Quartic anharmonic oscillator and non-Hermiticity

Jing-Ling Chen,[1,2,*] L. C. Kwek,[2,3,†] and C. H. Oh[2,‡]
[1]*Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, P.O. Box 8009(26), Beijing 100088, People's Republic of China*
[2]*Department of Physics, Faculty of Science, National University of Singapore, Lower Kent Ridge, Singapore 119260, Republic of Singapore*
[3]*National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore 639798*
(Received 21 May 2001; revised manuscript received 22 October 2002; published 6 January 2003)

Using a group-theoretic approach, we investigate some new peculiar features of a general quartic anharmonic oscillator. When the coefficient of the quartic term is positive and the potential is differentiable, we find that continuity of the derivative of the potential forces the nonexistence of an analytic wave function. For the case in which the coefficient of the quartic term is negative, we find that normalizability of the wave function requires non-Hermiticity of the Hamiltonian. Finally, we apply our method to gain some insight on the double well potential.

DOI: 10.1103/PhysRevA.67.012101 PACS number(s): 03.65.Fd, 34.20.Gj

I. INTRODUCTION AND MOTIVATION

One of the basic problems of nonrelativistic quantum mechanics is to find the energy spectrum and wave functions of a physical system governed by the Schrödinger equation $H\Psi(x)=E\Psi(x)$ with an appropriate potential $V(x)$. For a special class of potentials [1] such as the harmonic oscillator (HO), the Coulomb potential, the Morse potential and so forth, exact solutions can be found. However, there exist a large class of potentials with no known exact solutions. One such potential is the anharmonic oscillator [2–13]. While the harmonic oscillator has provided invaluable insights into the investigation of many physical systems, the anharmonic oscillator (AHO) has played a pivotal role in helping us understand and model more realistic physical systems since real-world problems certainly deviate from the idealized picture of harmonic oscillators. Indeed, it serves as a basic tool for checking different approximate and perturbative methods in quantum mechanics [14–30], the simplified counterpart of field-theoretical models and so forth. In spite of its apparent simplicity, it has been difficult to extract the energy spectrum and wave functions of physical systems endowed with an anharmonic interaction. Consequently, it has become a standard norm to approach this problem via perturbation theory.

The energy E of a physical system must necessarily be a real and finite number. Moreover, the associated wave function in quantum mechanics, Ψ, has to satisfy the following three conditions:

(i) Continuity,

$$\Psi^{-}(x)|_{x=0}=\Psi^{+}(x)|_{x=0};$$

(ii) normalizable,

$$\lim_{x\to+\infty}\Psi^{+}(x)=0,\quad \lim_{x\to-\infty}\Psi^{-}(x)=0,$$

$$\text{or}\quad \int_{-\infty}^{+\infty}|\Psi(x)|^2dx=\text{finite number};$$

(iii)

$$\left.\frac{d\Psi^{-}(x)}{dx}\right|_{x=0}=\left.\frac{d\Psi^{+}(x)}{dx}\right|_{x=0},$$

if $\mathrm{d}V(x)/dx$ is continuous everywhere.

It is well known that any perturbation theory can provide us with some approximate eigenvalues and wave functions. It is also easy to verify conditions (i) and (ii) for these approximate wave functions. However, it is difficult to verify condition (iii) without knowing the exact form of $\Psi(x)$. While literature on anharmonic oscillator regarding the convergent properties of eigenvalue and wave function abounds, condition (iii) has seldom been discussed and it is in fact often even ignored. Nevertheless, the eigenvalue E of a function $\Psi(x)$ that satisfies the Schrödinger equation $H\Psi(x)=E\Psi(x)$ without fulfilling condition (iii) is not a true solution and should not be regarded as the correct energy of the physical system.

One anharmonic oscillator system that has defied exact solutions for many years has been case of the harmonic oscillator with quartic terms. The aim of this paper is to investigate the most general quartic anharmonic oscillator based essentially on the SU(2) group. There has been extensive literature on the application of Lie groups to the study of harmonic oscillators and other solvable models [31]. The Schrödinger equation of a quartic anharmonic oscillator reads ($\gamma\neq 0$)

$$H\Psi(x)=\left(-\frac{d^2}{dx^2}+V(x)\right)\Psi(x)=E\Psi(x),$$

$$V(x)=\rho x+\alpha x^2+\beta x^3+\gamma x^4. \tag{1}$$

*Email address: jinglingchen@eyou.com
†Email address: lckwek@nie.edu.sg
‡Email address: phyohch@nus.edu.sg

Our result shows that there does not seem to exist a *correct* wave function for a Hermitian Hamiltonian H. More specifically, our result shows that the above Schrödinger equation cannot possess an analytic wave function unless H is non-Hermitian (i.e., ρ, α, β, and γ are not all real numbers). Indeed, however, the theory of $\mathcal{PT}$-symmetric Hamiltonians which are generally non-Hermitian but possesses real eigenvalues is currently an active area of research [32]. In Sec. II, we discuss the situation for the case in which $\gamma>0$. In fact, we found that in this case, it is not possible to have an analytic wave function unless the potential is nondifferentiable. In Sec. III, we present the case of $\gamma<0$. In this case, we found that a ground-state wave function can be found, provided we insist on non-Hermiticity for the Hamiltonian. Finally in Sec. IV, we briefly discuss the case of the double-well potential.

