

**Green functions for generalized point interactions in one dimension: A scattering approach**

Alexandre G. M. Schmidt,\* Bin Kang Cheng, and M. G. E. da Luz†

*Departamento de Física, Universidade Federal do Paraná, Caixa Postal 19044, 81531-990 Curitiba, Paraná, Brazil*

(Received 23 August 2002; published 31 December 2002)

Recently, general point interactions in one dimension has been used to model a large number of different phenomena in quantum mechanics. Such potentials, however, require some sort of regularization to lead to meaningful results. The usual ways to do so rely on technicalities that may hide important physical aspects of the problem. In this work we present a method to calculate the exact Green functions for general point interactions in one dimension. Our approach differs from previous ones because it is based only on physical quantities, namely, the scattering coefficients  $R$  and  $T$  to construct  $G$ . Renormalization or particular mathematical prescriptions are not invoked. The simple formulation of the method makes it easy to extend to more general contexts, such as for lattices of  $N$  general point interactions, on a line, on a half-line, under periodic boundary conditions, and confined in a box.

DOI: 10.1103/PhysRevA.66.062712

PACS number(s): 03.65.Nk, 03.65.Ge, 03.65.Db

**I. INTRODUCTION**

Point interactions in one or more dimensions have been of great interest in quantum physics and one can regard their relevance as being threefold. First, to model different phenomena such as energy-band structure in ordered lattices [1], emergence of quantum chaos [2], Aharonov-Bohm effect in spin-1/2 particles [3,4], duality between fermionic and bosonic systems [5], etc. Second, to allow exact closed analytical solutions in quantum mechanics [6], which are usually rare but quite useful. For instance, one of the few exactly solved many-body quantum problems is one-dimensional (1D) identical particles interacting by pairwise  $\delta$ -function potentials [7]. More recently, progress in extending such solutions to general point interactions has also been reported [8,9].

The third relevance of such potentials is to provide relative simple situations where development of regularization procedures are in order. This is important not only due the applications mentioned above, but also because such techniques may be extended to more complex and general contexts [10], e.g., anyons statistics, vortices and topological structures in scattering, quantum field theory, etc.

We can state the nonrelativistic quantum problem of a point interaction as the following. Consider a  $d$ -dimensional Hamiltonian, written formally as  $H(\mathbf{r}) = H_0(\mathbf{r}) + \Xi(\mathbf{r}; \mathbf{r}_0)$ .  $H_0 = -\nabla_{\mathbf{r}}^2 + V(\mathbf{r})$  is a well-defined self-adjoint “unperturbed” Hamiltonian and  $\Xi$  represents a general point interaction potential located at  $\mathbf{r}_0$ . One may have interest in the wave function, the Green function, or its Fourier transform, the propagator, which satisfy  $H(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r})$ ,  $[E - H(\mathbf{r}_f)]G(\mathbf{r}_f, \mathbf{r}_i; E) = \delta(\mathbf{r}_f - \mathbf{r}_i)$ , and  $K(\mathbf{r}_f, \mathbf{r}_i; t) = (2\pi i)^{-1} \int dE \exp(-iEt)G(\mathbf{r}_f, \mathbf{r}_i; E)$ , respectively. The whole issue is to ask if  $H$  is a self-adjoint operator, and the above equations lead to physical meaningful  $\psi$ ,  $G$ , and  $K$ , for instance, the quantum state has a unique time evolution.

If the point interaction is the usual delta function,  $\gamma\delta(\mathbf{r} - \mathbf{r}_0)$ , the answer is negative for  $d$  equal to 2 and 3. In one dimension, this is also the case for more singular point interactions such as delta-prime function,  $\gamma\delta'(x - x_0)$ . Different approaches are then used to regularize  $H$ . Methods such as formal self-adjoint extensions (see, for instance, Refs. [4,6,11]), series expansion [12], and renormalization [13] have been used in two or more dimensions. For the one-dimensional case the situation is far more rich. This is so because in one dimension there is a four-parameter family solution [6,14] for the problem (the following section), so different types of discontinuities for point interactions are possible. In studying such a family solution, functional methods [15], the nonrelativistic limit of Dirac’s equation [16], and self-adjoint extension [17] have been used to calculate Green functions and propagators.

From the mathematical point of view, all the mentioned procedures are quite ingenious. However, they rely on technicalities which may hide important aspects of the problem, making it hard to understand the physical meaning of relevant quantities, such as potential strengths and scattering amplitudes. In fact, in some cases the regularization of  $H$  leads to a renormalization of some parameters related to  $\Xi$ , which then become dependent on the energy and the spatial position [17]. So, one may find it difficult to interpret scattering by point interactions in terms of the so-called inverse scattering problem [18]. Furthermore, some of the above methods are cumbersome to apply to more general cases, e.g., for many point interactions of different types, or for certain  $V$ ’s.

Hence, it would be desirable to have simpler treatment for the regularization of single point interactions, as well as to calculate quantum objects, such as Green functions, directly from concrete physical quantities, instead of renormalized “bare” parameters (difficult to identify in a real system). Moreover, the method should be simple enough to be extended to more general situations. Actually, such a point of view has already been used to discuss  $\delta$  interactions in two dimensions [19]. In the present work we show how to do that for  $N$  general point interactions in one dimension under different boundary conditions.

\*Electronic address: schmidt@fisica.ufpr.br

†Electronic address: luz@fisica.ufpr.br

Our paper is organized as follows. In Sec. II we show how to characterize any general point interaction through its scattering amplitudes. Then, with the help of some results known in the literature we are able to readily write down the exact Green function for the problem. In Sec. III, from a multiple-scattering approach, we extend the calculations and obtain the exact  $G$  for an array of  $N$  general point interactions under different conditions, namely, on a line, on a half-line, confined in a box (with different boundary conditions at the walls), and finally for periodic boundary conditions. Possible physical applications for all these systems are briefly discussed. Finally, in Sec. IV we draw our final remarks and conclusion.

