Macroscopic entanglement jumps in model spin systems

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In this paper, we consider some frustrated spin models for which the ground states are known exactly. The concurrence, a measure of the amount of entanglement can be calculated exactly for entangled spin pairs. Quantum phase transitions involving macroscopic magnetization changes at critical values of the magnetic field are accompanied by macroscopic jumps in the $T=0$ entanglement. A specific example is given in which magnetization plateaus give rise to a plateau structure in the amount of entanglement associated with nearest-neighbor bonds. We further show that macroscopic entanglement changes can occur in quantum phase transitions brought about by the tuning of exchange interaction strengths.

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Entanglement is a characteristic feature of quantum-mechanical systems which has no classical analog [1]. The state of a pair (or more than a pair) of quantum systems is entangled if the corresponding wave function does not factorize, i.e., is not a product of the wave functions of the individual systems. A well-known example of an entangled state is the singlet state of two spin-$\frac{1}{2}$ particles, $(1/\sqrt{2})(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle)$, which cannot be written as a product of the spin states of individual spins. Measurement on one component of an entangled pair fixes the state of the other implying nonlocal correlations. Interest in quantum entanglement is extensive because of its fundamental role in quantum communication and information processing such as quantum teleportation [2], superdense coding [3], quantum cryptographic key distribution [4], etc. Experimental implementation of some of the protocols has so far been achieved in simple physical settings. Solid-state devices, specially, spin systems have been proposed as possible candidates for large scale realizations [5,6]. In particular, the Heisenberg spin-spin exchange interaction gives rise to entangled states in spin systems and has been shown to provide the basis for universal quantum computation [7,8]. Examples of other interacting many-body systems in which entanglement properties have been studied include the harmonic chain [9], the one-dimensional Kondo Necklace model [10] and the BCS condensate [11].

Entanglement in a state like its energy is quantifiable and has been computed both at $T=0$ and at finite T (thermal entanglement) for a variety of spin models in both zero and finite external magnetic fields. The models include the Heisenberg *XX*, *XY*, *XXX*, *XXZ*, and transverse Ising models in one dimension (1D) [12–18]. The computational studies show that the amount of entanglement between two spins in a multispin state can be modified by changing the temperature and/or the external magnetic field. Some recent studies have explored the relations between entanglement and quantum phase transitions in the ferromagnetic (FM) *XY* model in a transverse magnetic field and in a special case of the model, namely, the transverse Ising model [16,17]. A quantum phase transition (QPT) can take place at $T=0$ by changing some parameter of the system or an external variable like the magnetic field [19]. The ground-state wave function undergoes qualitative changes in a QPT and model studies indicate that entanglement develops special features in the vicinity of a quantum critical point.

The studies carried out so far have been confined to 1D systems with only nearest-neighbor (NN) exchange interactions. There are several quasi-1D and -2D antiferromagnetic (AFM) spin models which describe frustrated spin systems [20]. Frustration arises if conflicting interactions are present in the system, i.e., when all the interactions between spins cannot be simultaneously satisfied. A good example of a frustrated spin system is the AFM Ising model on the triangular lattice. An elementary plaquette of the lattice is a triangle. The Ising spin variables have two possible values, ± 1, corresponding to the up- and down-spin orientations. An antiparallel spin pair has the lowest interaction energy $-J$. A parallel spin pair has the energy $+J$. In an elementary triangular plaquette, there are three interacting spin pairs. Due to the topology of the plaquette (odd number of NN bonds), all the three spin pairs cannot be simultaneously antiparallel in the ground state. There is bound to be one parallel spin pair giving rise to frustration in the system. Consider another example in which the spins in a linear chain interact via NN as well as next-nearest-neighbor (NNN) AFM interactions. In a triplet of successive spins, the two NN spin pairs and one NNN spin pair cannot be simultaneously antiparallel and the linear spin chain is frustrated. Frustration in a system may occur due to the topology of the lattice (triangular, kagomé, etc.) or due to the inclusion of further-neighbor interactions leading in most cases to spin-disordered ground states. The models exhibit QPTs as the exchange interaction parameters are tuned to particular ratios or at critical values of the external magnetic field. Some of the models exhibit the phenomenon of magnetization plateaus in which plateaus appear in the magnetization/site m versus the external magnetic field h curve at quantized values of m (m and h are chosen to be dimensionless) [21]. The condition for the appearance of plateaus is