II. CASE OF $\gamma>0$

A. For $x>0$

After performing the transformation

$$\Psi^{+}(x)=\mathcal{N}_0^{+}\exp\left[-\int W_{+}(x)dx\right]\phi(x),$$

$$W_{+}(x)=\mu x^2+\tau x+\nu, \tag{2}$$

we shall arrive at

$$H\phi=\left(-\frac{d^2}{dx^2}+2W_{+}\frac{d}{dx}-(W_{+}^2-W_{+}'-V)\right)\phi=E\phi, \tag{3}$$

or

$$H\phi=\left(-\frac{d^2}{dx^2}+2(\mu x^2+\tau x+\nu)\frac{d}{dx}-[(\mu^2-\gamma)x^4+(2\mu\tau-\beta)x^3+(\tau^2+2\mu\nu-\alpha)x^2+(2\tau\nu-2\mu-\rho)x+\nu^2-\tau]\right)\phi=E\phi. \tag{4}$$

To express the Hamiltonian in terms of SU(2) generators [31] realized by

$$j_{+}=2jx-x^2\frac{d}{dx},\quad j_0=-j+x\frac{d}{dx},\quad j_{-}=\frac{d}{dx}, \tag{5}$$

where j is the spin ($j=0,1/2,1,\ldots,$ and $j_{-}x^0=j_{+}x^{2j}=0$), we must have

$$\mu^2=\gamma,\quad 2\mu\tau=\beta,\quad \tau^2+2\mu\nu=\alpha, \tag{6}$$

and as a result,

$$H\phi=(Aj_{-}^2+Bj_{+}+Cj_0+Dj_{-}+K)\phi=E\phi, \tag{7}$$

i.e.,

$$H\phi=\left(A\frac{d^2}{dx^2}+(-Bx^2+Cx+D)\frac{d}{dx}+2jBx+K-jC\right)\phi=E\phi, \tag{8}$$

where

$$A=-1,\quad B=-2\mu,\quad C=2\tau,\quad D=2\nu,$$

$$K=-\nu^2+(j+1/2)C,\quad \nu=[\rho+2\mu(1+2j)]/2\tau. \tag{9}$$

One can find that $[H,\mathbf{j}^2]=0$, which implies that $\mathbf{j}^2$ is a constant of motion. Therefore, $\phi_j^{+}(x)=\Sigma_{m=0}^{2j}a_m x^m$ could be the common eigenfunction of H and $\mathbf{j}^2$, here the wave function $\phi_j^{+}(x)$ are characterized by the quantum number j. The ground-state energy should correspond to the lowest value of j, i.e., $j=0$, in this case $\phi_0^{+}(x)=1$, and

$$E_0^{+}=K=-\nu^2+\tau, \tag{10}$$

with the corresponding wave function

$$\Psi_0^{+}(x)=\mathcal{N}_0^{+}\exp\left(-\int W_{+}(x)dx\right)=\mathcal{N}_0^{+}\exp[-(\tfrac{1}{3}\mu x^3+\tfrac{1}{2}\tau x^2+\nu x)], \tag{11}$$

where $\mathcal{N}_0^{+}$ is a normalization constant.

Since

$$\lim_{x\to+\infty}\Psi_0^{+}(x)=0, \tag{12}$$

we then have

$$\mu=\sqrt{\gamma},\quad \tau=\frac{\beta}{2\sqrt{\gamma}},\quad \nu=\frac{\alpha-\dfrac{\beta^2}{4\gamma}}{2\sqrt{\gamma}}=\frac{\rho\sqrt{\gamma}+2\gamma}{\beta}. \tag{13}$$

B. For $x<0$

After performing the transformation

$$\Psi^{-}(x)=\mathcal{N}_0^{-}\exp\left[-\int W_{-}(x)dx\right]\phi(x),$$

$$W_{-}(x)=\mu' x^2+\tau' x+\nu', \tag{14}$$

we shall arrive at

$$H\phi=\left(-\frac{d^2}{dx^2}+2W_{-}\frac{d}{dx}-(W_{-}^2-W_{-}'-V)\right)\phi=E\phi, \tag{15}$$

or

$$H\phi=\left(-\frac{d^2}{dx^2}+2(\mu' x^2+\tau' x+\nu')\frac{d}{dx}-[(\mu'^2-\gamma)x^4+(2\mu'\tau'-\beta)x^3+(\tau'^2+2\mu'\nu'-\alpha)x^2+(2\tau'\nu'-2\mu'-\rho)x+\nu'^2-\tau']\right)\psi=E\phi. \quad (16)$$

