. As a matter of fact, if we get all unitary irreducible representations of the Poincaré group, or the Lorentz group, we do have a complete knowledge of relativistic free particle states and behavior [14].

In this paper, we follow Wigner's approach and focus on the Lorentz transformation properties of quantum states, then obtain the relativistic transformation of entangled quantum states. For convenience, we follow the Weinberg's notation [15] throughout the paper.

A multiparticle state vector is denote by

$$\Psi_{p_1\sigma_1;p_2\sigma_2;\ldots}=a^+(\vec{p}_1,\sigma_1)a^+(\vec{p}_2,\sigma_2)\cdots\Psi_0, \tag{2}$$

where p_i labels the four momentum, σ_i is the spin z component, $a^+(\vec{p}_i,\sigma_i)$ is the creation operator which adds a particle with momentum $\vec{p}_i$ and spin σ_i, and Ψ_0 is the Lorentz invariant vacuum state. The Lorentz transformation Λ induces unitary transformation on vectors in the Hilbert space,

$$\Psi\rightarrow U(\Lambda)\Psi, \tag{3}$$

and the operators U satisfies the composition rule

$$U(\bar{\Lambda})U(\Lambda)=U(\bar{\Lambda}\Lambda), \tag{4}$$

while the creation operator has the following transformation rule:

$$U(\Lambda)a^+(\vec{p},\sigma)U(\Lambda)^{-1}=\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\bar{\sigma}}\mathcal{D}^{(j)}_{\bar{\sigma}\sigma}(W(\Lambda,p))a^+(\vec{p}_\Lambda,\bar{\sigma}). \tag{5}$$

Here, $W(\Lambda,p)$ is Wigner's little group element given by

$$W(\Lambda,p)=L^{-1}(\Lambda p)\Lambda L(p), \tag{6}$$

with $\mathcal{D}^{(j)}(W)$ the representation of W for spin j, $p^\mu=(\vec{p},p^0)$, $(\Lambda p)^\mu=[\vec{p}_\Lambda,(\Lambda p)^0]$ with $\mu=1,2,3,0$, and $L(p)$ is the Lorentz transformation such that

$$p^\mu=L^\mu_\nu k^\nu, \tag{7}$$

where $k^\nu=(0,0,0,m)$ is the four-momentum taken in the particle's rest frame. One can also use the conventional ket notation to represent the quantum states as

$$\Psi_{p,\sigma}=a^+(\vec{p},\sigma)\Psi_0=|\vec{p},\sigma\rangle=|\vec{p}\rangle\otimes|\sigma\rangle. \tag{8}$$

We now give the derivation of the representation of the Wigner's little group $W(\Lambda,p)$ for spin-$\frac{1}{2}$ particles following the Halpern's approach [16]. From Eq. (6), the representation $\mathcal{D}^{(1/2)}(W(\Lambda,p))$ is written as

$$\mathcal{D}^{(1/2)}(W(\Lambda,p))=\mathcal{D}^{(1/2)-1}(L(\Lambda p))\mathcal{D}^{(1/2)}(\Lambda)\mathcal{D}^{(1/2)}(L(p)). \tag{9}$$

If we consider an arbitrary boost given by the velocity $\vec{v}$ with $\hat{e}$ as the normal vector in the boost direction, the Lorentz transformation Λ^μ_ν is [17]

$$\Lambda^i_j=\delta_{ij}+e_ie_j(\cosh\alpha-1),$$
$$\Lambda^i_0=\Lambda^0_i=e_i\sinh\alpha,$$
$$\Lambda^0_0=\cosh\alpha=\gamma=\frac{1}{\sqrt{1-\beta^2}}. \tag{10}$$

Then, for $p^\mu=(\vec{p},p^0)$ with $p^0=E_{\vec{p}}$,

$$\vec{p}\,'=\vec{p}_\Lambda=[\vec{p}-(\vec{p}\cdot\hat{e})\hat{e}]+[E_{\vec{p}}\sinh\alpha+(\vec{p}\cdot\hat{e})\cosh\alpha]\hat{e}, \tag{11}$$

$$(\Lambda p)^0=(p^0)'=E_{\vec{p}}\cosh\alpha+(\vec{p}\cdot\hat{e})\sinh\alpha, \tag{12}$$

and

$$\mathcal{D}^{(1/2)}(\Lambda)=I\cosh\frac{\alpha}{2}+(\vec{\sigma}\cdot\hat{e})\sinh\frac{\alpha}{2}. \tag{13}$$

From the two-component spinor representation (Appendix),

$$\begin{aligned}\phi_R(\vec{p})&=\left[\left(\frac{\gamma+1}{2}\right)^{1/2}+\vec{\sigma}\cdot\frac{\vec{p}}{|\vec{p}|}\left(\frac{\gamma-1}{2}\right)^{1/2}\right]\phi_R(0)\\&=\mathcal{D}^{(1/2)}(L(p))\phi_R(0),\end{aligned} \tag{14}$$

where ϕ_R is the two-component spinor, we obtain

$$\mathcal{D}^{(1/2)}(L(p))=\left(\frac{p^0+m}{2m}\right)^{1/2}I+\left(\frac{p^0-m}{2m}\right)^{1/2}\vec{\sigma}\cdot\frac{\vec{p}}{|\vec{p}|} \tag{15}$$

and

$$\mathcal{D}^{(1/2)}(\Lambda L(p))=\left(\frac{(\Lambda p)^0+m}{2m}\right)^{1/2}I+\left(\frac{(\Lambda p)^0-m}{2m}\right)^{1/2}\vec{\sigma}\cdot\frac{\vec{p}_\Lambda}{|\vec{p}_\Lambda|}. \tag{16}$$