## II. THE SCATTERING AMPLITUDES CHARACTERIZATION OF A GENERAL POINT INTERACTION AND THE GREEN FUNCTION

It is a well-known fact that solving the one-dimensional Schrödinger equation for a  $\delta$  potential located at the origin,  $\delta(x)$ , is equivalent to the boundary conditions [ $\psi'(x) \equiv d\psi/dx$ ]

$$\begin{pmatrix} \psi(0^+) \\ \psi'(0^+) \end{pmatrix} = \omega \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi(0^-) \\ \psi'(0^-) \end{pmatrix}, \quad (1)$$

where the parameter values are  $a=d=\omega=1$ ,  $b=0$ , and  $c=\gamma$ , with  $\gamma$  the potential's strength. This boundary condition can be obtained from the Schrödinger equation by imposing that the wave function is continuous at  $x=0$ . However, the same does not apply if the potential in question is  $\delta$  prime,  $\delta'(x)$ : the boundary condition satisfied by  $\psi(x)$  cannot be determined from the Schrödinger equation. The only condition known *a priori* is the one  $\psi'(x)$  fulfills, namely,  $\psi'(x)$  is continuous at  $x=0$ . For this very reason self-adjoint extension is invoked [6,20]. So, one can solve an equivalent problem to the Schrödinger equation with the  $\delta'$  potential imposing Eq. (1) with  $c=0$ ,  $a=d=\omega=1$ , and  $b=\gamma$ .

The above two examples do not represent all possible one-dimensional point interactions. In fact, through the self-adjoint extension technique it is shown that the most general case is the one in which

$$|\omega|=1 \quad \text{and} \quad ad-bc=1, \quad \text{with } a,b,c,d \text{ all real.} \quad (2)$$

An important aspect of this prescription is that it does not allow a concrete realization for generalized point interactions. In other words, it does not lead to a unique function depending on  $(a,b,c,d,\omega)$  which reproduces the whole boundary conditions given in Eqs. (1) and (2). So, we cannot write a Hamiltonian  $H=H_0+\Xi(x)$ , since one does not know a single form for the potential  $\Xi(x)$  (actually, a different procedure is to represent a generalized point interaction by making compositions of triple  $\delta$  functions and then taking certain limits [21], which, however, also cannot be put in the form of usual potential).

An alternative way to characterize the boundary conditions (1) is through the scattering amplitudes. Let a plane-

wave function, of wave number  $k$  and incident either from the left (+) or right (-), be written as

$$\psi^{(\pm)}(x) = \frac{1}{\sqrt{2\pi}} \times \begin{cases} \exp[\pm ikx] + R^{(\pm)} \exp[\mp ikx], & x \leq 0 \\ T^{(\pm)} \exp[\pm ikx], & x \geq 0. \end{cases} \quad (3)$$

$\psi$  satisfies  $-d^2\psi(x)/dx^2 = k^2\psi(x)$  for  $x \neq 0$ . Now, if we choose for the scattering amplitudes ( $\theta^{(+)} = ad - bc$  and  $\theta^{(-)} = 1$ )

$$R^{(\pm)} = \frac{c \pm ik(d-a) + bk^2}{-c + ik(d+a) + bk^2},$$

$$T^{(\pm)} = \frac{2ik\omega^{\pm 1}\theta^{(\pm)}}{-c + ik(d+a) + bk^2}, \quad (4)$$

we find that Eq. (3) satisfies the boundary conditions (1). Furthermore, by imposing [18]

$$|R|^2 + |T^{(\pm)}|^2 = 1, \quad R^{(+)*}T^{(+)} + T^{(-)*}R^{(-)} = 0,$$

$$R_k^{(\pm)*} = R_{-k}^{(\pm)}, \quad T_k^{(\pm)*} = T_{-k}^{(\mp)}, \quad (5)$$

the parameters must necessarily obey Eq. (2). The conditions in Eq. (5) assure important properties, e.g., probability conservation and the existence of the scattering inverse problem (see Ref. [18] for a detailed discussion). Furthermore, if one also requires time-reverse invariance, which is translated into the relation  $T^{(+)} = T^{(-)}$  (which we are not imposing in this work), then one should choose  $\omega = \pm 1$ .

From all these results we see that there is a complete equivalence in defining the most general point interaction through Eqs. (1) and (2) or by specifying its scattering amplitudes (4) and (5). We also observe that eventually we may have bounded states for a given point interaction potential depending on its parameters. In such a case the quantum amplitudes  $R$  and  $T$  have poles at the upper half of the complex plane  $k$ , corresponding to the eigenvalues. The eigenfunctions can then be obtained from an appropriate extension of the scattering states to those  $k$  values [22].

The exact Green function for arbitrary potentials of compact support have been obtained in Ref. [23], with an extension for more general potentials presented in Ref. [24]. For the derivations in Ref. [23], it is necessary that the  $R$  and  $T$  satisfy certain conditions, which indeed are those Eq. (5). Thus, based on Ref. [23] we can calculate the Green function for general point interactions by using the reflection and transmission coefficients, which for their very construction carry information on boundary conditions and are relevant quantities with a clear physical interpretation (being the measured quantities in real situations [25]).

So, from Ref. [23] we can readily write down the exact Green function as the following. By defining  $G_{+-}$  for  $x_f > 0 > x_i$ ,  $G_{-+}$  for  $x_i > 0 > x_f$ ,  $G_{++}$  for  $x_f, x_i > 0$ , and  $G_{--}$  for  $x_f, x_i < 0$ , we have

$$G_{\pm\mp} = \frac{1}{2ik} T^{(\pm)} \exp[ik|x_f - x_i|],$$

$$G_{\pm\pm} = \frac{1}{2ik} \{ \exp[ik|x_f - x_i|] + R^{(\pm)} \exp[ik(|x_f| + |x_i|)] \}. \quad (6)$$

The Green functions for  $\delta(x)$  and  $\delta'(x)$  were calculated, respectively, in Ref. [26] and [12,16,17]. As we show in Appendix A, the  $G$ 's in Eq. (6) reduce to those cases if we assume for the parameters the appropriate values as previously discussed.