To express the Hamiltonian in terms of SU(2) generators realized by

$$j_+=2jx-x^2\frac{d}{dx},\quad j_0=-j+x\frac{d}{dx},\quad j_-=\frac{d}{dx},\quad (17)$$

where j is the spin ($j=0,1/2,1,\ldots,$ and $j_-x^0=j_+x^{2j}=0$), we must have

$$\mu'^2=\gamma,\quad 2\mu'\tau'=\beta,\quad \tau'^2+2\mu'\nu'=\alpha, \quad (18)$$

and as a result,

$$H\phi=(A'j_-^2+B'j_++C'j_0+D'j_-+K')\phi=E\phi, \quad (19)$$

where

$$A'=-1,\quad B'=-2\mu',\quad C'=2\tau',\quad D'=2\nu',$$

$$K'=-\nu'^2+(j+1/2)C',\quad \nu'=[\rho+2\mu'(1+2j)]/2\tau'.$$

One can find that $[H,\mathbf{j}^2]=0$, which implies that $\mathbf{j}^2$ is a constant of motion. Therefore $\phi_j^-(x)=\Sigma_{m=0}^{2j}a_m x^m$ could be the common eigenfunction of H and $\mathbf{j}^2$, here the wave function $\phi_j^-(x)$ is characterized by the quantum number j. The ground-state energy should correspond to $j=0$, in this case $\phi_0^-(x)=1$, and

$$E_0^-=K'=-\nu'^2+\tau', \quad (20)$$

with the corresponding wave function

$$\Psi_0^-(x)=\mathcal{N}_0^-\exp\left(-\int W_-(x)dx\right)=\mathcal{N}_0^-\exp[-(\tfrac{1}{3}\mu' x^3+\tfrac{1}{2}\tau' x^2+\nu' x)], \quad (21)$$

where $\mathcal{N}_0^-$ is a normalization constant.

Since

$$\lim_{x\to-\infty}\Psi_0^-(x)=0, \quad (22)$$

we then have

$$\mu'=-\sqrt{\gamma},\quad \tau'=-\frac{\beta}{2\sqrt{\gamma}},$$

$$\nu'=-\frac{\alpha-\dfrac{\beta^2}{4\gamma}}{2\sqrt{\gamma}}=\frac{-\rho\sqrt{\gamma}+2\gamma}{\beta}. \quad (23)$$

C. Remarks

(1) The continuity of the wave function $\Psi_0^-(x)|_{x=0}=\Psi_0^+(x)|_{x=0}$ implies that

$$\mathcal{N}_0^+=\mathcal{N}_0^-\equiv\mathcal{N}_0. \quad (24)$$

(2) The quartic anharmonic potential $V(x)=\rho x+\alpha x^2+\beta x^3+\gamma x^4$ does not contain a term like $V_0\delta(x-a)$, therefore the continuity of the derivation for the potential implies that $[d\Psi_0^-(x)/dx]|_{x=0}=[d\Psi_0^+(x)/dx]|_{x=0}$, i.e.,

$$\nu=\nu', \quad (25)$$

or

$$\alpha-\frac{\beta^2}{4\gamma}=0,\quad \rho\sqrt{\gamma}+2\gamma=0,\quad \text{and}\quad -\rho\sqrt{\gamma}+2\gamma=0, \quad (26)$$

however, the last two relations of the above equation contradict with each other. Consequently, when $\gamma>0$, there does not exist a suitable wave function for a differentiable quartic harmonic potential $V(x)=\rho x+\alpha x^2+\beta x^3+\gamma x^4$.

(3) For a nondifferentiable quartic anharmonic potential $V_1(x)=\rho|x|+\alpha x^2+\beta x^3+\gamma x^4$, by using the similar procedure developed in Sec. I A and Sec. I B, one will obtain

$$\nu=\frac{\alpha-\dfrac{\beta^2}{4\gamma}}{2\sqrt{\gamma}}=\frac{\rho\sqrt{\gamma}+2\gamma}{\beta},\quad \nu'=-\frac{\alpha-\dfrac{\beta^2}{4\gamma}}{2\sqrt{\gamma}}=-\frac{\rho\sqrt{\gamma}+2\gamma}{\beta}. \quad (27)$$

In the case, $\Psi_0^-(x)|_{x=0}=\Psi_0^+(x)|_{x=0}$ is satisfied provided that $\mathcal{N}_0^+=\mathcal{N}_0^-$. For the condition $[d\Psi_0^-(x)/dx]|_{x=0}=[d\Psi_0^+(x)/dx]|_{x=0}$ to be satisfied, we must have $\nu=\nu'$, i.e.,

$$\alpha-\frac{\beta^2}{4\gamma}=0,\quad \text{and}\quad \rho\sqrt{\gamma}+2\gamma=0, \quad (28)$$

or

$$\alpha=\frac{\beta^2}{4\gamma},\quad \rho=-2\sqrt{\gamma}. \quad (29)$$

Furthermore, as an eigenvalue of Hamiltonian, E_0^- should be equal to E_0^+, and hence we have

$$\tau=\tau', \quad (30)$$

which implies that

$$\beta=0, \quad (31)$$

thus

$$\alpha=0,\quad E_0^-=E_0^+\equiv E_0=0. \quad (32)$$

Hence, when $\gamma>0$, for the nondifferentiable quartic anharmonic potential $V_1(x)=\rho|x|+\gamma x^4$, its exact groundstate and energy are

$$\Psi_0(x)=\mathcal{N}_0\exp\left(-\frac{\sqrt{\gamma}}{3}|x|^3\right),\quad E_0=0. \tag{33}$$

