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Properties of a beam-splitter entangler with Gaussian input states

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An explicit formula is given for the quantity of entanglement in the output state of a beam splitter, given the squeezed vacuum states' input in each mode.

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I. INTRODUCTION

As one of the few quantum devices that may act as the entangler, beam splitters have been extensively studied for their entangler related properties [1–9]. In the laboratory, coherent states and squeezed states are two practically existing robust states. It is well known that no entanglement is produced if the input states are coherent states. Therefore it is important to know the entanglement property when squeezed states are used as the input. The output entanglement quantity is studied in Ref. [4], given the squeezed state input. In particular, an explicit formula expressing the output state in the form of two mode squeezed states are given. However, the result there is limited to a type of rather specific case. For example, the beam splitter there is limited to the 50:50 beam splitter, the input squeezed states can only have the real squeezing parameters, and so on. In this paper, we shall investigate this problem in a rather general background. We will give an explicit formula for the entanglement quantity of the output state.

It has been shown in Refs. [8,9] that in order to obtain an entangled output state, a necessary condition is that the input state should be nonclassical. More generally, it was shown in Ref. [9] that an arbitrary multimode classical state is still classical after an arbitrary multimode rotation transformation. This means, for an arbitrary linear optical system including passive devices such as beam splitters, polarizing beam splitters, phase shifters, polarization rotators, and so on, the output multimode state must be classical (therefore separable) if the input is classical. However, this is only a necessary condition to obtain the entangled output state, it is not a sufficient condition in general. In certain cases one may have the interest to know the exact amount of entanglement in the output state of the beam splitter and how to maximize it through adjusting the parameters in the passive linear optical system. Here we make an explicit calculation with the input of two single-mode squeezed states.

Consider a lossless beam splitter (see Fig. 1 in Ref. [9]). We can distinguish the field mode a and mode b by the different propagating directions. Most generally, the property of a beam-splitter operator $\hat{B}$ in the Schrödinger picture can be summarized by the following equations (see, e.g., Ref. [10]):

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$$\rho_{out}=\hat{B}\rho_{in}\hat{B}^{-1}, \tag{1}$$

$$\hat{B}^{\dagger}=\hat{B}^{-1}, \tag{2}$$

$$\hat{B}\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix}\hat{B}^{-1}=M_B\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix}, \tag{3}$$

$$M_B=\begin{pmatrix}\cos\theta e^{i\phi_0} & \sin\theta e^{i\phi_1}\\ -\sin\theta e^{-i\phi_1} & \cos\theta e^{-i\phi_0}\end{pmatrix}, \tag{4}$$

$$\hat{B}|00\rangle=|00\rangle. \tag{5}$$

Here ρ_{in} and ρ_{out} are the density operators for the input and output states, respectively. Both of them are two mode states including mode a and mode b. The elements in the matrix M_B are determined by the beam splitter itself, $\hat{a},\hat{b}$ are the annihilation operators for mode a and mode b, respectively; $|00\rangle$ is the vacuum state for both modes. Equation (5) is due to the simple fact of no input no output.

II. INSEPARABILITY QUANTITY WITH SQUEEZED STATES' INPUT

Suppose the input states are the squeezed vacuum states in each mode, i.e.,

$$\rho_{in}=\hat{S}_a(\zeta_a)\hat{S}_b(\zeta_b)|00\rangle\langle 00|\hat{S}^{\dagger}(\zeta_a)\hat{S}^{\dagger}(\zeta_b), \tag{6}$$

where

$$\hat{S}_a(\zeta_a)=\exp(\tfrac{1}{2}\zeta_a^*\hat{a}^2-\tfrac{1}{2}\zeta_a\hat{a}^{\dagger 2}),$$

$$\hat{S}_b(\zeta_b)=\exp(\tfrac{1}{2}\zeta_b^*\hat{b}^2-\tfrac{1}{2}\zeta_b\hat{b}^{\dagger 2}). \tag{7}$$