### III. ARBITRARY NUMBER OF POTENTIALS AND DIFFERENT BOUNDARY CONDITIONS

The advantage of the present method is that it can be easily used to calculate the exact Green function for arbitrary (finite) number of different point interactions, both on the infinite line or for periodic boundary conditions. Also, we can obtain the  $G$ 's for restricted systems such as  $N$  potentials on a half-line or confined inside an infinite box, with different boundary conditions at the border walls.

There are many reasons to study such problems. For instance, for  $N$  disordered general point interactions on a line one may have interest in the propagation of wave packets through the lattice in order to analyze the influence of more singular potentials in the usual scale theory of localization [27]. For periodic boundary conditions, we recall recent experiments using a waveguide filled with localized scatterers in a circular setup [28]. The dynamics of the microwaves is analogous to our 1D quantum system. They measure the transmission amplitudes along the waveguide and observe the Hofstadter butterfly [29], which has a fractal structure (a Cantor set). Similar systems are used to study the quantum dynamics of eigenvalues on changing of some external parameter [30]. Also, periodic boundary conditions were used [5] to study duality properties of point interactions in systems of bosons and fermions.

#### A. Green function for arbitrary number of point interactions on the line

Consider  $N$  point interaction potentials, each characterized by its location  $y_n$  (with  $y_{n-1} < y_n$ ) and the set of parameters  $\{a_n, b_n, c_n, d_n, \omega_n\}$ . The quantum coefficients are  $R_n^{(\pm)}(y_n)$  and  $T_n^{(\pm)}(y_n)$ . For a potential located at  $y_n$ , it is easy to see that  $R_n^{(\pm)}(y_n) = R_n^{(\pm)} \exp(\pm 2iky_n)$  and  $T_n^{(\pm)}(y_n) = T_n^{(\pm)}$ , where  $R_n^{(\pm)}$  and  $T_n^{(\pm)}$  are given by Eq. (4). In other words, the reflection coefficients change by a phase factor while transmission coefficients remain the same.

Now, we can obtain the exact  $G_{jm}(x_f, x_i; k)$  for  $y_m < x_i < y_{m+1}$  and  $y_j < x_f < y_{j+1}$ , (for arbitrary  $m$  and  $j$ ) by using the multiple-scattering approach introduced in Refs. [23,24]. We only discuss the case of  $x_i < x_f$ , see Fig. 1. The idea is simple,  $G$  is given by  $(2ik)^{-1} \sum_{\text{SP}} W_{\text{SP}} \exp[iS_{\text{SP}}(x_f, x_i; k)]$ . The sum is performed over all possible scattering paths (sp) joining the end points, with  $S_{\text{SP}}$  and  $W_{\text{SP}}$  their actions and amplitudes (or weights). For each scattering path,  $S$  is calcu-

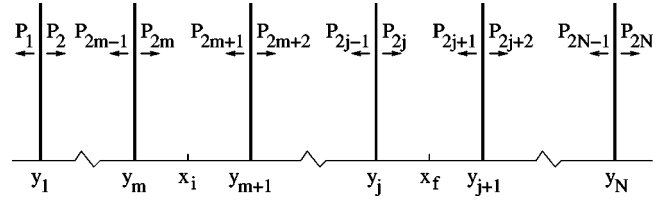


FIG. 1.  $N$  arbitrary point interactions on the line and located at the arbitrary positions  $y$ 's.

lated from the free propagations between the potentials.  $W$  is the product of the quantum coefficients (up to the phase factors) gained each time the particle is scattered off by a point interaction. When the particle hits the potential  $n$ , it can be reflected (getting an amplitude factor  $R_n$ ) or transmitted (getting an amplitude factor  $T_n$ ) from the position  $y_n$ . Once scattered the particle can go either to the left, along the path  $P_{2n-1}$ , or to the right, along the path  $P_{2n}$  (see Fig. 1). To calculate the Green function we have to classify and sum up all the scattering trajectories, which always can be done in a closed form as shown in Refs. [23,24]. Following such references, we find

$$G_{jm}^N(x_f, x_i; k) = \left( \frac{1}{2ik} \right) \frac{T_{j,m+1}^{(+)} \exp(-ik(y_j - y_{m+1}))}{D_{\text{line}}} \times (\exp[-ikx_i] + R_{m,1}^{(-)} \exp[ikx_i] \times \exp[-2iky_m]) (\exp[ikx_f] + R_{N,j+1}^{(+)} \times \exp[-ikx_f] \exp[2iky_{j+1}]), \quad (7)$$

where

$$D_{\text{line}} = \{ 1 - R_{m,1}^{(-)} R_{j,m+1}^{(+)} \exp[2ik(y_{m+1} - y_m)] \} \times \{ 1 - R_{N,j+1}^{(+)} R_{j,m+1}^{(-)} \exp[2ik(y_{j+1} - y_j)] \} - R_{m,1}^{(-)} R_{N,j+1}^{(+)} T_{j,m+1}^{(+)} T_{j,m+1}^{(-)} \times \exp[-2ik(y_j - y_{m+1} + y_m - y_{j+1})]. \quad (8)$$

The factors  $R_{n,l}$  and  $T_{n,l}$  are derived in Appendix B.

#### B. Green function for arbitrary number of point interactions on the half-line

Recently, an interesting effect, which is related to the so-called atomic mirrors, has been proposed [31]. The idea is to place a wave packet initially with a zero mean momentum near a given barrier potential, e.g., an infinite hard wall or a  $\delta$  potential. The wave packet spreads out and due to energy quantum fluctuations its mean momentum value increases. This behavior is associated with an effective ‘‘quantum repulsive force,’’ which in principle, can be measured using ultracold atoms. The problem is to reach the necessary extreme conditions in the laboratory. Thus, it would be helpful to enhance the phenomenon, which eventually can be done by considering more singular potentials. This motivates us to look at the problem of general point interactions near by an infinite hard wall, i.e.,  $N$  point interactions on the half-line.