III. CASE OF $\gamma<0$

A. The forms of ground state and ground-state energy

After performing the transformation $(-\infty<x<+\infty)$

$$\Psi(x)=\mathcal{N}_0'\exp\left[-\int W(x)dx\right]\phi(x),\quad W(x)=\mu x^2+\tau x+\nu, \tag{34}$$

we arrive at

$$H\phi=\left(-\frac{d^2}{dx^2}+2W\frac{d}{dx}-(W^2-W'-V)\right)\phi=E\phi, \tag{35}$$

or

$$H\phi=\left(-\frac{d^2}{dx^2}+2(\mu x^2+\tau x+\nu)\frac{d}{dx}-[(\mu^2-\gamma)x^4+(2\mu\tau-\beta)x^3+(\tau^2+2\mu\nu-\alpha)x^2+(2\tau\nu-2\mu-\rho)x+\nu^2-\tau]\right)\psi=E\phi, \tag{36}$$

To express the Hamiltonian in terms of SU(2) generators, we must have

$$\mu^2=\gamma,\quad 2\mu\tau=\beta,\quad \tau^2+2\mu\nu=\alpha, \tag{37}$$

and as a result,

$$H\phi=(Aj_-^2+Bj_++Cj_0+Dj_-+K)\phi=E\phi, \tag{38}$$

where

$$A=-1,\quad B=-2\mu,\quad C=2\tau,\quad D=2\nu,$$

$$K=-\nu^2+(j+1/2)C,\quad \nu=[\rho+2\mu(1+2j)]/2\tau. \tag{39}$$

$[H,\mathbf{j}^2]=0$ implies that $\mathbf{j}^2$ is a constant of motion. Therefore $\phi_j(x)=\Sigma_{m=0}^{2j}a_m x^m$ could be the common eigenfunction of H and $\mathbf{j}^2$, here we characterize the wave function $\phi_j(x)$ by the quantum number j. The ground-state energy should correspond to the lowest value of j, i.e., $j=0$, in this case $\phi_0(x)=1$, and

$$E_0=K=-\nu^2+\tau, \tag{40}$$

with the corresponding wave function

$$\Psi_0(x)=\mathcal{N}_0'\exp\left(-\int W(x)dx\right)=\mathcal{N}_0'\exp[-(\tfrac{1}{3}\mu x^3+\tfrac{1}{2}\tau x^2+\nu x)], \tag{41}$$

where $\mathcal{N}_0'$ is the normalization constant.

B. Remarks

(1) It is easy to verify that $\Psi_0(x)$ and $[d\Psi_0(x)/dx]$ are continuous everywhere.

(2) As a wave function, $\Psi_0(x)$ should be normalizable. From

$$\int_{-\infty}^{+\infty}|\Psi_0(x)|^2dx=\text{finite number},$$

we must have

$$\mu+\mu^*=0,\quad \tau+\tau^*>0, \tag{42}$$

or more precisely,

$$\gamma^{1/2}+(\gamma^{1/2})^*=0,\quad \frac{\beta}{2\gamma^{1/2}}+\left(\frac{\beta}{2\gamma^{1/2}}\right)^*>0. \tag{43}$$

From the above equation, we must have

$$\gamma<0,\quad \mathrm{Re}\left(\frac{\beta}{2\gamma^{1/2}}\right)>0,\quad \beta\neq\text{real number}. \tag{44}$$

Equation (43) implies that β cannot be a real number if $\Psi_0(x)$ is normalizable. Thus, if we insist on the normalizability of a wave function, the Hamiltonian with a quartic anharmonic potential $V(x)=\rho x+\alpha x^2+\beta x^3+\gamma x^4$ must be non-Hermitian.

(3) For simplicity, let us denote $\tau=\tau_1+i\tau_2$, $\nu=\nu_1+i\nu_2$. E_0 could be expressed as

$$E_0=-\nu^2+\tau=-\nu_1^2+\nu_2^2+\tau_1+i(\tau_2-2\nu_1\nu_2). \tag{45}$$

The energy E_0 should be a real number, one must have

$$\tau_2=2\nu_1\nu_2,\quad\text{or}\quad \mathrm{Im}\left(\frac{\beta}{2\gamma^{1/2}}\right)=2\nu_1\nu_2, \tag{46}$$

which yields

$$E_0=-\nu_1^2+\nu_2^2+\tau_1=-\frac{\left[\mathrm{Im}\left(\frac{\beta}{2\gamma^{1/2}}\right)\right]^2}{4\nu_2^2}+\nu_2^2+\mathrm{Re}\left(\frac{\beta}{2\gamma^{1/2}}\right). \tag{47}$$