Then obviously we get

$$[\mathcal{D}^{(1/2)}(\Lambda L(p))]^{-1}=\left(\frac{(\Lambda p)^0+m}{2m}\right)^{1/2}I-\left(\frac{(\Lambda p)^0-m}{2m}\right)^{1/2}\vec{\sigma}\cdot\frac{\vec{p}_\Lambda}{|\vec{p}_\Lambda|}. \tag{17}$$

Here, $\vec{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ and σ_i is the Pauli matrix. If we put Eqs. (13), (15), and (16) into Eq. (9), we obtain

$$\begin{aligned}\mathcal{D}^{(1/2)}(W(\Lambda,p)) = {} & \alpha^{+}_{\vec{p}_\Lambda}\alpha^{+}_{\vec{p}}\cosh\frac{\alpha}{2} - \alpha^{-}_{\vec{p}_\Lambda}\alpha^{+}_{\vec{p}}\cosh\frac{\alpha}{2}(\vec{\sigma}\cdot\hat{p}_\Lambda) \\ & + \alpha^{+}_{\vec{p}_\Lambda}\alpha^{+}_{\vec{p}}\sinh\frac{\alpha}{2}(\vec{\sigma}\cdot\hat{e}) \\ & - \alpha^{-}_{\vec{p}_\Lambda}\alpha^{+}_{\vec{p}}\sinh\frac{\alpha}{2}(\vec{\sigma}\cdot\hat{p}_\Lambda)(\vec{\sigma}\cdot\hat{e}) \\ & + \alpha^{+}_{\vec{p}_\Lambda}\alpha^{-}_{\vec{p}}\cosh\frac{\alpha}{2}(\vec{\sigma}\cdot\hat{p}) \\ & - \alpha^{-}_{\vec{p}_\Lambda}\alpha^{-}_{\vec{p}}\cosh\frac{\alpha}{2}(\vec{\sigma}\cdot\hat{p}_\Lambda)(\vec{\sigma}\cdot\hat{p}) \\ & + \alpha^{+}_{\vec{p}_\Lambda}\alpha^{-}_{\vec{p}}\sinh\frac{\alpha}{2}(\vec{\sigma}\cdot\hat{e})(\vec{\sigma}\cdot\hat{p}) \\ & - \alpha^{-}_{\vec{p}_\Lambda}\alpha^{-}_{\vec{p}}\sinh\frac{\alpha}{2}(\vec{\sigma}\cdot\hat{p}_\Lambda)(\vec{\sigma}\cdot\hat{e})(\vec{\sigma}\cdot\hat{p}),\end{aligned} \tag{18}$$

where $\alpha^{\pm}_{\vec{p}} = [(p^0 \pm m)/2m]^{1/2}$, $\alpha^{\pm}_{\vec{p}_\Lambda} = \{[(\Lambda p)^0 \pm m]/2m\}^{1/2}$, $\hat{p} = (\vec{P}/|\vec{p}|)$, and $\hat{p}_\Lambda = (\vec{P}_\Lambda/|\vec{p}_\Lambda|)$. Equation (18) can be rearranged into the following form:

$$\mathcal{D}^{1/2}(W(\Lambda,p)) = A + B\vec{\sigma}\cdot\hat{p} + C\vec{\sigma}\cdot\hat{e} + iD\vec{\sigma}\cdot(\hat{p}\times\hat{e}), \tag{19}$$

by using the relations

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b}) \tag{20}$$

and

$$\begin{aligned}(\vec{\sigma}\cdot\hat{p}_\Lambda)(\vec{\sigma}\cdot\hat{e})(\vec{\sigma}\cdot\hat{p}) = {} & (\hat{p}_\Lambda\cdot\hat{e})(\vec{\sigma}\cdot\hat{p}) \\ & + i(\hat{p}_\Lambda\times\hat{e})\cdot\hat{p} + \vec{\sigma}\cdot\{\hat{p}\times(\hat{p}_\Lambda\times\hat{e})\}.\end{aligned} \tag{21}$$

The coefficients A, B, C, and D are obtained after lengthy mathematical manipulations [16,18]. They are

$$A = \frac{1}{\{(p^0+m)[(\Lambda p)^0+m]\}^{1/2}}\left\{(p^0+m)\cosh\frac{\alpha}{2} + (\vec{p}\cdot\hat{e})\sinh\frac{\alpha}{2}\right\}, \tag{22}$$

$$B = C = 0, \tag{23}$$

$$D = -\frac{1}{\{(p^0+m)[(\Lambda p)^0+m]\}^{1/2}}\sinh\frac{\alpha}{2}. \tag{24}$$