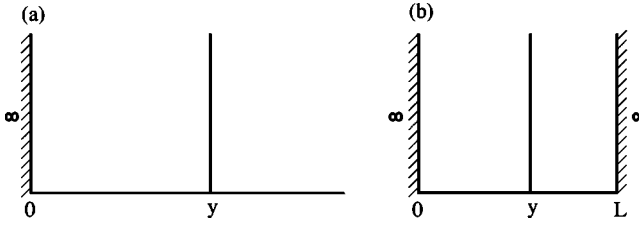


FIG. 2. A general point interaction at  $y$  (a) on the half-line and (b) within an infinite well. In both cases we may have Dirichlet or Neumann boundary conditions at the walls.

The exact Green function for this case is easily obtained from our previous results. Actually, by setting  $y_1=0$  in our general expression (7) and taking  $R_1^{(-)}$  to be  $-1$  or  $+1$ , we have, respectively, Dirichlet or Neumann boundary conditions at  $x=0$ . We observe that it is simple to consider that the first point interaction is either  $\gamma\delta(x)$  or  $\gamma\delta'(x)$ , where the limit  $\gamma\rightarrow\infty$  is taken, i.e., the point interaction becomes an infinite hard wall with different reflection properties. Just as a simple example, consider the situation in Fig. 2(a), an infinite hard wall plus a general point interaction at  $x=y$ . Denoting  $G_{--}$  for  $x_i, x_f < y$  and  $G_{+-}$  for  $x_i < y < x_f$ , we have after straightforward simplifications (here,  $s=1$  for Dirichlet and  $s=-1$  for Neumann boundary conditions)

$$\begin{aligned}
 G_{+-}(x_f, x_i; k) &= \frac{1}{2ik} \frac{T^{(+)}}{(1+sR^{(+)})\exp[2iky]} \\
 &\quad \times \{\exp[ik(x_f-x_i)] - s \exp[ik(x_f+x_i)]\} \\
 &\quad \times \exp[iky], \\
 G_{--}(x_f, x_i; k) &= \frac{1}{2ik} \frac{1}{(1+sR^{(+)})\exp[2iky]} \\
 &\quad \times \{\exp[ik|x_f-x_i|] - sR^{(+)}\exp[2iky] \\
 &\quad \times \exp[-ik|x_f-x_i|] - s \exp[ik(x_f+x_i)] \\
 &\quad + R^{(+)}\exp[2iky]\exp[-ik(x_f+x_i)]\}.
 \end{aligned} \tag{9}$$

### C. Green function for arbitrary number of point interactions inside an infinite well

A system so simple as a short-range potential placed inside a 1D infinite well can sometimes present unusual dynamics [32]. In fact, it has been shown that such a system can exhibit chaoticlike behavior. Another interesting property, which can be studied in 2D boxes [33] as well in systems of the present type, is the revival time. Initially, a wave packet is placed in one side of the box. Then it evolves, spreading over the whole configuration space it is allowed to fulfill. After a certain time (the revival time) all the ‘‘pieces’’ of the wave packet return to the initial situation, reconstructing the original state. Such applications may lead one to try to calculate the Green function for an arbitrary number of point interactions inside an infinite well.

As before, the exact  $G$  for this case can be obtained from Eq. (7). For that, we just set  $y_1=0$ ,  $y_N=L$  and then choose for  $R_1^{(-)}$  and  $R_N^{(+)}$  the values  $-1$  or  $+1$ . This will lead to all the four possible combinations of Dirichlet and Neumann boundary conditions for the walls at  $x=0$  and  $x=L$ .

As an example, consider the potential in Fig. 2(b), a single point interaction at  $x=y$  inside an infinite well of length  $L$ . We restrict the discussion to  $G_{--}$  for  $x_i, x_f < y$  and  $G_{+-}$  for  $x_i < y < x_f$ . After lengthy but simple manipulations, we find the Green functions (again,  $s=1$  and  $s=-1$  lead to Dirichlet and Neumann boundary conditions at the corresponding walls)

$$\begin{aligned}
 G_{--}(x_f, x_i; k) &= \frac{1}{2ik} \frac{\{1+s_L R^{(-)}\exp[2ik(L-y)]\}}{D} \\
 &\quad \times \left\{ \exp[ik|x_f-x_i|] - s_0 \left( R^{(+)}\exp[2iky] \right. \right. \\
 &\quad \left. \left. - \frac{s_L T^{(+)} T^{(-)}\exp[2ikL]}{1+s_L R^{(-)}\exp[2ik(L-y)]} \right) \right. \\
 &\quad \times \exp[-ik|x_f-x_i|] - s_0 \exp[ik(x_f+x_i)] \\
 &\quad \left. + \left( R^{(+)}\exp[2iky] \right. \right. \\
 &\quad \left. \left. - \frac{s_L T^{(+)} T^{(-)}\exp[2ikL]}{1+s_L R^{(-)}\exp[2ik(L-y)]} \right) \right. \\
 &\quad \left. \times \exp[-ik(x_f+x_i)] \right\}, \\
 G_{+-}(x_f, x_i; k) &= \frac{1}{2ik} \frac{T^{(+)}}{D} \{(\exp[-ikx_i] - s_0 \exp[ikx_i]) \\
 &\quad \times (\exp[ikx_f] - s_L \exp[2ikL]) \\
 &\quad \times \exp[-ikx_f]\},
 \end{aligned} \tag{10}$$

with

$$\begin{aligned}
 D &= (1+s_0 R^{(+)}\exp[2iky])\{1+s_L R^{(-)}\exp[2ik(L-y)]\} \\
 &\quad - s_0 s_L T^{(+)} T^{(-)}\exp[2ikL].
 \end{aligned} \tag{11}$$

The poles of  $G$  give the system eigenvalues. In our case, they come from  $D=0$ .