(4) For $\Psi_0(x)$, $\Delta x=(\overline{x^2}-\overline{x}^2)^{1/2}$ and $\Delta p=(\overline{p^2}-\overline{p}^2)^{1/2}$ should satisfy the *Heisenberg uncertainty relation*,

$$\Delta x\Delta p\geqslant\frac{1}{2}. \tag{48}$$

From $\int_{-\infty}^{+\infty}|\Psi_0(x)|^2dx=1$, we have the normalization constant

$$\mathcal{N}_0'=\left(\frac{\tau_1}{\pi}\right)^{1/4}\exp\left(-\frac{\nu_1^2}{2\tau_1}\right).$$

On one hand, one has

$$\overline{x}=\int_{-\infty}^{+\infty}x|\Psi_0(x)|^2dx=-\frac{\nu_1}{\tau_1},$$

$$\overline{x^2}=\int_{-\infty}^{+\infty}x^2|\Psi_0(x)|^2dx=\frac{1}{2\tau_1}+\frac{\nu_1^2}{\tau_1^2}, \tag{49}$$

so that

$$\Delta x=(\overline{x^2}-\overline{x}^2)^{1/2}=\frac{1}{\sqrt{2}}\frac{1}{\sqrt{\tau_1}}. \tag{50}$$

On the other hand, one has

$$\overline{p}=-i\int_{-\infty}^{+\infty}\Psi_0^*(x)\frac{d\Psi_0(x)}{dx}dx$$
$$=i\left[\mu\left(\frac{1}{2\tau_1}+\frac{\nu_1^2}{\tau_1^2}\right)-\tau\frac{\nu_1}{\tau_1}+\nu\right], \tag{51}$$

$$\overline{p^2}=-\int_{-\infty}^{+\infty}\Psi_0^*(x)\frac{d^2\Psi_0(x)}{dx^2}dx$$
$$=\int_{-\infty}^{+\infty}\Psi_0^*(x)[E_0-(\rho x+\alpha x^2+\beta x^3+\gamma x^4)]\Psi_0(x)dx$$
$$=-\gamma\left[\frac{3}{4}\frac{1}{\tau_1^2}-\left(\frac{\nu_1^2}{\tau_1^2}\right)^2\right]+\beta\frac{\nu_1}{\tau_1}\left[\frac{3}{2\tau_1}+\frac{\nu_1^2}{\tau_1^2}\right]-\alpha\left[\frac{1}{2\tau_1}+\frac{\nu_1^2}{\tau_1^2}\right]$$
$$+\rho\frac{\nu_1}{\tau_1}+[-\nu^2+\tau]. \tag{52}$$

By using Eq. (37), we have

$$(\Delta p)^2=\overline{p^2}-\overline{p}^2$$
$$=-\gamma\frac{1}{\tau_1^2}\left[1+\frac{\nu_1^2}{\tau_1}\right]+\beta\frac{\nu_1}{\tau_1^2}-\tau^2\frac{1}{2\tau_1}-2\mu\frac{\nu_1}{\tau_1}+\tau$$
$$=-\gamma\frac{1}{\tau_1^2}\left[1+\frac{\nu_1^2}{\tau_1}\right]+(2\mu\tau_1+i2\mu\tau_2)\frac{\nu_1}{\tau_1^2}-(\tau_1^2-\tau_2^2$$
$$+i\tau_1\tau_2)\frac{1}{2\tau_1}-2\mu\frac{\nu_1}{\tau_1}+\tau_1+i\tau_2$$
$$=-\gamma\frac{1}{\tau_1^2}\left[1+\frac{\nu_1^2}{\tau_1}\right](\sqrt{-\gamma})^2+2i\mu\frac{\nu_1\tau_2}{\tau_1^2}+\frac{\tau_2^2}{2\tau_1}+\frac{\tau_1}{2}$$
$$=\frac{1}{\tau_1^2}\left[1+\frac{\nu_1^2}{\tau_1}\right](\sqrt{-\gamma})^2\pm2\frac{\nu_1\tau_2}{\tau_1^2}\sqrt{-\gamma}+\frac{\tau_2^2}{2\tau_1}+\frac{\tau_1}{2}, \tag{53}$$

where we have used $i\mu=\pm\sqrt{-\gamma}$. For given admissible values of γ and β, since $\gamma<0$ and $\mathrm{Re}(\beta/2\mu)>0$, we will have $\mu=i\sqrt{-\gamma}$ when $\mathrm{Im}\beta>0$, and $\mu=-i\sqrt{-\gamma}$ when $\mathrm{Im}\beta<0$.

The *Heisenberg uncertainty relation* requires

$$\frac{1}{\tau_1^2}\left[1+\frac{\nu_1^2}{\tau_1}\right](\sqrt{-\gamma})^2\pm2\frac{\nu_1\tau_2}{\tau_1^2}\sqrt{-\gamma}+\frac{\tau_2^2}{2\tau_1}\geqslant0, \tag{54}$$

which is a quadratic algebraic equation of the variable $\sqrt{-\gamma}$. Because $\tau_1>0$, it is obvious that the above equation is valid if $\nu_1=0$ and $\tau_2=2\nu_1\nu_2=0$. If $\nu_1\neq0$, since $(1/\tau_1^2)[1+\nu_1^2/\tau_1]>0$, one must require that

$$\Delta=4\left(\frac{\nu_1\tau_2}{\tau_1^2}\right)^2-4\frac{1}{\tau_1^2}\left[1+\frac{\nu_1^2}{\tau_1}\right]\frac{\tau_2^2}{2\tau_1}=2\frac{\tau_2^2}{\tau_1^4}[\nu_1^2-\tau_1]\leqslant0, \tag{55}$$

from which we have $\nu_1^2\leqslant\tau_1$, i.e.,

$$|\nu_1|\leqslant\sqrt{\mathrm{Re}\left(\frac{\beta}{2\gamma^{1/2}}\right)}. \tag{56}$$