Then the Wigner representation of the Lorentz group for the spin-$\frac{1}{2}$ becomes

$$\begin{aligned}\mathcal{D}^{(1/2)}(W(\Lambda,p)) = {} & \frac{1}{\{(p^0+m)[(\Lambda p)^0+m]\}^{1/2}} \\ & \times\left\{(p^0+m)\cosh\frac{\alpha}{2} + (\vec{p}\cdot\hat{e})\sinh\frac{\alpha}{2} - i\sinh\frac{\alpha}{2}\vec{\sigma}\cdot(\vec{p}\times\hat{e})\right\}\end{aligned} \tag{25}$$

$$= \cos\frac{\Omega_{\vec{p}}}{2} + i\sin\frac{\Omega_{\vec{p}}}{2}(\vec{\sigma}\cdot\hat{n}), \tag{26}$$

with

$$\cos\frac{\Omega_{\vec{p}}}{2} = \frac{\cosh\frac{\alpha}{2}\cosh\frac{\delta}{2} + \sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}(\hat{e}\cdot\hat{p})}{\left[\frac{1}{2} + \frac{1}{2}\cosh\alpha\cosh\delta + \frac{1}{2}\sinh\alpha\sinh\delta(\hat{e}\cdot\hat{p})\right]^{1/2}} \tag{27}$$

and

$$\sin\frac{\Omega_{\vec{p}}}{2}\hat{n} = \frac{\sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}(\hat{e}\times\hat{p})}{\left[\frac{1}{2} + \frac{1}{2}\cosh\alpha\cosh\delta + \frac{1}{2}\sinh\alpha\sinh\delta(\hat{e}\cdot\hat{p})\right]^{1/2}}, \tag{28}$$

where $\cosh\delta = (p^0/m)$. We note that Eq. (26) indicates the Lorentz group can be represented by the pure rotation by axis $\hat{n} = \hat{e}\times\hat{p}$ for the two-component spinor.

As an example, we consider the case of momentum vector in the z direction and the boost in the x direction. In this case, we have

$$\cos\frac{\Omega_{\vec{p}}}{2} = \frac{\cosh\frac{\alpha}{2}\cosh\frac{\delta}{2}}{\left[\frac{1}{2} + \frac{1}{2}\cosh\alpha\cosh\delta\right]^{1/2}}, \tag{29}$$

$$\sin\frac{\Omega_{\vec{p}}}{2}\hat{n} = \frac{-\hat{y}\sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}}{\left[\frac{1}{2} + \frac{1}{2}\cosh\alpha\cosh\delta\right]^{1/2}}, \tag{30}$$

and

$$\mathcal{D}^{1/2}(W(\Lambda,p)) = \cos\frac{\Omega_{\vec{p}}}{2} - i\sigma_y\sin\frac{\Omega_{\vec{p}}}{2} = \begin{pmatrix}\cos\frac{\Omega_{\vec{p}}}{2} & -\sin\frac{\Omega_{\vec{p}}}{2} \\ \sin\frac{\Omega_{\vec{p}}}{2} & \cos\frac{\Omega_{\vec{p}}}{2}\end{pmatrix}. \tag{31}$$

The Wigner angle $\Omega_{\vec{p}}$ is defined by

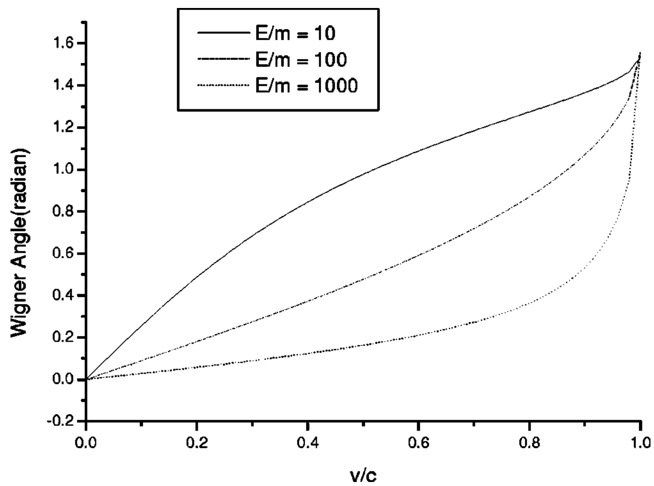

FIG. 1. Wigner angle for different energy to mass ratios for (i) $E_{\vec{p}}/m=10$ (solid line), (ii) $E_{\vec{p}}/m=100$ (dashed line), and (iii) $E_{\vec{p}}/m=1000$ (dotted line) as a function of $\beta=(v/c)$.

$$\tan\Omega_{\vec{p}}=\frac{\sinh\alpha\sinh\delta}{\cosh\alpha+\cosh\delta}. \tag{32}$$

In Fig. 1, we plot the Wigner angle given by Eq. (32) as a function of $\beta=(v/c)$ for (i) $E_{\vec{p}}/m=10$ (solid line), (ii) $E_{\vec{p}}/m=100$ (dashed line), and (iii) $E_{\vec{p}}/m=1000$ (dotted line). It is interesting to note that the higher energy (at the rest frame) of a particle for a given mass, the smaller the rotation angle at lower β. On the other hand, $\Omega_{\vec{p}}$ goes to $(\pi/2)$ as β becomes 1 when the momentum is highly relativistic.