### D. Green function for periodic boundary conditions

Here we consider periodic boundary conditions for the wave function, or  $\psi(-L/2)=\psi(L/2)$  and  $\psi'(-L/2)=\psi'(L/2)$ . So, the Green function satisfies exactly these same relations in both  $x_i$  and  $x_f$ . For simplicity, let us start with one point potential at  $x=0$ , see Fig. 3(a). We need to consider all the scattering paths starting from  $x_i$  and arriving at  $x_f$ . We can think of the points  $-L/2$  and  $L/2$  as being connected, so we have trajectories on a circle. The paths are



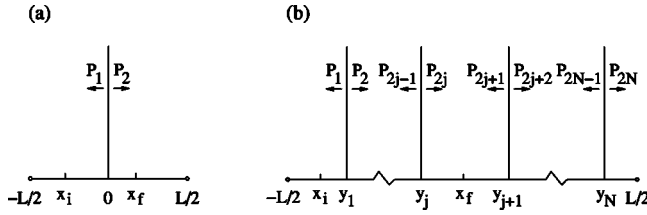


FIG. 3. Periodic boundary conditions, i.e., the point  $x = -L/2$  is equivalent to the point  $x = L/2$ . (a) A single general point interaction at  $x=0$ . (b)  $N$  general point interactions located at arbitrary positions  $y$ 's.

then given by arbitrary number of rounds clockwise and counterclockwise on the circle. Each time the particle hits the point interaction at  $x=0$ , it is either reflected from or transmitted through the potential (with the amplitude corresponding to that trajectory getting the factor  $R$  or  $T$ , respectively). The paths are continuous and no extra factor to construct  $W_{\text{SP}}$  is gained when the particle crosses from  $-L/2$  to  $L/2$  or  $L/2$  to  $-L/2$ . By classifying and summing up all the scattering paths, we finally obtain the exact Green function as

$$G(x_f, x_i; k) = \frac{1}{2ik} \frac{1}{D_{\text{circle}}} \{ \{ T^{(+)} + (R^{(+)}R^{(-)} - T^{(+)}T^{(-)}) \times \exp[ikL] \} \exp[ik(x_f - x_i)] + (1 - T^{(+)}\exp[ikL]) \exp\{ik[L - (x_f - x_i)]\} + R^{(+)}\exp\{ik[L - (x_f + x_i)]\} + R^{(-)}\exp\{ik[L + (x_f + x_i)]\} \}, \quad (12)$$

with

$$D_{\text{circle}} = (1 - T^{(+)}\exp[ikL])(1 - T^{(-)}\exp[ikL]) - R^{(+)}R^{(-)}\exp[2ikL]. \quad (13)$$

The zeros of  $D_{\text{circle}}$  are the energy eigenvalues.

The same reasoning to construct the scattering paths can be used for the more general case shown in Fig. 3(b). The final result is

$$G(x_f, x_i; k) = \frac{1}{D_{\text{pbc}}} \{ \{ T_{j,1}^{(+)} \exp[-ik(y_j - y_1)] + T_{N,j+1}^{(-)} K_{j,1} \exp[-ik(y_N - y_{j+1})] \} \exp[ikL] \} \exp[ik(x_f - x_i)] + \{ R_{N,j+1}^{(-)} T_{j,1}^{(+)} \exp[ik(L - 2y_N - y_j + y_1)] + R_{j,1}^{(-)} T_{N,j+1}^{(-)} \exp[-ik(2y_j + y_N - y_{j+1})] \} \exp[ikL] \times \exp[ik(x_f + x_i)] + \{ R_{j,1}^{(+)} T_{N,j+1}^{(-)} \exp[ik(L + 2y_1 - y_N + y_{j+1})] + R_{N,j+1}^{(+)} T_{j,1}^{(+)} \exp[ik(2y_{j+1} - y_j + y_1)] \} \times \exp[-ik(x_f + x_i)] + \{ T_{N,j+1}^{(-)} \exp[-ik(y_N - y_{j+1})] + T_{j,1}^{(+)} K_{N,j+1} \exp[-ik(y_j - y_1)] \} \exp[ikL] \} \exp[ikL] \times \exp[-ik(x_f - x_i)], \quad (14)$$

where  $K_{b,a} = R_{b,a}^{(+)} R_{b,a}^{(-)} \exp[2ik(y_a - y_b)] - T_{b,a}^{(+)} T_{b,a}^{(-)} \exp[-2ik(y_b - y_a)]$  and

$$D_{\text{pbc}} = -1 + R_{j,1}^{(+)} R_{N,j+1}^{(-)} \exp[2ik(L - y_N + y_j)] + R_{j,1}^{(-)} R_{N,j+1}^{(+)} \exp[2ik(y_{j+1} - y_j)] - (T_{j,1}^{(-)} T_{N,j+1}^{(-)} + T_{j,1}^{(+)} T_{N,j+1}^{(+)} \exp[-ik(y_N - y_{j+1} + y_j - y_1)]) - K_{j,1} K_{N,j+1} \exp[2ikL]. \quad (15)$$

For some of the above quantities see Appendix B.

#### IV. REMARKS AND CONCLUSION

Here we have presented a way to calculate the Green functions for generalized point interactions in one dimension. Our approach uses a totally different approach than most theoretical treatments. The method is based mostly on physical grounds. First, we show that one can define a general point interaction through its scattering properties, i.e., the reflection and transmission amplitudes. Then, we discuss

how to construct the exact Green function from such coefficients. It helps to keep track of the relevant physical quantities what may not happen in some other methods due to their subtle renormalization procedures.