Owning to $\tau_2=2\nu_1\nu_2$, the condition required by the Heisenberg uncertainty relation can be expressed as

$$|\nu_2|\geqslant\left|\frac{\tau_2}{2\sqrt{\tau_1}}\right|=\frac{\left|\mathrm{Im}\left(\frac{\beta}{2\gamma^{1/2}}\right)\right|}{2\sqrt{\mathrm{Re}\left(\frac{\beta}{2\gamma^{1/2}}\right)}}. \tag{57}$$

C. Analysis: Admissible values of γ, β, α, and ρ

(1) Admissible values of γ. $\gamma<0$, $\gamma\in$ real numbers.

(2) Admissible values of β. $\beta\in$ complex numbers, but $\mathrm{Im}\beta\neq0$.

(3) Admissible values of $\alpha=(\alpha_x,\alpha_y)$. From Eq. (37), we have

$$\alpha=\tau^2+2\mu\nu=\tau_1^2-\tau_2^2+2i\tau_1\tau_2+2\gamma^{1/2}(\nu_1+i\nu_2)$$
$$=[\tau_1^2-\tau_2^2+2i\gamma^{1/2}\nu_2]+i\left[2\tau_1\tau_2-2i\gamma^{1/2}\frac{\tau_2}{2\nu_2}\right]. \tag{58}$$

Writing $\alpha=\alpha_x+i\alpha_y$, where α_x and α_y denote the real and imaginary parts of α, respectively, we have

$$\alpha_x=\tau_1^2-\tau_2^2+2i\gamma^{1/2}\nu_2=\left[\mathrm{Re}\left(\frac{\beta}{2\gamma^{1/2}}\right)\right]^2-\left[\mathrm{Im}\left(\frac{\beta}{2\gamma^{1/2}}\right)\right]^2$$
$$+2i\gamma^{1/2}\nu_2,$$

$$\alpha_y=2\tau_1\tau_2-2i\gamma^{1/2}\frac{\tau_2}{2\nu_2}$$

$$=2\mathrm{Re}\left(\frac{\beta}{2\gamma^{1/2}}\right)\mathrm{Im}\left(\frac{\beta}{2\gamma^{1/2}}\right)-i\gamma^{1/2}\frac{\mathrm{Im}\left(\frac{\beta}{2\gamma^{1/2}}\right)}{\nu_2}. \quad (59)$$

Thus the points (α_x,α_y) satisfy the following algebraic equation:

$$\alpha_y-y_0=\frac{4\gamma\mathrm{Im}\left(\frac{\beta}{2\gamma^{1/2}}\right)}{\alpha_x-x_0},$$

$$x_0=\left[\mathrm{Re}\left(\frac{\beta}{2\gamma^{1/2}}\right)\right]^2-\left[\mathrm{Im}\left(\frac{\beta}{2\gamma^{1/2}}\right)\right]^2,$$

$$y_0=2\mathrm{Re}\left(\frac{\beta}{2\gamma^{1/2}}\right)\mathrm{Im}\left(\frac{\beta}{2\gamma^{1/2}}\right). \quad (60)$$

Equation (60) corresponds to a hyperbola under $|\nu_2|\geqslant|\mathrm{Im}(\beta/2\gamma^{1/2})|/2\sqrt{\mathrm{Re}(\beta/2\gamma^{1/2})}$, for given admissible values of γ and β.

(4) Admissible values of $\rho=(\rho_x,\rho_y)$. From Eq. (39), we have

$$\rho=2\tau\nu-2\mu=2[\tau_1\nu_1-\tau_2\nu_2]+i2[\tau_1\nu_2+\tau_2\nu_1+i\gamma^{1/2}]. \quad (61)$$

Again writing $\rho=\rho_x+i\rho_y$, where ρ_x and ρ_y denote the real and imaginary parts of ρ, respectively, we get

$$\rho_x=2\left[\tau_1\frac{\tau_2}{2\nu_2}-\tau_2\nu_2\right]$$

$$=\mathrm{Re}\left(\frac{\beta}{2\gamma^{1/2}}\right)\mathrm{Im}\left(\frac{\beta}{2\gamma^{1/2}}\right)\frac{1}{\nu_2}-2\mathrm{Im}\left(\frac{\beta}{2\gamma^{1/2}}\right)\nu_2,$$

$$\rho_y=2\left[\tau_1\nu_2+\frac{\tau_2^2}{2\nu_2}+i\gamma^{1/2}\right]$$

$$=\left[\mathrm{Im}\left(\frac{\beta}{2\gamma^{1/2}}\right)\right]^2\frac{1}{\nu_2}+2\mathrm{Re}\left(\frac{\beta}{2\gamma^{1/2}}\right)\nu_2+2i\gamma^{1/2}. \quad (62)$$