$$S_U-m_U=\text{integer}, \tag{1}$$

where S_U and m_U are the total spin and magnetization in unit period of the ground state. The plateaus indicate that the spin excitation spectrum is gapped so that the magnetization remains constant. In this paper, we consider some spin models for which the ground states are known exactly in certain parameter regimes. An exact measure of the entanglement between two spins can be obtained in these states in a straightforward manner because of the simple structure of the ground states. We report two major results. In the presence of an external magnetic field, macroscopic magnetization jumps at critical values of the magnetic fields (first-order QPTs) are accompanied by macroscopic changes in the amount of pairwise entanglement. Furthermore, some examples are given in which the entanglement structure is modified due to QPTs (again first order) brought about by the tuning of exchange interaction strengths.

A measure of entanglement between the spins A and B is given by a quantity called concurrence [12,13]. To calculate this, a knowledge of the reduced density matrix ρ_{AB} is required. This is obtained from the ground-state wave function by tracing out all the spin degrees of freedom except those of the spins A and B. Let ρ_{AB} be defined as a matrix in the standard basis $\{|\uparrow\uparrow\rangle,|\uparrow\downarrow\rangle,|\downarrow\uparrow\rangle,|\downarrow\downarrow\rangle\}$. One can define the spin-reversed density matrix as $\tilde{\rho}=(\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y)$, where σ_y is the Pauli matrix. The concurrence C is given by $C=\max\{\lambda_1-\lambda_2-\lambda_3-\lambda_4,0\}$, where $\lambda_i's$ are the square roots of the eigenvalues of the matrix $\rho\tilde{\rho}$ in descending order. The spins A and B are entangled if C is nonzero, $C=0$ implies an unentangled state, and $C=1$ corresponds to maximum entanglement. The models, we consider in this paper belong to the Majumdar-Ghosh (MG) [22] and the Affleck-Kennedy-Lieb-Tasaki (AKLT) [23] families of models. The MG model is the simplest frustrated model in 1D. The spins have magnitude $\frac{1}{2}$ and the Hamiltonian is given by

$$H_{MG}=J\sum_{i=1}^{N}\vec{S}_i\cdot\vec{S}_{i+1}+\frac{J}{2}\sum_{i=1}^{N}\vec{S}_i\cdot\vec{S}_{i+2}\,. \tag{2}$$

N is the total number of spins and the strength of the NNN exchange interaction is half that of the NN interaction. The boundary condition is periodic. The exact ground state is doubly degenerate with the wave functions

$$\phi_1=[12][34]\,.\,.\,.\,.\,.\,.\,.[N-1N],$$
$$\phi_2=[23][45]\,.\,.\,.\,.\,.\,.\,.\,.[N1], \tag{3}$$

where $[lm]$ denotes a singlet [valence bond (VB)] of two spins located at the lattice sites l and m. Thus in the ground-state NN spin pairs in singlet spin configurations are maximally entangled with concurrence $C=1$. The value of C for all the other NN or further-neighbor pairs is zero. This is consistent with the special property of entanglement that in a set of three spins A, B, and C, if A and B are maximally entangled, the entanglement between A and C is zero [12]. Translational invariance in the ground states can be restored by taking linear combinations of the ground states, $[(\phi_1\pm\phi_2)/\sqrt{2}]$. In this case, all NN pairs are entangled with $C=0.5$. The value of C in the case of the $S=\frac{1}{2}$ Heisenberg AFM spin chain is $C=0.386$ [12]. Frustration appears to increase the NN entanglement in a spin system.

We now give examples of some spin models in the presence of an external magnetic field for which macroscopic magnetization jumps are accompanied by macroscopic jumps in entanglement. The first model is a frustrated two-leg $S=\frac{1}{2}$ ladder model [24] (Fig. 1) the Hamiltonian of which is given by

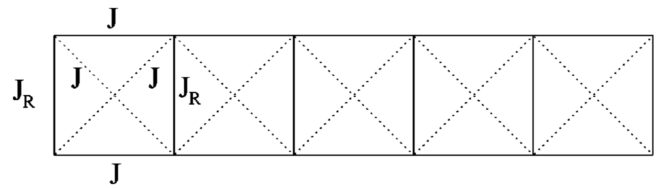


FIG. 1. A two-leg frustrated spin ladder. The spin-spin exchange interaction strength along the rung is J_R. The interleg NN and the diagonal exchange interactions are of equal strength J.