We do not invoke any kind of regularization, such as renormalization, series expansion, or self-adjoint extension. The advantage in not using specific mathematical prescriptions is that, in general, such techniques may not have a clear physical meaning (differently from some other contexts, where there are solid principles as a guide for regularization, such as in the case of the Mandelstam-Leibbrandt prescription for the light-cone gauge in field theory) or are not unique. In fact, one can find in the literature the Green function for a same general point interaction based on calculations from either a single  $\delta'$  potential [17] or a combination of  $\delta$  and  $\delta'$  potentials [34]. This apparent contradictory result is just due to the fact that the authors use different prescriptions to regularize their Hamiltonians.

The second advantage of this simpler method is that we can easily extend our calculations to more general cases.

Indeed, we have derived the Green functions for  $N$  generalized point interactions for several situations; (i) on a line, (ii) restricted to the half-line, and (iii) within in an infinite box. For the later two cases we can impose different boundary conditions at the border walls. Also, (iv) we have obtained  $G$  for periodic boundary conditions (the circlelike case). As far as we know, explicit expressions for  $G$  for all these systems were not known in the literature and our aim was to fulfill this gap.

A way to construct generalized point interactions has been recently proposed in a series of papers [5,21]. The idea is to consider different usual  $\delta$  functions, all separated by a distance  $y_0$ , with appropriate values for their strengths. In the limit  $y_0 \rightarrow 0$ , one obtains the desired potential. Although the exact limit cannot be taken in practice, if we can approximate each  $\delta$  by some short-range potential and then have  $y_0$  finite but small, we may in principle obtain a physical realization of a general point interaction. This opens a great possibility of experiments in order to test fundamental and interesting phenomena in quantum mechanics as applications for the systems discussed in this work. In particular, of great interest would be the calculation of the time evolution of wave packets,

$$\Psi(x_f, t) = i/(2\pi) \int \int dx_i dE \exp(-iEt) G(x_f, x_i; E) \Psi(x_f, 0),$$

or of the density of states,  $\rho(E) = -(1/\pi) \Im \int dx G(x, x; E)$ . Due to the form of our Green function, the integration with respect to the position can be solved analytical for  $\rho$ . For  $\Psi$ , it can also be done for simple initial wave packets such as Gaussians. The integral on the energy, on its turn, can be easily carried out using numerical methods such as the fast Fourier transform.

Regarding extensions of the present results we comment the following. In Ref. [35], both the local description of point interaction (which we used in this work) and a global one based on the  $U(2)$  group (which contains the former) were discussed. In this  $U(2)$  context, one can also obtain the reflection and transmission coefficients. So, our results are valid in the global description of point interactions. We emphasize that the key point to obtain all the Green functions calculated here is to know the  $R$  and  $T$  coefficients of the potentials. In fact, our results apply for every one-dimensional  $V(x)$  that satisfies the assumptions given in Refs. [23,24]. Thus, a ‘‘mixing’’ 1D lattice including both general point interactions and usual potentials (which decay at least exponentially) can be calculated from our method.

As a final remark we mention that it would be interesting to extend the present approach to higher dimensions. In a recent work [36], it has been discussed how to calculate the wave function and also the Green function for boundary walls of arbitrary shapes and with very general boundary conditions. The method is based on the calculation of a  $T$  matrix, which plays a role similar to 1D quantum amplitudes. By taking appropriate limits, we could construct pointlike interactions in two and three dimensions for which the Green function is already regularized. Such work is in progress and will be reported in the due course.

## ACKNOWLEDGMENT

A.G.M.S. and M.G.E.dL. gratefully acknowledge CNPq for financial support.

## APPENDIX A: SPECIAL CASES

In order to exemplify our general result for a single generalized point interaction on the line, Eq. (6), we show how to obtain from it the well-known Green functions for  $\delta$  [26] and  $\delta'$  [12,16,17] potentials.

The  $\delta$  potential is a particular case of Eq. (1), where  $a = d = \omega = 1, b = 0, c = \gamma$ . Substituting these parameters into Eqs. (4) and (6), one obtains

$$G_{\pm\mp} = \frac{\exp[ik|x_f - x_i|]}{2ik - \gamma},$$

$$G_{\pm\pm} = \frac{1}{2ik} \left[ \exp[ik|x_f - x_i|] + \left( \frac{\gamma}{2ik - \gamma} \right) \exp[ik(|x_f| + |x_i|)] \right]. \quad (A1)$$

Now, recalling the meaning of the subscripts for  $G$  (see Sec. II) in the above equation, it is not difficult to realize that we can write Eq. (A1) as the following single formula:

$$G = \frac{1}{2ik} \left[ \exp[ik|x_f - x_i|] + \left( \frac{\gamma}{2ik - \gamma} \right) \exp[ik(|x_f| + |x_i|)] \right], \quad (A2)$$

which agrees with Eq. (17) of Ref. [26] if we identify in the coefficients which multiply the exponentials,  $\gamma \leftrightarrow -Z$  and  $2ik \leftrightarrow ik$  (this last relation is due to the fact that in Ref. [26] the author uses  $m = 1$  instead of  $m = 1/2$  as in our case).

The  $\delta'$  potential is defined by  $a = d = \omega = 1, b = \gamma, c = 0$ , so one gets the Green functions

$$G_{\pm\mp} = \frac{\exp[ik|x_f - x_i|]}{(2i + \gamma k)k},$$

$$G_{\pm\pm} = \frac{1}{2ik} \left[ \exp[ik|x_f - x_i|] + \left( \frac{\gamma k}{2i + \gamma k} \right) \exp[ik(|x_f| + |x_i|)] \right]. \quad (A3)$$

Again, it is easy to show that the above expressions can be summarized as  $[\text{sign}(\cdot)]$  is the signal function

$$G = \frac{1}{2ik} \left[ \exp[ik|x_f - x_i|] + \left( \frac{\gamma k}{2i + \gamma k} \right) \text{sign}(x_f) \text{sign}(x_i) \exp[ik(|x_f| + |x_i|)] \right]. \quad (A4)$$

Equation (A4) agrees with Eq. (12) of Grosche in Ref. [16] if we identify  $\gamma \leftrightarrow -\beta$ ,  $-ik \leftrightarrow \sqrt{-E}$  (because in such refer-

ence a Wick rotation is used) and  $G \leftrightarrow -G$  (due to an opposite sign used in the definition of the Green function).