Thus the point (ρ_x,ρ_y) satisfies the following algebraic equation:

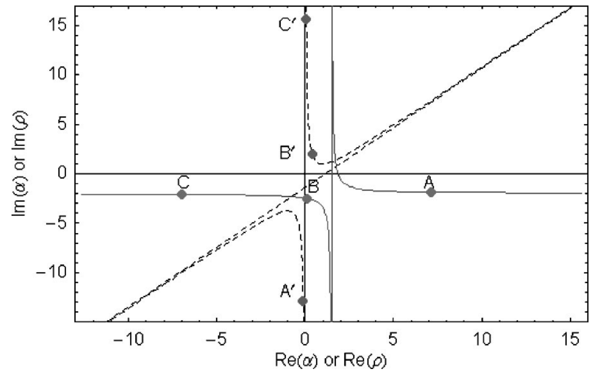

FIG. 1. Parametric plot of α (bold curve) and ρ (dashed curve) in the Argand plane for $\beta=1+2i$. For simplicity, a particular branch of $\gamma^{(1/2)}$ has been chosen for the plots, specifically $\gamma^{(1/2)}=(i/\sqrt{2})$. The points *A*, *B*, and *C* on the parametric plot of α (bold curve) correspond to the values of $\nu_2=-4,\ 1$, and 6. A similar choice of values of ν_2 has also been chosen for the parametric plot of ρ (dashed curve) for the points A', B', and C'.

$$\rho_x^2-(\rho_y-2i\gamma^{1/2})^2-\frac{\left[\mathrm{Re}\left(\frac{\beta}{2\gamma^{1/2}}\right)\right]^2-\left[\mathrm{Im}\left(\frac{\beta}{2\gamma^{1/2}}\right)\right]^2}{\mathrm{Re}\left(\frac{\beta}{2\gamma^{1/2}}\right)\mathrm{Im}\left(\frac{\beta}{2\gamma^{1/2}}\right)}$$

$$\times\rho_x(\rho_y-2i\gamma^{1/2})+\frac{2\left|\frac{\beta}{2\gamma^{1/2}}\right|^4}{\mathrm{Re}\left(\frac{\beta}{2\gamma^{1/2}}\right)}=0. \quad (63)$$

Indeed, Eq. (63) also corresponds to a hyperbola under $|\nu_2|\geqslant|\mathrm{Im}(\beta/2\gamma^{1/2})|/2\sqrt{\mathrm{Re}(\beta/2\gamma^{1/2})}$.

For a particular value of $\gamma=-\xi$ (ξ being real and positive), there are two solutions for $\gamma^{1/2}$, namely $\gamma^{1/2}=\pm i\sqrt{\xi}$. In Fig. 1, we have plotted possible values of α and ρ, parametrized by different values of ν_2. For simplicity, we have chosen the value of $\gamma^{1/2}$ as $i\sqrt{\xi}$ with $\xi=0.5$ and the value of $\beta=1+2i$. For the chosen values of β and γ, we see that depending on the values of ν_2, the admissible values of α and ρ lies on the appropriate branch of the hyperbolas in the Argand plane. Consequently we have shown the location of α (corresponding to the points *A*, *B*, and *C*) and ρ (corresponding to the points A', B', and C') in the Argand plane for $\nu_2=-4,\ 1$, and 6.

(5) The constraint on ρ, α, β, *and* γ. For a value ν_2, we have one corresponding point $\alpha=(\alpha_x,\alpha_y)$ and one corresponding point $\rho=(\rho_x,\rho_y)$ [see Eqs. (59) and (62)]. From $\alpha=\tau^2+2\mu\nu$ and $\nu=[\rho+2\mu(1+2j)]/2\tau$, one obtains a constraint on ρ, α, β, and γ as follows (when $j=0$):

$$\alpha=\frac{\beta^2}{4\gamma}+\frac{4\gamma^{3/2}}{\beta}+\frac{2\gamma}{\beta}\rho, \quad (64)$$

hence, for fixed values of γ and β, the point $\rho=(\rho_x,\rho_y)$ and the point $\alpha=(\alpha_x,\alpha_y)$ are in one-to-one correspondence.

(6) If $\gamma=-1$, $\beta\equiv 2ia$ is a pure imaginary. $\gamma=-1$ implies that $\mu=i$ or $\mu=-i$. If $\beta\equiv 2ia$ is a pure imaginary, for $a>0$ we have $\mu=i$, $\tau=\tau_1+i\tau_2=(\beta/2\mu)\equiv a$ is a real positive number (for $a<0$, we have $\mu=-i$ and $\tau=-a$), based on which we have $\tau_2=\mathrm{Im}(\beta/2\gamma^{1/2})=0$. Since $\tau_2=2\nu_1\nu_2$, one must have $\nu_1=0$ or $\nu_2=0$.