$$H_{ladder}=\sum_{\langle ij\rangle}J_{ij}\vec{S}_i\cdot\vec{S}_j-h\sum_{i=1}^{N}S_i^z\,. \tag{4}$$

The exchange interaction strength $J_{ij}=J_R$ along the rungs. The intrachain NN and the diagonal exchange interactions are of equal strength J. When $h=0$ and $(J_R/J)>\lambda$ ($\lambda\simeq1.401$), the exact ground state consists of singlets (VBs) along the rungs. The spin pairs along the rungs are thus maximally entangled and all other spin pairs are unentangled. We define a quantity

$$f=\frac{2}{N}\sum_{i=1}^{N/2}C(i), \tag{5}$$

which is the average concurrence per rung of the ladder. The summation is over the rung index with $N/2$ being the total number of rungs. Since all the other spin pairs are unentangled, the average concurrence per NN bond, $f_{NN}=\frac{1}{3}$ (the NN bonds include $N/2$ rung bonds and N intrachain NN bonds). The magnetization properties of the frustrated ladder model are simple [25]. Figure 2(a) shows the magnetization per rung M as a function of the magnetic field h. For $0<h<h_{c_1}=J_R$, $M=0$. For $h_{c_1}<h<h_{c_2}=J_R+2J$, the rungs are alternately in singlet and $S^z=1$ triplet spin configurations in the ground state. The value of M is now $\frac{1}{2}$. For $h>h_{c_2}$,

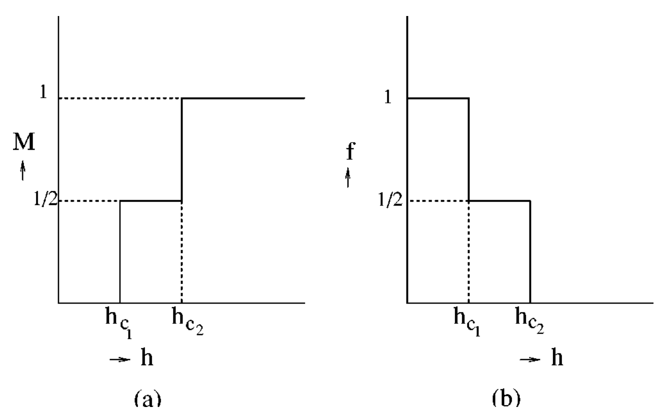


FIG. 2. The frustrated spin-ladder model (Fig. 1) exhibits plateaus in (a) magnetization/rung M versus an external magnetic field h and (b) average concurrence per rung f vs h.

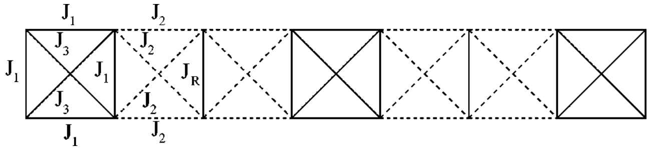


FIG. 3. Two-chain ladder model consisting of four-spin plaquettes coupled to two-spin rungs. The exchange interaction strengths are as shown in the figure.

saturation magnetization is obtained with $M=1$. One can verify that the quantization condition (1) is obeyed at each plateau. Figure 2(b) shows the plot of the average concurrence per rung *f* versus the magnetic field *h*. A similar plot is obtained for f_{NN} with the difference that $f_{NN}=\frac{1}{3}$ for $0<h<h_{c_1}$ and $f_{NN}=\frac{1}{6}$ for $h_{c_1}<h<h_{c_2}$. For $h_{c_1}<h<h_{c_2}$, half the total number of rungs are in $S^z=1$ triplet spin configurations in which the two spins of a rung point up. The parallel spin pairs are unentangled with $C=0$. The other rungs are in singlet spin configurations so that $f=\frac{1}{2}$. In the fully polarized state $f=0$. Figure 2(b) provides an example of a changing magnetic field giving rise to macroscopic changes in entanglement. The ground state of the frustrated spin-ladder model for $h=0$ has a simple structure and can be expressed as a product over rung singlet states. The reduced density matrix ρ_{AB}, needed to calculate the entanglement between the spins *A* and *B*, can be calculated exactly and analytically in a straightforward manner. Once the matrix elements of ρ_{AB} are known, the concurrence *C* can be calculated. Because of the uncomplicated structure of the ground state, *C* can alternatively be calculated using simple arguments. Each spin in the ladder belongs to a rung singlet or VB, so that it is maximally entangled with its partner spin. Thus, the entanglement between a spin *j* and any other spin, belonging to other rung singlets, is zero. The concurrence *C* of a maximally entangled pair is 1 so that *f*, the average concurrence per rung, is 1 for $0<h<h_{c_1}$. Similar arguments hold true for $h_{c_1}<h<h_{c_2}$, when the ladder has a different ground state, again with a simple structure.