They have the following properties:

$$\hat{S}_a^{\dagger}(\zeta_a)\begin{pmatrix}a^{\dagger}\\ a\end{pmatrix}\hat{S}_a(\zeta_a)=\begin{pmatrix}\cosh r_a & -e^{-i\chi_a}\sinh r_a\\ -e^{i\chi_a}\sinh r_a & \cosh r_a\end{pmatrix}\begin{pmatrix}b^{\dagger}\\ b\end{pmatrix}, \tag{8}$$

$$\hat{S}_b^{\dagger}(\zeta_b)\begin{pmatrix}b^{\dagger}\\ b\end{pmatrix}\hat{S}_b(\zeta_b)=\begin{pmatrix}\cosh r_b & -e^{-i\chi_b}\sinh r_b\\ -e^{i\chi_b}\sinh r_b & \cosh r_b\end{pmatrix}\begin{pmatrix}b^{\dagger}\\ b\end{pmatrix}, \tag{9}$$

where $r_{a,b}=|\zeta_{a,b}|$ and $\chi_{a,b}=\tanh^{-1}\zeta_{a,b}/r_{a,b}$.

For simplicity, we use the characteristic function for the input state ρ_{in} and the output state ρ_{out}:

$$C_{in}(\xi_a,\xi_b)=\mathrm{tr}[\exp\,(\xi_a\hat{a}-\xi_a^*\hat{a}^\dagger+\xi_b\hat{b}-\xi_b^*\hat{b}^\dagger)\rho_{in}]$$
$$=\mathrm{tr}\{\exp\,[i\sqrt{2}(\xi_a^I\hat{x}_a+\xi_a^R\hat{p}_a+\xi_b^I\hat{x}_b+\xi_b^R\hat{p}_b)]\rho_{in}\},\tag{10}$$

where the parameters $\xi_{a,b}=\xi_{a,b}^R+i\xi_{a,b}^I$, $(\hat{x}_a,\hat{p}_a)=N(\hat{a}^\dagger,\hat{a})^T$, $(\hat{x}_b,\hat{p}_b)=N(\hat{b}^\dagger,\hat{b})^T$, and $N=(1/\sqrt{2})\left(\begin{smallmatrix}1 & 1\\ -i & i\end{smallmatrix}\right)$. For convenience, we denote $\hat{D}(\xi_a,\xi_b)=\exp\,(\xi_a\hat{a}-\xi_a^*\hat{a}^\dagger+\xi_b\hat{b}-\xi_b^*\hat{b}^\dagger)$. In the case of squeezed states input, the characteristic function for the output state is

$$C_{out}(\xi_a,\xi_b)=\mathrm{tr}[\hat{D}(\xi_a,\xi_b)\hat{B}\hat{S}_a(\zeta_a)\hat{S}_b(\zeta_b)|00\rangle$$
$$\times\langle 00|\hat{S}_a^\dagger(\zeta_a)\hat{S}_b^\dagger(\zeta_b)\hat{B}^\dagger]$$
$$=\mathrm{tr}[\hat{S}_a^\dagger(\zeta_a)\hat{S}_b^\dagger(\zeta_b)B^\dagger\hat{D}(\xi_a,\xi_b)\hat{B}\hat{S}_a(\zeta_a)$$
$$\times\hat{S}_b(\zeta_b)|00\rangle\langle 00|].\tag{11}$$

Suppose the output state of mode a is ρ_{oa}. The quantity of entanglement for the output state between mode a and mode b is

$$E(\rho_{oa})=\mathrm{tr}(\rho_{oa}\,\ln\rho_{oa}).\tag{12}$$

Using Eq. (11), we can calculate the characteristic function for the output state in mode a explicitly:

$$C_{oa}(\xi_a)=C_{out}(\xi_a,\xi_b=0)$$
$$=\exp[-\tfrac{1}{2}\,\cos^2\theta|\xi_a^*e^{i\phi_0}\cosh r_a$$
$$+\xi_a e^{-i\phi_0+i\chi_a}\sinh r_a|^2]$$
$$\times\exp[-\tfrac{1}{2}\sin^2\theta|\xi_a e^{-i\phi_1}\cosh r_b$$
$$+\xi_a^* e^{i\phi_1-i\chi_b}\sinh r_b|^2].\tag{13}$$

In obtaining the above equation, we have used Eqs. (3), (4) and (8), (9) to reduce the part

$$\hat{S}_a^\dagger(\zeta_a)\hat{S}_b^\dagger(\zeta_b)B^\dagger\hat{D}(\xi_a,\xi_b)\hat{B}\hat{S}_a(\zeta_a)\hat{S}_b(\zeta_b).$$