### APPENDIX B: RECURRENCE FORMULAS

Here we show how to obtain some coefficients used in the exact expressions for the Green functions by means of recurrence formulas. The idea is to face a series of  $n-l+1$  potentials located at  $y_l, y_{l+1}, \dots, y_{n-1}, y_n$  as a single block and then to associate with it the amplitudes  $R_{n,l}$  and  $T_{n,l}$  (for a detailed discussion, see Ref. [23]).

Assume first a potential composed by two point interactions, placed at  $y_l$  and  $y_{l+1}$ , and let  $x_i, x_f < y_l < y_{l+1}$ . By using the same approach developed throughout this paper, we obtain for the Green function

$$G_{--} = \exp[ik|x_f - x_i|] + R_l^{(+)} \exp[-ik(x_f + x_i - 2y_l)] \\ + \frac{R_{l+1}^{(+)} T_l^{(+)} T_l^{(-)} \exp[-ik(x_f + x_i - 2y_l)]}{1 - R_l^{(-)} R_{l+1}^{(+)} \exp[2ik(y_{l+1} - y_l)]}. \quad (\text{B1})$$

We define a reflection coefficient for this block (made by two potentials) as

$$R_{l+1,l}^{(+)} = R_l^{(+)} + \frac{R_{l+1}^{(+)} T_l^{(+)} T_l^{(-)} \exp[2ik(y_{l+1} - y_l)]}{1 - R_l^{(-)} R_{l+1}^{(+)} \exp[2ik(y_{l+1} - y_l)]}. \quad (\text{B2})$$

In analogy, we can define  $R_{l+1,l}^{(-)}$  for this same block by calculating  $G$  for  $x_i, x_f > y_{l+1} > y_l$ .

Now, consider  $x_i < y_l < y_{l+1} < x_f$ , we have

$$G_{+-} = \frac{T_l^{(+)} T_{l+1}^{(+)} \exp[ik(y_{l+1} - y_l)]}{1 - R_l^{(-)} R_{l+1}^{(+)} \exp[2ik(y_{l+1} - y_l)]} \\ \times \exp\{ik[x_f - x_i - (y_{l+1} - y_l)]\}, \quad (\text{B3})$$

thus, we can again define

$$T_{l+1,l}^{(+)} = \frac{T_l^{(+)} T_{l+1}^{(+)} \exp[ik(y_{l+1} - y_l)]}{1 - R_l^{(-)} R_{l+1}^{(+)} \exp[2ik(y_{l+1} - y_l)]}. \quad (\text{B4})$$

Similarly, we derive  $T_{l+1,l}^{(-)}$  by calculating  $G$  for  $x_i > y_{l+1} > y_l > x_f$ .

In order to get recurrence formulas, consider a third potential located at  $y_{l+2}$  (recall that our two point interactions block has its end points at  $y_l$  and  $y_{l+1}$ ). Let  $x_i, x_f < y_l < y_{l+1} < y_{l+2}$  and consider  $R_{l+1,l}$  and  $T_{l+1,l}$  of the two potentials block as obtained before. By a direct inspection of Eq. (B2), we can readily infer the reflection coefficient  $R_{l+2,l}^{(+)}$  (for the new block formed by the three potentials) as

$$R_{l+2,l}^{(+)} = R_{l+1,l}^{(+)} + \frac{T_{l+1,l}^{(+)} T_{l+1,l}^{(-)} R_{l+2}^{(+)} \exp[2ik(y_{l+2} - y_{l+1})]}{1 - R_{l+1,l}^{(-)} R_{l+2}^{(+)} \exp[2ik(y_{l+2} - y_{l+1})]}; \quad (\text{B5})$$

the generalization is then straightforward,

$$R_{n,l}^{(+)} = R_{n-1,l}^{(+)} + \frac{T_{n-1,l}^{(+)} T_{n-1,l}^{(-)} R_n^{(+)} \exp[2ik(y_n - y_{n-1})]}{1 - R_{n-1,l}^{(-)} R_n^{(+)} \exp[2ik(y_n - y_{n-1})]}. \quad (\text{B6})$$

For the reflection coefficient  $R_{n-1,l}^{(-)}$  one finds

$$R_{n,l}^{(-)} = R_n^{(-)} + \frac{T_n^{(+)} T_n^{(-)} R_{n-1,l}^{(-)} \exp[2ik(y_n - y_{n-1})]}{1 - R_{n-1,l}^{(-)} R_n^{(+)} \exp[2ik(y_n - y_{n-1})]}. \quad (\text{B7})$$

Transmission coefficients can be written in terms of recurrence relations as well. The final results are

$$T_{n,l}^{(+)} = \frac{T_{n-1,l}^{(+)} T_n^{(+)} \exp[ik(y_n - y_{n-1})]}{1 - R_{n-1,l}^{(-)} R_n^{(+)} \exp[2ik(y_n - y_{n-1})]} \quad (\text{B8})$$

and

$$T_{n,l}^{(-)} = \frac{T_{n-1,l}^{(-)} T_n^{(-)} \exp[ik(y_n - y_{n-1})]}{1 - R_{n-1,l}^{(-)} R_n^{(+)} \exp[2ik(y_n - y_{n-1})]}. \quad (\text{B9})$$