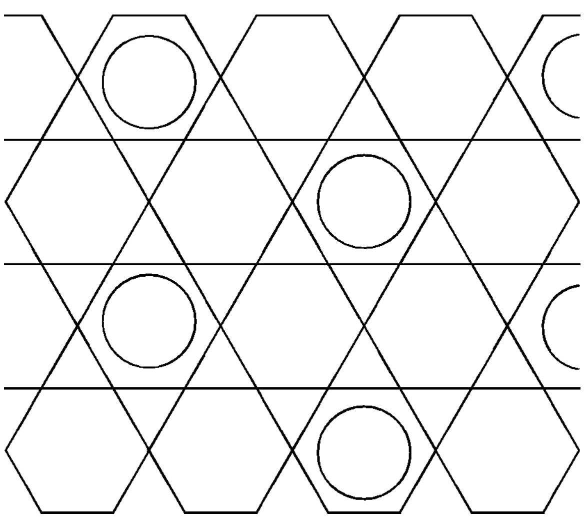
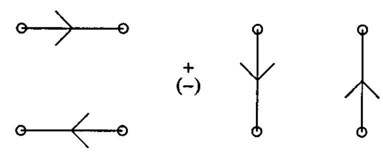


FIG. 4. Kagomé lattice with *N* spins. The circles mark the *N*/9 hexagons in which independent magnons can be localized.

Figure 3 shows another spin-ladder model with modulated exchange interactions [26]. The model consists of four-spin plaquettes coupled to two-spin rungs (solid lines) through NN and diagonal exchange interactions (dotted lines) of strength J_2. Within each plaquette, the NN and diagonal exchange interactions are of strength J_1 and J_3, respectively. The rung exchange interaction strength is J_R. In a wide parameter regime, the exact ground-state has a simple product form: the plaquettes and the rungs are individually in their ground-state spin configurations. The ground state of a plaquette is a resonating valence bond (RVB) state with wave function given by ψ_{RVB1} (ψ_{RVB2}) for $J_3<J_1$ ($J_3>J_1$). The ground state of a rung is a singlet. The RVB states are given by

$\psi_{RVB1(RVB2)} =$



(6)