The right-hand side of Eq. (13) can be written in the form ξ_R and ξ_I, where $\xi_R+i\xi_I=\xi_a$, i.e.,

$$C_{oa}=\exp[-\tfrac{1}{2}(\xi_R,\xi_I)M_{oa}(\xi_R,\xi_I)^T].\tag{14}$$

Here M_{oa} is the 2×2 covariance matrix as

$$M_{oa}=\begin{pmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}.$$

After calculation we obtain the matrix elements

$$m_{11}=\Sigma_a\,\cos^2\theta+\Sigma_b\,\sin^2\theta+2x_a\,\cos^2\theta\cos\Delta_a$$
$$+2x_b\,\sin^2\theta\cos\Delta_b,\tag{15}$$

$$m_{12}=m_{21}=2x_a\,\cos^2\theta\sin\Delta_a+2x_b\,\sin^2\theta\sin\Delta_b,\tag{16}$$

and

$$m_{22}=\Sigma_a\,\cos^2\theta+\Sigma_b\,\sin^2\theta-2x_a\,\cos^2\theta\cos\Delta_a$$
$$-2x_b\,\sin^2\theta\cos\Delta_b,\tag{17}$$

where $\Sigma_a=\cosh^2 r_a+\sinh^2 r_a$, $\Sigma_b=\cosh^2 r_b+\sinh^2 r_b$, $x_a=\sinh r_a\cosh r_a$, $x_b=\sinh r_b\cosh r_b$, $\Delta_a=2\phi_0-\chi_a$, and $\Delta_b=2\phi_1-\chi_b$. We can choose an appropriate unitary transformation to ρ_{oa} to obtain another density operator ρ'_{oa} whose characteristic function is

$$C'_{oa}(\xi_a)=\exp\left[-\frac{1}{2}(\xi_R,\xi_I)\begin{pmatrix}\delta & 0\\ 0 & \delta\end{pmatrix}(\xi_R,\xi_I)^T\right]\tag{18}$$

and

$$\delta=\sqrt{m_{11}m_{22}-m_{12}^2}.\tag{19}$$

We know that the Wigner characteristic function for a thermal state $(1-e^{-\beta})e^{-\beta a^\dagger a}$ is [11]

$$C_{th}(\xi)=\exp\left[-\frac{1}{2}(\xi_R,\xi_I)\begin{pmatrix}\dfrac{1+e^{-\beta}}{1-e^{-\beta}} & 0\\ 0 & \dfrac{1+e^{-\beta}}{1-e^{-\beta}}\end{pmatrix}(\xi_R,\xi_I)\right].\tag{20}$$

This is to say, the state defined by the characteristic function in Eq. (18) is a thermal state of the form

$$\rho'_{oa}=(1-e^{-\beta})e^{-\beta a^\dagger a},\tag{21}$$

with the parameter β satisfying

$$e^{-\beta}=\frac{\delta-1}{\delta+1}.\tag{22}$$

Since the trace value does not change under any unitary transformation, the entanglement quantity defined in Eq. (12) is

$$E(\rho_{oa})=\mathrm{tr}\;\rho'_{ao}\,\ln\rho'_{ao}.\tag{23}$$

For the thermal state defined by Eq. (21), calculation for the quantity tr $\rho'_{ao}\ln\,\rho'_{ao}$ is straightforward. Thus we have the following result for the quantity of entanglement for the output state, given the squeezed state input in each mode:

$$E(\rho_{out})=\ln(1-e^{-\beta})+\frac{\beta e^{-\beta}}{1-e^{-\beta}}=\ln\frac{2}{\delta+1}-\frac{\delta-1}{2}\ln\frac{\delta-1}{\delta+1},\tag{24}$$

with δ being defined by Eq. (19) and Eqs. (15)–(17). The above equation together with the previous equations for the definition of δ gives a direct calculation formula for the entanglement quantity given the independent squeezed state as the input to each mode. This is to say, the maximum value of det M_{oa} gives the largest entanglement. In order to maximize the entanglement, we should maximize the value of δ. After calculation, we can see that

$$\delta^2 = (\Sigma_a + \Sigma_b + \sinh 2r_a + \sinh 2r_b)(\sin^4\theta + \cos^4\theta) + \tfrac{1}{2}\Sigma_a\Sigma_b \sin^2 2\theta - 2x_a x_b \sin^2 2\theta \cos(\Delta_b - \Delta_a). \quad (25)$$