III. RELATIVISTIC ENTANGLEMENT OF QUANTUM STATES AND BELL'S INEQUALITY

We define the momentum-conserved entangled Bell states for spin-$\frac{1}{2}$ particles in the rest frame as follows:

$$\Psi_{00}=\frac{1}{\sqrt{2}}\{a^{+}(\vec{p},\tfrac{1}{2})a^{+}(-\vec{p},\tfrac{1}{2})+a^{+}(\vec{p},-\tfrac{1}{2})a^{+}(-\vec{p},-\tfrac{1}{2})\}\Psi_0, \tag{33a}$$

$$\Psi_{01}=\frac{1}{\sqrt{2}}\{a^{+}(\vec{p},\tfrac{1}{2})a^{+}(-\vec{p},\tfrac{1}{2})-a^{+}(\vec{p},-\tfrac{1}{2})a^{+}(-\vec{p},-\tfrac{1}{2})\}\Psi_0, \tag{33b}$$

$$\Psi_{10}=\frac{1}{\sqrt{2}}\{a^{+}(\vec{p},\tfrac{1}{2})a^{+}(-\vec{p},-\tfrac{1}{2})+a^{+}(\vec{p},-\tfrac{1}{2})a^{+}(-\vec{p},\tfrac{1}{2})\}\Psi_0, \tag{33c}$$

$$\Psi^{11}=\frac{1}{\sqrt{2}}\{a^{+}(\vec{p},\tfrac{1}{2})a^{+}(-\vec{p},-\tfrac{1}{2})-a^{+}(\vec{p},-\tfrac{1}{2})a^{+}(-\vec{p},\tfrac{1}{2})\}\Psi_0, \tag{33d}$$

where Ψ_0 is the Lorentz invariant vacuum state.

For an observer in another reference frame S' described by an arbitrary boost Λ, the transformed Bell states are given by

$$\Psi_{ij}\rightarrow U(\Lambda)\Psi_{ij}. \tag{34}$$

For example, from Eqs. (5) and (33a), $U(\Lambda)\Psi_{00}$ becomes

$$\begin{aligned}U(\Lambda)\Psi_{00}=&\frac{1}{\sqrt{2}}\{U(\Lambda)a^{+}(\vec{p},\tfrac{1}{2})U^{-1}(\Lambda)U(\Lambda)a^{+}(-\vec{p},\tfrac{1}{2})U^{-1}(\Lambda)\\&+U(\Lambda)a^{+}(\vec{p},-\tfrac{1}{2})U^{-1}(\Lambda)U(\Lambda)a^{+}(-\vec{p},-\tfrac{1}{2})U^{-1}(\Lambda)\}U(\Lambda)\Psi_0\\&=\frac{1}{\sqrt{2}}\sum_{\sigma,\sigma'}\left\{\sqrt{\frac{(\Lambda p)^0}{p^0}}\mathcal{D}^{(1/2)}_{\sigma 1/2}(W(\Lambda,p))\sqrt{\frac{(\Lambda\mathcal{P}p)^0}{(\mathcal{P}p)^0}}\mathcal{D}^{(1/2)}_{\sigma' 1/2}(W(\Lambda,\mathcal{P}p))a^{+}(\vec{p}_\Lambda,\sigma)a^{+}(-\vec{p}_\Lambda,\sigma')\right.\\&\left.+\sqrt{\frac{(\Lambda p)^0}{p^0}}\mathcal{D}^{(1/2)}_{\sigma -1/2}(W(\Lambda,p))\sqrt{\frac{(\Lambda\mathcal{P}p)^0}{(\mathcal{P}p)^0}}\mathcal{D}^{(1/2)}_{\sigma' -1/2}(W(\Lambda,\mathcal{P}p))a^{+}(\vec{p}_\Lambda,\sigma)a^{+}(-\vec{p}_\Lambda,\sigma')\right\}\Psi_0\end{aligned} \tag{35}$$

and so on. For simplicity, we assume that $\vec{p}$ is in the z direction, $\vec{p}=(0,0,p)$, and the boost Λ is in the x direction.

Then from Eqs. (31) and (35), we obtain

$$U(\Lambda)\Psi_{00}=\frac{(\Lambda p)^0}{p^0}\cos\Omega_{\vec{p}}\frac{1}{\sqrt{2}}\{a^+(\vec{p}_\Lambda,\tfrac{1}{2})a^+(-\vec{p}_\Lambda,\tfrac{1}{2})+a^+(\vec{p}_\Lambda,-\tfrac{1}{2})a^+(-\vec{p}_\Lambda,-\tfrac{1}{2})\}\Psi_0-\frac{(\Lambda p)^0}{p^0}\sin\Omega_{\vec{p}}\frac{1}{\sqrt{2}}\{a^+(\vec{p}_\Lambda,\tfrac{1}{2})a^+(-\vec{p}_\Lambda,-\tfrac{1}{2})-a^+(\vec{p}_\Lambda,-\tfrac{1}{2})a^+(-\vec{p}_\Lambda,\tfrac{1}{2})\}\Psi_0$$