The solid lines represent singlets and the arrow signs are drawn according to the phase convention that in a VB between the sites *i* and *j*, if the arrow points away from the site *i*, then the spin configuration is $(1/\sqrt{2})[|\uparrow(i)\downarrow(j)\rangle-|\downarrow(i)\uparrow(j)\rangle]$. It is easy to check that the NN spins in $|\psi_{RVB1}\rangle$ are entangled with concurrence $C=0.5$. The NNN spins along the diagonals are unentangled. On the other hand, the NN spins in $|\psi_{RVB2}\rangle$ are unentangled and the NNN spin pairs are entangled with $C=1$. At $J_1=J_3$, there is a QPT from the ground state in which the plaquette spin configurations are described by $|\psi_{RVB1}\rangle$ to the ground state in which the same are described by $|\psi_{RVB2}\rangle$. In both the phases, the rungs are in singlet spin configurations. The QPT is accompanied by macroscopic changes in the amount of NN and NNN entanglements. The average concurrence per NN bond, f_{NN}, in the full ladder is $\frac{1}{3}$ for $J_3<J_1$ and $\frac{1}{9}$ for $J_3>J_1$. The average concurrence per NNN bond, f_D, is 0 for $J_3<J_1$ and $\frac{1}{3}$ for $J_3>J_1$. In a finite magnetic field $h\neq 0$, the exact ground state maintains its product form in an extended parameter regime. This gives rise to magnetization plateaus in the magnetization/site *m* versus *h* curves. Again, the jumps in the magnetization are accompanied by jumps in the amount of entanglement. To give one specific example, consider the case $J_3<J_1$. The average concurrence/NNN bond in the full ladder is zero for $0<h<h_{c_1}=J_1$ and $\frac{1}{6}$ for $h_{c_1}<h<h_{c_2}=2J_2$. At h_{c_1}, *m* jumps from zero to the value $\frac{1}{6}$. Each plaquette is in the ground-state spin configuration $\frac{1}{2}(|\uparrow\uparrow\uparrow\downarrow\rangle-|\uparrow\uparrow\downarrow\uparrow\rangle-|\uparrow\downarrow\uparrow\uparrow\rangle+|\downarrow\uparrow\uparrow\uparrow\rangle)$ in the field range $h_{c_1}<h<h_{c_2}$. Schulenberg *et al.* [27] have constructed exact eigenstates which consist of independent, localized one-magnon states in a class of frustrated spin lattices and have shown that these are the ground states in high magnetic fields. The eigenstates are obtained provided certain conditions are satisfied. In a kagomé lattice, the magnon states are localized in *N*/9 hexagons, where *N* is the total number of spins. The hexagons in which the magnon excitations occur are isolated from each other (Fig. 4). Above the saturation

magnetic field, all the spins in the lattice are in a FM spin configuration $|0\rangle$, and each spin pair is unentangled. The state $|0\rangle$ is the vacuum for magnon excitations. At a critical field below the saturation magnetic field, the exact eigenstate consisting of $N/9$ localized one-magnon excitations becomes the ground state. The localized magnon excitation has the wave function

$$|1\rangle=\frac{1}{\sqrt{6}}\sum_{l=1}^{6}(-1)^l S_l^-|0\rangle. \tag{7}$$

This gives rise to a macroscopic change in the magnetization curve. In each of the $N/9$ hexagons, a spin is equally entangled with all the other five spins and the magnitude of the concurrence is $C=\frac{1}{3}$. This signifies a macroscopic change in the amount of entanglement. The magnetization jump occurring at a critical value of the magnetic field signifies a first-order QPT. We have shown through specific examples that such QPTs give rise to macroscopic jumps in the amount of entanglement associated with NN and/or further-neighbor spin pairs.

QPTs can also be brought about by tunning exchange interaction strengths. We have already given an example of this in the case of the ladder model shown in Fig. 3. For $h=0$, a QPT occurs at $J_3/J_1=1$. In the frustrated two-leg ladder model shown in Fig. 1, a QPT takes place at the critical value $([J_R/J])_c\simeq 1.401$. Below the critical point, the ground state is that of an effective $S=1$ chain with the $S=1$ spins forming out of the pairs of $S=\frac{1}{2}$ rung spins. The spin ladder is now in the Haldane phase of a $S=1$ chain. Above the critical point, the ladder is in the rung-singlet (RS) phase in which the rung spins are in singlet spin configurations. Kolezhuk and Mikeska [28] have proposed a generalized frustrated ladder model in which the first-order QPT between the RS and the Haldane phases can be studied in an exact manner. This model includes biquadratic interactions besides NN and NNN (diagonal) ones. The Hamiltonian is given by

$$H=\sum_i h_{i,i+1},$$

$$\begin{aligned} h_{i,i+1}=&\frac{y_1}{2}(\vec{S}_{1,i}\cdot\vec{S}_{2,i}+\vec{S}_{1,i+1}\cdot\vec{S}_{2,i+1})+(\vec{S}_{1,i}\cdot\vec{S}_{1,i+1}\\ &+\vec{S}_{2,i}\cdot\vec{S}_{2,i+1})+y_2(\vec{S}_{1,i}\cdot\vec{S}_{2,i+1}+\vec{S}_{2,i}\cdot\vec{S}_{1,i+1})\\ &+x_1(\vec{S}_{1,i}\cdot\vec{S}_{1,i+1})(\vec{S}_{2,i}\cdot\vec{S}_{2,i+1})+x_2(\vec{S}_{1,i}\cdot\vec{S}_{2,i+1})\\ &\times(\vec{S}_{2,i}\cdot\vec{S}_{1,i+1}), \end{aligned} \tag{8}$$

$$\text{where}\quad x_1=\tfrac{4}{5}(3-2y_2),\quad x_2=\tfrac{4}{5}(3y_2-2). \tag{9}$$