- 
- [1] R.L. Kronig and W.G. Penny, Proc. R. Soc. London, Ser. A **130**, 499 (1931).  
[2] P. Seba and K. Zyczkowski, Phys. Rev. A **44**, 3457 (1991); T. Shigehara, Phys. Rev. E **50**, 4357 (1994); M. Van Vessen, Jr., M.C. Santos, B.K. Cheng, and M.G.E. da Luz, *ibid.* **64**, 026201 (2001).  
[3] Ph. de Sousa Gerbert, Phys. Rev. D **40**, 1346 (1989); C.R. Hagen, Phys. Rev. Lett. **64**, 503 (1990); Int. J. Mod. Phys. A **6**, 3119 (1991).  
[4] D.K. Park, J. Math. Phys. **36**, 5453 (1995).  
[5] T. Cheon and T. Shigehara, Phys. Rev. Lett. **82**, 2536 (1999).  
[6] S. Albeverio, F. Gesztesy, R. Höegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics* (Springer-Verlag, Berlin, 1988).  
[7] J.B. McGuire, J. Math. Phys. **5**, 622 (1964); **6**, 432 (1965); C.N. Yang, Phys. Rev. Lett. **19**, 1312 (1967); Phys. Rev. **168**, 1920 (1968).  
[8] S. Albeverio, S-M. Fei, and P. Kurasov, Oper. Th.: Adv. and Appl. **132**, 67 (2002).  
[9] F.A.B. Coutinho, Y. Nogami, and L. Tomio, J. Phys. A **32**, 4931 (1999).  
[10] A.Z. Capri, Am. J. Phys. **45**, 823 (1977); M.A. Bég and R.C. Furlong, Phys. Rev. D **31**, 1370 (1985); R. Jackiw, in *M. A. Bég Memorial Volume*, edited by A. Ali and P. Hoodbhoy (World Scientific, Singapore, 1991); C. Manuel, and R. Tarrach, Phys. Lett. B **328**, 113 (1994); P. Exner and P. Seba, Phys. Lett. A **245**, 35 (1998); C.R. Hagen, Phys. Rev. A **61**, 032715 (2000).

- [11] J. Zorbas, J. Math. Phys. **21**, 840 (1980).
- [12] C. Grosche, J. Phys. A **23**, 5205 (1990).
- [13] D.R. Phillips, S.R. Beane, and T.D. Cohen, Ann. Phys. (N.Y.) **263**, 255 (1998); H.E. Camblong and C.R. Ordoñez, Phys. Rev. A **65**, 052123 (2002).
- [14] M. Carreau, J. Phys. A **26**, 427 (1993); F.A.B. Coutinho, Y. Nogami, and J.F. Perez, *ibid.* **30**, 3937 (1997).
- [15] T.E. Clark, R. Menikoff, and D.H. Sharp, Phys. Rev. D **22**, 3012 (1980); M. Carreau, E. Farhi, and S. Gutmann, *ibid.* **42**, 1194 (1990); C. Grosche, Phys. Rev. Lett. **71**, 1 (1993); M.G.E. da Luz and B.K. Cheng, Phys. Rev. A **51**, 1811 (1995).
- [16] C.L. Roy, Phys. Rev. A **47**, 3417 (1993); C. Grosche, J. Phys. A **28**, L99 (1995).
- [17] D.K. Park, J. Phys. A **29**, 6407 (1996).
- [18] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd ed. (Springer, Berlin, 1989).
- [19] R.L. Weaver and D. Sornette, Phys. Rev. E **52**, 3341 (1995).
- [20] G. Bonneau, J. Faraut, and G. Valent, Am. J. Phys. **69**, 322 (2001).
- [21] T. Cheon and T. Shigehara, Phys. Lett. A **243**, 111 (1998); T. Cheon, T. Shigehara, and K. Takayanagi, J. Phys. Soc. Jpn. **69**, 345 (2000); T. Shigehara, H. Mizoguchi, T. Mishima, and T. Cheon, IEICE Trans. Fundamentals **E82-A**, 1708 (1999); quant-ph/9812006.
- [22] L.I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1955).
- [23] M.G.E. da Luz, E.J. Heller, and B.K. Cheng, J. Phys. A **31**, 2975 (1998).
- [24] M.G.E. da Luz, B.K. Cheng, and M.W. Beims, J. Phys. A **34**, 5041 (2001).
- [25] H.J. Stöckmann and J. Stein, Phys. Rev. Lett. **64**, 2215 (1990); H. Schanze, E.R.P. Alves, C.H. Lewenkopf, and H.J. Stöckmann, Phys. Rev. E **64**, 065201 (2001).
- [26] S.M. Blinder, Phys. Rev. A **37**, 973 (1988).
- [27] R. Landauer, Philos. Mag. **21**, 863 (1970); P.W. Anderson, D.J. Thouless, E. Abrahams, and D.S. Fisher, Phys. Rev. B **22**, 3519 (1980).
- [28] F.M. Izrailev and A.A. Krokhin, Phys. Rev. Lett. **82**, 4062 (1999); U. Kuhl and H.-J. Stöckmann, *ibid.* **80**, 3232 (1998); U. Kuhl, F.M. Izrailev, A.A. Krokhin, and H.-J. Stöckmann, Appl. Phys. Lett. **77**, 633 (2000); U. Kuhl and H.-J. Stöckmann, Physica E (Amsterdam) **9**, 384 (2001).
- [29] D.R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).
- [30] M. Barth, U. Kuhl, and H.-J. Stöckmann, Phys. Rev. Lett. **82**, 2026 (1999).
- [31] V.V. Dodonov and M.A. Andreatta, Phys. Lett. A **275**, 173 (2000); Laser Phys. **12**, 57 (2002).
- [32] Y. Ashkenazy, L.P. Horwitz, J. Levitan, M. Lewkowicz, and Y. Rothschild, Phys. Rev. Lett. **75**, 1070 (1995).
- [33] R.W. Robinett and S. Hoppellmann, Phys. Rev. A **65**, 062103 (2002).
- [34] S. Alberverio, Z. Brzezniak, and L. Dabrowski, J. Phys. A **27**, 4933 (1994).
- [35] T. Cheon, T. Fülöp, and I. Tsutsui, Ann. Phys. (San Diego) **294**, 1 (2001).
- [36] M.G.E. da Luz, A.S. Lupu-Sax, and E.J. Heller, Phys. Rev. E **56**, 2496 (1997).