(i) If $\nu_1=0$, Eq. (56) is always satisfied, Eq. (57) implies that ν_2 can be an arbitrary real number, thus $\nu\equiv ib$ is an arbitrary pure imaginary. One then obtains

$$\alpha=\tau^2+2\mu\nu=a^2-2b, \quad \rho=2\tau\nu-2\mu=2i(ab-1). \tag{65}$$

The above result recovers the non-Hermitian quartic anharmonic potential suggested in Bender's paper [33] with $J=j+1=1$. In the case, the energy $E_0=b^2+a$.

(ii) If $\nu_2=0$, Eq. (54) is always satisfied, Eq. (56) implies that

$$|\nu_1|\leqslant\sqrt{\mathrm{Re}\left(\frac{\beta}{2\gamma^{1/2}}\right)}=\sqrt{a}. \tag{66}$$

Thus $\nu\equiv b$ is a real number ($|b|\leqslant a$). One then obtains [34–36]

$$\alpha=\tau^2+2\mu\nu=a^2+2ib, \quad \rho=2\tau\nu-2\mu=2(ab-i). \tag{67}$$

In this case, the energy is $E_0=-b^2+a$, $|b|\leqslant a$. Finally, one can verify that $\gamma=-1$, $\beta=2ia$, α, and ρ as shown in Eq. (65) or Eq. (67) satisfy the constraint (64).

IV. DISCUSSION: ON THE DOUBLE-WELL POTENTIAL

The double-well potential

$$H\Psi(x)=\left(-\frac{d^2}{dx^2}+V(x)\right)\Psi(x)=E\Psi(x),$$
$$V(x)=\alpha x^2+\gamma x^4, \tag{68}$$

with real α and γ has been investigated for a long time by many authors. However, up till now, its exact ground state has not been found. It is therefore interesting to find out if we can get some insight into this problem using our SU(2) realization shown in Eq. (5) for the case of the double-well potential.

Similarly, after performing the transformation ($-\infty<x<+\infty$)

$$\Psi(x)=\mathcal{N}_0\exp\left[-\int W(x)dx\right]\phi(x), \quad W(x)=\mu x^2+\nu, \tag{69}$$

we shall arrive at

$$H\phi=\left(-\frac{d^2}{dx^2}+2W\frac{d}{dx}-(W^2-W'-V)\right)\phi=E\phi, \tag{70}$$

or

$$H\phi=\left(-\frac{d^2}{dx^2}+2(\mu x^2+\nu)\frac{d}{dx}-[(\mu^2-\gamma)x^4+(2\mu\nu-\alpha)x^2-2\mu x+\nu^2]\right)\phi$$
$$=E\phi. \tag{71}$$

To express the Hamiltonian in terms of SU(2) generators, we must have

$$\mu^2=\gamma, \quad 2\mu\nu=\alpha, \tag{72}$$

and as a result,

$$H\phi=(Aj_-^2+Bj_++Cj_0+Dj_-+K)\phi=E\phi, \tag{73}$$

i.e.,

$$H\phi=\left(A\frac{d^2}{dx^2}+(-Bx^2+Cx+D)\frac{d}{dx}+2jBx+K-jC\right)\phi$$
$$=E\phi, \tag{74}$$

where

$$A=-1, \quad B=-2\mu, \quad C=0, \quad D=2\nu,$$
$$2jB=2\mu, \quad K=-\nu^2+jC. \tag{75}$$

However, $B=-2\mu$ and $2jB=2\mu$ inevitably lead to $j=-1/2$. According to quantum mechanics, there has not been any physical meaning for $j=-1/2$.

V. CONCLUDING REMARKS

In this paper, we have investigated the nature of the energy spectrum, particularly the wave function, of the quartic anharmonic oscillator using group-theoretic approach. Although such approach is not new [37–39], it has allowed us to explore the relation of the associated wave function to hermiticity of the Hamiltonian.

In our analysis, we distinguish between the case in which the coefficient attached to the quartic term γ is positive with the case in which γ is negative. For $\gamma>0$, we find that it is not possible to find suitable wave function for differentiable quartic harmonic potential, i.e., $V(x)$ is differentiable everywhere. However, if we relax differentiability of $V(x)$, we can find the form of the ground-state wave function with zero energy.

For negative γ, we find that the coefficient of the cubic term must necessarily be complex. This immediately implies that the Hamiltonian of the system is no longer Hermitian. Moreover, the other coefficients α (coefficient of x^2) and ρ (coefficient of x) cannot assume arbitrary values for consistency and the admissible values in fact lie on hyperbolae in the complex plane.

Finally, we extend our analysis briefly to the case of double-well potential. In this case, if we express the Hamiltonian in terms of generators of SU(2), we require $j=-1/2$ for consistency. However, we should note that we can avoid this problem by performing the same analysis with generators of SU(1,1).

ACKNOWLEDGMENTS

This work has been supported by NUS Research Grant No. R-144-000-020-112. We would also like to thank Professor A. Turbiner, Professor V.I. Lyakhovsky, and Professor C.M. Bender for their invaluable comments. J.L.C. was supported in part by the NSF of China (No. 10201015).

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