The indices 1 and 2 correspond to the lower and upper legs of the ladder, respectively, and i is the rung index. The exact phase boundary between the RS and the Haldane phases is given by $y_1=\frac{4}{5}(1+y_2)$. The ground state in each phase is known exactly. In the Haldane phase the ground state is the AKLT state [23]. The ground-state energy/rung in the RS and the Haldane phases are $E_{RS}=-\frac{3}{4}y_1+\frac{3}{20}(1+y_2)$ and $E_{AKLT}=\frac{1}{4}y_1-\frac{13}{20}(1+y_2)$, respectively.

As mentioned before, in the RS phase the spin pairs along the rungs are perfectly entangled with concurrence $C=1$. All the other spin pairs are unentangled. The amount of entanglement in the AKLT phase can be computed in the following manner. For $[H,S^z]=0$, where S^z is the z component of the total spin, the reduced density matrix of a spin pair located at the sites i and j has the form [12]

$$\rho_{ij}=\begin{pmatrix} u_+ & 0 & 0 & 0\\ 0 & w_1 & z & 0\\ 0 & z & w_2 & 0\\ 0 & 0 & 0 & u_- \end{pmatrix}. \tag{10}$$

The concurrence quantifying the entanglement is given by

$$C=2\max[0,|z|-\sqrt{u_+u_-}]. \tag{11}$$

Wang and Zanardi [29] have shown that the matrix element of ρ_{ij} can be expressed in terms of the various correlation functions $G_{\alpha\beta}=\langle\sigma_{i\alpha}\sigma_{j\beta}\rangle=\mathrm{Tr}(\sigma_{i\alpha}\sigma_{j\beta}\rho)$, $(\alpha=x,y,z)$, where ρ is the density operator. The magnetization/site $m=(1/N)\mathrm{Tr}(\Sigma_{i=1}^N\sigma_i^z\rho)$. In particular, the following relations hold true:

$$u_\pm=\tfrac{1}{4}(1\pm 2m+G_{zz}),$$

$$z=\tfrac{1}{4}(G_{xx}+G_{yy}). \tag{12}$$

The correlation functions in the AKLT phase can be calculated in an exact manner using the transfer-matrix method in the matrix product (MP) formalism [30]. The AKLT ground state can be written in the MP form as

$$\psi_{AKLT}=\mathrm{Tr}\prod_{i=1}^{N/2} g_i \tag{13}$$

$$\text{with}\quad g_i=\begin{bmatrix} |t_0\rangle_i & -\sqrt{2}|t_+\rangle_i\\ \sqrt{2}|t_-\rangle_i & -|t_0\rangle_i \end{bmatrix}. \tag{14}$$

The product in Eq. (13) is over the $(N/2)$ rungs of the ladder. The two spin $\frac{1}{2}$'s of each rung are in a triplet spin configuration in the AKLT phase giving rise to an effective spin 1. In Eq. (14), the states $|t_\mu\rangle$ with $\mu=+1$, 0, and -1 represent a spin state with $S^z=+1$, 0 and -1, respectively. Calculation of the correlation functions $G_{\alpha\alpha}$ $(\alpha=x,y,z)$ of the spin $\frac{1}{2}$ pairs in the AKLT state can be carried out following standard procedure [30]. We quote the final results. The various correlation functions are

$$\begin{aligned}\langle\sigma_{1,n\alpha}\sigma_{1,n+1\alpha}\rangle&=\langle\sigma_{2,n\alpha}\sigma_{2,n+1\alpha}\rangle=\langle\sigma_{1,n\alpha}\sigma_{2,n+1\alpha}\rangle\\ &=\langle\sigma_{1,n+1\alpha}\sigma_{2,n\alpha}\rangle=-\tfrac{4}{9}\quad(\alpha=x,y,z),\end{aligned} \tag{15}$$

$$\langle\sigma_{1,n\alpha}\sigma_{2,n\alpha}\rangle=\tfrac{1}{3}\quad(\alpha=x,y,z). \tag{16}$$