$$=\frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda,-\vec{p}_\Lambda\rangle\otimes\left\{\cos\Omega_{\vec{p}}\frac{1}{\sqrt{2}}(|\tfrac{1}{2},\tfrac{1}{2}\rangle+|-\tfrac{1}{2},-\tfrac{1}{2}\rangle)\right\}-\frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda,-\vec{p}_\Lambda\rangle\otimes\left\{\sin\Omega_{\vec{p}}\frac{1}{\sqrt{2}}(|\tfrac{1}{2},-\tfrac{1}{2}\rangle-|-\tfrac{1}{2},\tfrac{1}{2}\right\}$$

$$=\frac{(\Lambda p)^0}{p^0}\{\cos\Omega_{\vec{p}}\Psi'_{00}-\sin\Omega_p\Psi'_{11}\},\tag{36a}$$

where Ψ'_{ij} is the Bell states in the moving frame S' whose momenta are transformed as $\vec{p}\to\vec{p}_\Lambda, -\vec{p}\to-\vec{p}_\Lambda$. Likewise, we have

$$U(\Lambda)\Psi_{01}=\frac{(\Lambda p)^0}{p^0}\Psi'_{01},\tag{36b}$$

$$U(\Lambda)\Psi_{10}=\frac{(\Lambda p)^0}{p^0}\Psi'_{10},\tag{36c}$$

and

$$U(\Lambda)\Psi_{11}=\frac{(\Lambda p)^0}{p^0}\{\sin\Omega_{\vec{p}}\Psi'_{00}+\cos\Omega_p\Psi'_{11}\}.\tag{36d}$$

If we regard Ψ'_{ij} as Bell states in the moving frame S', then to an observer in S', the effects of the Lorentz transformation on entangled Bell states among themselves should appear as rotations of Bell states in the frame S'. We are now ready to check whether the Lorentz-transformed Bell states violate Bell's inequality by calculating the average of the Bell observable defined in Eq. (1). Before we proceed, we note that the Bell states can be categorized into two groups. The first group is the subset $\{\Psi_{00},\Psi_{11}\}$ which transforms among themselves by the Lorentz transformation, and the second group is the set $\{\Psi_{01},\Psi_{10}\}$ which remains invariant in forms under the boost except the momentum change. Normalized relativistic spin observables $\hat{a},\hat{b}$ are given by [1]

$$\hat{a}=\frac{(\sqrt{1-\beta^2}\vec{a}_\perp+\vec{a}_\parallel)\cdot\vec{\sigma}}{\sqrt{1+\beta^2[(\hat{e}\cdot\vec{a})^2-1]}}\tag{37}$$

and

$$\hat{b}=\frac{(\sqrt{1-\beta^2}\vec{b}_\perp+\vec{b}_\parallel)\cdot\vec{\sigma}}{\sqrt{1+\beta^2[(\hat{e}\cdot\vec{b})^2-1]}},\tag{38}$$

where the subscripts $\perp$ and $\parallel$ denote the components of $\vec{a}$ (or $\vec{b}$) which are perpendicular and parallel to the boost direction, respectively. Moreover, $|\vec{a}|=|\vec{b}|=1$.

Case I. $\Psi_{00}\to U(\Lambda)\Psi_{00}$. From Eq. (36a), we have

$$U(\Lambda)\Psi_{00}=\frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda,-\vec{p}_\Lambda\rangle\otimes\left[\frac{1}{\sqrt{2}}\cos\Omega_{\vec{p}}(|\tfrac{1}{2},\tfrac{1}{2}\rangle+|-\tfrac{1}{2},-\tfrac{1}{2}\rangle)-\frac{1}{\sqrt{2}}\sin\Omega_{\vec{p}}(|\tfrac{1}{2},-\tfrac{1}{2}\rangle-|-\tfrac{1}{2},\tfrac{1}{2}\rangle)\right].\tag{39}$$

Then, after some mathematical manipulations, we get

$$\hat{a}\otimes\hat{b}|\tfrac{1}{2},\tfrac{1}{2}\rangle=\frac{1}{\sqrt{[1+\beta^2(a_x^2-1)][1+\beta^2(b_x^2-1)]}}\{(1-\beta^2)a_zb_z|\tfrac{1}{2},\tfrac{1}{2}\rangle+\sqrt{1-\beta^2}a_z(b_x+ib_y\sqrt{1-\beta^2})|\tfrac{1}{2},-\tfrac{1}{2}\rangle+\sqrt{1-\beta^2}b_z(a_x+ia_y\sqrt{1-\beta^2})|-\tfrac{1}{2},\tfrac{1}{2}\rangle+(a_x+ia_y\sqrt{1-\beta^2})(b_x+ib_y\sqrt{1-\beta^2})|-\tfrac{1}{2},-\tfrac{1}{2}\rangle\},\tag{40a}$$

$$\hat{a}\otimes\hat{b}|-\tfrac{1}{2},-\tfrac{1}{2}\rangle=\frac{1}{\sqrt{[1+\beta^2(a_x^2-1)][1+\beta^2(b_x^2-1)]}}\times\{(a_x-ia_y\sqrt{1-\beta^2})(b_x-ib_y\sqrt{1-\beta^2})|\tfrac{1}{2},\tfrac{1}{2}\rangle-\sqrt{1-\beta^2}b_z(a_x-ia_y\sqrt{1-\beta^2})|\tfrac{1}{2},-\tfrac{1}{2}\rangle-\sqrt{1-\beta^2}a_z(b_x-ib_y\sqrt{1-\beta^2})|-\tfrac{1}{2},\tfrac{1}{2}\rangle+(1-\beta^2)a_zb_z|-\tfrac{1}{2},-\tfrac{1}{2}\rangle\},\tag{40b}$$