Obviously the following condition is required to maximize the value of δ^2 for the maximum entanglement:

$$\Delta_b - \Delta_a = 2(\phi_1 + \phi_0) - (\chi_b - \chi_a) = (2k+1)\pi, \quad (26)$$

where k is an arbitrary integer. And we know that the values of both $\chi_b - \chi_a$ and $\phi_1 + \phi_0$ are practically detectable and controllable in a beam-splitter experiment. This constraint is independent of θ or r_a, r_b. In particular, taking the special case $\phi_0 = 0$ and $|\cos\theta| = 1/2$ it is just the result given by Kim *et al.* [12]. However, our result is more general than that of Ref. [4]. Reference [4] has only given the maximum point in the case of 50:50 beam splitter with $\phi_0 = 0$. No explicit formula for the quantity of entanglement is given there [4]. Our result is more general, in that it can not only be used for the exact amount of entanglement but also to find the maximum point of entanglement for the output state of a beam splitter with arbitrary transmission rate and with arbitrary phase values of ϕ_0, ϕ_1, χ_a, and χ_b.

III. CONCLUDING REMARK

In summary, we have studied the entanglement quantity for the output state of a beam splitter, given a squeezed vacuum state as the input state in each mode. Different from the previous result [4], our result is not limited to the 50:50 result. We do not know how to obtain the more general result, given the general Gaussian state, since so far there is no good entanglement for the impure Gaussian state. It has been shown in Ref. [4] that a nonclassical separable input state can be changed to an entangled state in the output. The inverse of such a process exemplifies that even though the input state is nonclassical, the output could be still separable. Some specific examples are given in Ref. [7]. The necessary and sufficient condition for an inseparable output state is not given so far. It is possible to obtain the necessary and sufficient condition for the inseparability of the output state, given the Gaussian input state. We give this condition explicitly in this paper. However, one may still easily find the criterion on whether the output state is inseparable through the inseparability criterion [13]:

$$M_{out} + i\tilde{\sigma} \geqslant 0, \quad (27)$$

where M is the correlation matrix of the output state, the 4×4 matrix $\tilde{\sigma} = J_A^T \oplus J_B$, $J_A = J_B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. A detailed calculation of this is given in Ref. [14].

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[1] S.M. Tan, D.F. Walls, and M.J. Collett, Phys. Rev. Lett. **77**, 285 (1990).
[2] B.C. Sanders, Phys. Rev. A **45**, 6811 (1992); B.C. Sanders, K.S. Lee, and M.S. Kim, *ibid.* **52**, 735 (1995); S. Scheel *et al.*, *ibid.* **62**, 043803 (2000).
[3] M.G.A. Paris, Phys. Rev. A **59**, 1615 (1999).
[4] M.S. Kim, W. Son, V. Buzek, and P.L. Knight, Phys. Rev. A **65**, 032323 (2002).
[5] M.S. Kim and J. Lee, e-print quant-ph/0203151.
[6] H. Huang and G.S. Agarwal, Phys. Rev. A **49**, 52 (1994).
[7] R.A. Campos, Phys. Rev. A **62**, 013809 (2000).
[8] Arvind and N. Mukunda, Phys. Lett. A **259**, 421 (1999).
[9] Wang Xiang-bin, Phys. Rev. A **66**, 024303 (2002); e-print quant-ph/0204039.
[10] R.A. Campos, B.E.A. Saleh, and M.C. Teich, Phys. Rev. A **40**, 1371 (1989).
[11] M. Orszag, *Quantum Optics* (Springer-Verlag, Berlin, 2000).
[12] To see this point, we should first observe Eq. (13) in Ref. [4]. In our notation, Eq. (13) of Ref. [4] reads $\hat{B}(\theta,\phi_1)\hat{S}_a(\zeta_a),(\zeta_b)|00\rangle = \hat{R}(-\chi_a/2)\hat{R}(\chi_b/2)\hat{B}(\theta,\phi_1 - \chi_b/2 + \chi_a/2)\hat{S}_a(r_a)\hat{S}_b(r_b)|00\rangle$ with the special value of $\phi_0 = 0$. The condition" $\phi = \pi/2$" in Ref. [4] reads $\phi_1 - \chi_b/2 + \chi_a/2 = \pi/2$ in our notation. This is in agreement with Eq. (26) in this paper.
[13] L. Duan, G. Giedke, J.I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000); R. Simon, *ibid.* **84**, 2726 (2000); R.F. Werner and M.M. Wolf *ibid.* **86**, 3658 (2001).
[14] Wang Xiang-bin, e-print quant-ph/0204082, v3.