The correlation functions, in which the distance l separating the rungs on which the spins are located is $\geqslant 1$ (in Eq. (15) $l=1$), involve the factor $4(-1)^l 3^{-l-1}$. With the knowledge of the correlation functions, the concurrence can be determined using the relations (11) and (12) (m in the AKLT phase is zero). One finds that in the AKLT phase, the spin pairs along the rungs are unentangled. In the RS phase, the same spins are maximally entangled. The intraleg NN and the NNN (diagonal) spin pairs are entangled in the AKLT phase with concurrence $C=\frac{1}{6}$ in each case. In the RS phase, these spin pairs are unentangled. Again, the first-order QPT from the RS phase to the AKLT phase is accompanied by macroscopic changes in the entanglement structure.

The generalized ladder model studied by Kolezhuk and Mikeska has a rich phase diagram [28] describing both first-order and second-order QPTs. At the second-order phase boundary, the gap in the excitation spectrum goes to zero. There are five possible phases: RS, AKLT, FM, $D1$, and $D2$. The phases D1 and D2 have spontaneously broken symmetry but the exact ground states are not known in these phases. The phase boundaries separating the FM-D1, AKLT-D1, AKLT-D2, RD-D1, and RD-D2 are known exactly and have been determined in the MP formalism. The corresponding QPTs are second-order transitions. The two first-order phase boundaries separating the RD-AKLT and AKLT-FM phase are also known exactly. Preliminary calculations [31] near the second-order phase boundaries suggest that the pairwise entanglement does not extend beyond the NNN distance. The same is true in the case of first-order QPTs. Osterloh *et al.* [16] have considered the range of entanglement in the vicinity of the quantum critical point of the transverse Ising model in 1D. They have found that even at the critical point, where the spin-spin correlations are long ranged, the concurrence is zero for spin pairs separated by more than NNN distance. There is, however, one crucial difference between this model and the AKLT-type models described by MP states. The MP states are finitely correlated, i.e., the spin-spin correlation function $\langle S_i^z S_{i+l}^z \rangle = A_S e^{-l/\xi}$ decays exponentially with the correlation length ξ equal to a few lattice spacings. As the transition point $\tau=\tau_c$ (τ is some model parameter) is approached, the correlation length ξ either does not exhibit any singularity or diverges in a power-law fashion as $(\tau-\tau_c)^2$. In the latter case, however, the prefactor $A_s \propto (\tau-\tau_c)$ becomes zero at the transition point [28]. Thus, long range spin-spin correlations cannot develop in the system.

In summary, we have shown through specific examples that first-order QPTs can bring about macroscopic changes in the amount of pairwise entanglement in spin systems. We have given the specific example of a spin model in which magnetization plateaus give rise to a plateau structure in the average concurrence per rung as well as NN bond. There are several low-dimensional AFM compounds in which magnetization plateaus have been observed experimentally [32]. The appearance of magnetization plateaus has been explained in terms of metal-insulator transitions of magnetic excitations driven by a magnetic field [33]. In the insulating (plateau) phase, the magnetic excitations give rise to crystalline order and in the metallic (nonplateau) phase they are itinerant. It will be of interest to explore the possibility of a plateau structure in the amount of $T=0$ as well as finite T entanglements in such systems. Our study shows that an external magnetic field can be employed to give rise to large changes in the amount of entanglement and provides the basics for the construction of an entanglement ‘‘amplifier’’ or ‘‘switch.’’ There is a large number of spin models which exhibit QPTs of significant interest. Some of these models describe 2D systems. Examples include the Shastry-Sutherland model [34], the $S=\frac{1}{2}$ AFM model on the (1/5)-depleted square lattice [35], a lattice of weakly-coupled two-leg ladders [36], etc. Some of these models are reviewed in Refs. [20,37]. All these models exhibit second-order QPTs from a spin-disordered gapped phase to a gapless phase with long range spin-spin correlations. Osterloh *et al.* [16] and Osborne and Nielsen [17] have found evidence of entanglement showing scaling behavior in the vicinity of the quantum critical point. Similar studies should be carried out for the spin models mentioned to augment our knowledge of the relationship between QPTs and entanglement.

Note added in Proof. Our attention has been drawn to an earlier work by Y. Shi [38] on entanglement properties of quantum spin liquids in two dimensions.

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