$$\hat{a}\otimes\hat{b}|\tfrac{1}{2},-\tfrac{1}{2}\rangle=\frac{1}{\sqrt{[1+\beta^2(a_x^2-1)][1+\beta^2(b_x^2-1)]}}\times\{\sqrt{1-\beta^2}a_z(b_x-ib_y\sqrt{1-\beta^2})|\tfrac{1}{2},\tfrac{1}{2}\rangle-(1-\beta^2)a_zb_z|\tfrac{1}{2},-\tfrac{1}{2}\rangle+(a_x+ia_y\sqrt{1-\beta^2})(b_x-ib_y\sqrt{1-\beta^2})|-\tfrac{1}{2},\tfrac{1}{2}\rangle-\sqrt{1-\beta^2}b_z(a_x+ia_y\sqrt{1-\beta^2})|-\tfrac{1}{2},-\tfrac{1}{2}\rangle\},\tag{40c}$$

$$\hat{a}\otimes\hat{b}|-\tfrac{1}{2},\tfrac{1}{2}\rangle = \frac{1}{\sqrt{[1+\beta^2(a_x^2-1)][1+\beta^2(b_x^2-1)]}} \times\{\sqrt{1-\beta^2}b_z(a_x-ia_y\sqrt{1-\beta^2})|\tfrac{1}{2},\tfrac{1}{2}\rangle + (a_x-ia_y\sqrt{1-\beta^2})(b_x+ib_y\sqrt{1-\beta^2})|\tfrac{1}{2},-\tfrac{1}{2}\rangle - (1-\beta^2)a_zb_z|-\tfrac{1}{2},\tfrac{1}{2}\rangle - \sqrt{1-\beta^2}a_z(b_x+ib_y\sqrt{1-\beta^2})|-\tfrac{1}{2},-\tfrac{1}{2}\rangle\} \quad (40d)$$

for the boost in the x direction. The calculation of $\langle\hat{a}\otimes\hat{b}\rangle$ is straightforward and is given by

$$\langle\hat{a}\otimes\hat{b}\rangle = \frac{1}{\sqrt{[1+\beta^2(a_x^2-1)][1+\beta^2(b_x^2-1)]}} \times\{[a_xb_x+(1-\beta^2)a_zb_z]\cos(2\Omega_{\vec{p}}) - (1-\beta^2)a_yb_y - \sqrt{1-\beta^2} \times(a_zb_x-b_za_x)\sin(2\Omega_{\vec{p}})\}. \quad (41)$$

It is interesting to note that in the ultrarelativistic limit $\beta\rightarrow 1$, Eq. (41) becomes

$$\langle\hat{a}\otimes\hat{b}\rangle \rightarrow \frac{a_x}{|a_x|}\frac{b_x}{|b_x|}\cos(2\Omega_{\vec{p}}), \quad (42)$$

implying that the joint measurements are not correlated at all. As a result, one might suspect that the entangled state satisfies Bell's inequality. We now consider the vectors $\vec{a}=[(1/\sqrt{2}),-(1/\sqrt{2}),0]$, $\vec{a}'=[-(1/\sqrt{2}),-(1/\sqrt{2}),0]$, $\vec{b}=(0,1,0)$, $\vec{b}'=(1,0,0)$, which lead to the maximum violation of the Bell's inequality in the nonrelativistic domain $\Omega_p=0$ and $\beta=0$. Then the Bell observable for the four relevant joint measurements becomes

$$\langle\hat{a}\otimes\hat{b}\rangle+\langle\hat{a}\otimes\hat{b}'\rangle+\langle\hat{a}'\otimes\hat{b}\rangle-\langle\hat{a}'\otimes\hat{b}'\rangle = \frac{2}{\sqrt{2-\beta^2}}(\sqrt{1-\beta^2}+\cos\Omega_p). \quad (43)$$

In the ultrarelativistic limit where $\beta=1$, the Eq. (43) gives the maximum value of 2 satisfying Bell's inequality as expected.

Case II. $\Psi_{10}\rightarrow U(\Lambda)\Psi_{10}$. This case is interesting because the Lorentz transformation leaves the Bell state invariant in form, which is

$$U(\Lambda)\Psi_{10} = \frac{(\Lambda p)^0}{p^0}|\vec{p}_\Lambda,-\vec{p}_\Lambda\rangle\otimes\frac{1}{\sqrt{2}}(|\tfrac{1}{2},-\tfrac{1}{2}\rangle+|-\tfrac{1}{2},\tfrac{1}{2}\rangle). \quad (44)$$

From Eqs. (40a)–(40d), we obtain

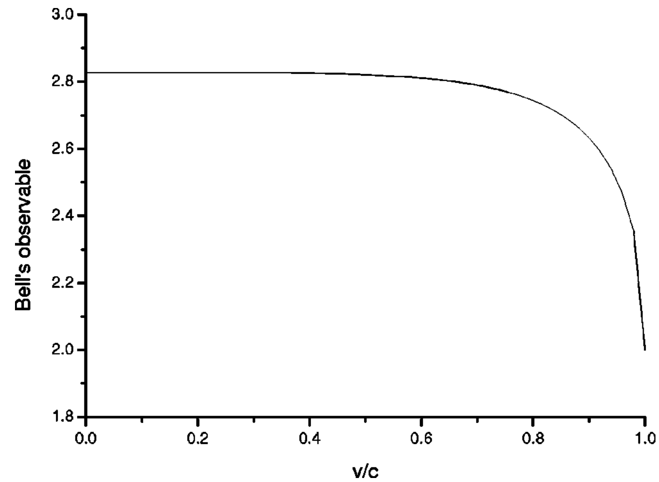

FIG. 2. Plot of the Bell observables as a function of $\beta = (v/c)$.

$$\langle\hat{a}\otimes\hat{b}\rangle = \frac{1}{\sqrt{[1+\beta^2(a_x^2-1)][1+\beta^2(b_x^2-1)]}} \times\{a_xb_x+(1-\beta^2)(a_yb_y-a_zb_z)\}. \quad (45)$$

Then, in the ultrarelativistic limit $\beta\rightarrow 1$, we have

$$\langle\hat{a}\otimes\hat{b}\rangle \rightarrow \frac{a_x}{|a_x|}\frac{b_x}{|b_x|}, \quad (46)$$

again, indicating the joint measurements, become uncorrelated in this limit. We consider the vectors $\vec{a}=[(1/\sqrt{2}),(1/\sqrt{2}),0]$, $\vec{a}'=[-(1/\sqrt{2}),-(1/\sqrt{2}),0]$, $\vec{b}=(0,1,0)$, $\vec{b}'=(1,0,0)$, which lead to the maximum violation of Bell's inequality in the nonrelativistic regime. Then the Bell observable for the four relevant joint measurements becomes

$$\langle\hat{a}\otimes\hat{b}\rangle+\langle\hat{a}\otimes\hat{b}'\rangle+\langle\hat{a}'\otimes\hat{b}\rangle-\langle\hat{a}'\otimes\hat{b}'\rangle = \frac{2}{\sqrt{2-\beta^2}}(\sqrt{1-\beta^2}+1), \quad (47)$$

thus giving same maximum value as in case I. The above results show that it may be irrelevant whether the form of the entanglement is invariant or not in calculating the Bell's inequality. It can also be shown that one can obtain the same value for the Bell observables given by Eq. (47) for $U(\Lambda)\Psi_{01}$ and $U(\Lambda)\Psi_{11}$ implying Eq. (47) is the universal result. In Fig. 2, the universal Bell observable [Eq. (47)] is calculated as a function of $\beta=(v/c)$.

We note that the Bell observable [Eq. (47)] is a monotonically decreasing function of β, approaching the limit value of 2 from above when $\beta=1$ which indicates the degree of violation of Bell's inequality is decreasing with increasing velocity of the observer. It is interesting to note that if one simply rotates the spin directions of $\vec{a},\vec{b},\vec{a}',\vec{b}'$ instead of using the Pauli-Lubanski vectors, then the entanglement between the spins of the Bell states would be unchanged and the results of the spin measurements will be exactly the same as if they were done in the rest frame. There still remains a question of why Bell's inequality is satisfied at the ultrarelativistic limit. One plausible explanation seems that the shape of the entangled pair seen by the observer becomes more deformed as boost speed increases and as a consequence, both spins of the pair are tilted toward the boost axis [19]. Then such a spin has, for $\beta=1$, commuting components and behaves like a classical observable.

IV. SUMMARY

In this work, we studied the Lorentz-transformed entangled Bell states and the Bell observables to investigate whether Bell's inequality is violated in all regime. We first present the quantum state transformation and the Wigner representation of Lorentz group from the heuristic point of view. The calculated Wigner angle as a function of $\beta=(v/c)$ shows that it depends on the energy-mass ratio. We have calculated the Bell observable for the joint four measurements and found that the results are universal for all entangled states:

$$c(\vec{a},\vec{a}',\vec{b},\vec{b}')=\langle\hat{a}\otimes\hat{b}\rangle+\langle\hat{a}\otimes\hat{b}'\rangle+\langle\hat{a}'\otimes\hat{b}\rangle-\langle\hat{a}'\otimes\hat{b}'\rangle$$

$$=\frac{2}{\sqrt{2-\beta^2}}(1+\sqrt{1-\beta^2}),$$

where $\hat{a},\hat{b}$ are the relativistic spin observables derived from the Pauli-Lubanski pseudovector. It turns out that the Bell observable is a monotonically decreasing function of β and approaches the limit value of 2 when $\beta=1$ indicating that the Bell's inequality is satisfied in the ultrarelativistic limit.

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APPENDIX: DERIVATION OF EQ. (14)

Let $\vec{J}=(J_1,J_2,J_3)$ and $\vec{K}=(K_1,K_2,K_3)$ be generators for the rotation and the boost, respectively, and define

$$\vec{A}=\frac{1}{2}(\vec{J}+i\vec{K}) \quad \text{and} \quad \vec{B}=\frac{1}{2}(\vec{J}-i\vec{K}). \tag{A1}$$

Then it is easy to show that

$$[A_i,A_j]=i\epsilon_{ijk}A_k,$$
$$[B_i,B_j]=i\epsilon_{ijk}B_k,$$
$$[A_i,B_j]=0. \tag{A2}$$

We now define the unitary transformation corresponding to the homogeneous Lorentz transformation as

$$U(\Lambda)=1+\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}=e^{i/2\omega_{\mu\nu}J^{\mu\nu}}. \tag{A3}$$

The antisymmetric tensor $\omega_{\mu\nu}$ and the generator $J^{\mu\nu}$ can be written as

$$\omega_{\mu\nu}=\begin{bmatrix}0&\omega_{01}&\omega_{02}&\omega_{03}\\-\omega_{01}&0&\omega_{12}&\omega_{13}\\-\omega_{02}&-\omega_{12}&0&\omega_{23}\\-\omega_{03}&-\omega_{13}&-\omega_{23}&0\end{bmatrix},$$
$$=\begin{bmatrix}0&\phi_1&\phi_2&\phi_3\\-\phi_1&0&\theta_3&-\theta_2\\-\phi_2&-\theta_3&0&\theta_1\\-\phi_3&\theta_2&-\theta_1&0\end{bmatrix}, \tag{A4}$$

and

$$J^{\mu\nu}=\begin{bmatrix}0&J^{01}&J^{02}&J^{03}\\-J^{01}&0&J^{12}&J^{13}\\-J^{02}&-J^{12}&0&J^{23}\\-J^{03}&-J^{13}&-J^{23}&0\end{bmatrix}$$
$$=\begin{bmatrix}0&K_1&K_2&K_3\\-K_1&0&J_3&-J_2\\-K_2&-J_3&0&J_1\\-K_3&J_2&-J_1&0\end{bmatrix}. \tag{A5}$$

Here,

$$\vec{J}=(J_1,J_2,J_3)=(J^{23},J^{21},J^{12}),$$
$$\vec{K}=(K_1,K_2,K_3)=(J^{01},J^{12},J^{13}),$$
$$\vec{\theta}=(\theta_1,\theta_2,\theta_3)=(\omega_{23},\omega_{31},\omega_{12}),$$
$$\vec{\phi}=(\phi_1,\phi_2,\phi_3)=(\omega_{01},\omega_{02},\omega_{03}). \tag{A6}$$

Then from Eqs. (A3)–(A6), we obtain

$$U(\Lambda)=\exp\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}=\exp[i(\omega_{01}J^{01}+\omega_{02}J^{02}+\omega_{03}J^{03}$$
$$+\omega_{12}J^{12}+\omega_{23}J^{23}+\omega_{31}J^{31})]$$
$$=\exp[i(\vec{\phi}\cdot\vec{K}+\vec{\theta}\cdot\vec{J})]=\exp\{i[\vec{\phi}\cdot(-i)(\vec{A}-\vec{B})$$
$$+\vec{\theta}\cdot(\vec{A}+\vec{B})]\}$$
$$=\exp[i(\vec{\theta}-i\vec{\phi})\cdot\vec{A}+i(\vec{\theta}+i\vec{\phi})\cdot\vec{B}]$$
$$=\exp[i(\vec{\theta}-i\vec{\phi})\cdot\vec{A}]\exp[i(\vec{\theta}+i\vec{\phi})\cdot\vec{B}], \tag{A7}$$

since $[\vec{A},\vec{B}]=0$. From Eq. (A7), it can be seen that $U(\Lambda)$ can be represented by $SU(2)\otimes SU(2)$ for spin-$\frac{1}{2}$ particle. From the relation $[\vec{A},\vec{B}]=0$, we can find the common eigenstate $\psi=\psi(a_j,b_j)$ which can be used in the representation of $U(\Lambda)$. As a special case, we consider the case of $b_j=0$ and $j=\frac{1}{2}$. Then $\vec{B}=\frac{1}{2}(\vec{J}-i\vec{K})=0,\vec{A}=\frac{1}{2}(\vec{J}+i\vec{K})=\vec{J}$, and as result, we get

$$U(\Lambda)=\exp[i(\vec{\theta}-i\vec{\phi})\cdot\vec{J}]=\exp\left[i(\vec{\theta}-i\vec{\phi})\cdot\frac{\vec{\sigma}}{2}\right]. \quad \text{(A8)}$$

For a given two-component spinor ϕ_R, we note that ϕ_R transforms under the homogenous Lorentz transformation as

$$\phi_R \rightarrow U(\Lambda)\phi_R=\exp(\tfrac{1}{2}\vec{\sigma}\cdot\vec{\phi})\phi_R . \quad \text{(A9)}$$

Using the relations, $\gamma=\cosh\phi$, $\gamma\beta=\sinh\phi$, and $\hat{p}=\hat{\phi}$, we finally obtain

$$\phi_R(\vec{p})=\left[\left(\frac{\gamma+1}{2}\right)^{1/2}+\vec{\sigma}\cdot\frac{\vec{p}}{|\vec{p}|}\left(\frac{\gamma-1}{2}\right)^{1/2}\right]\phi_R(0). \quad \text{(A10)}$$

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