Teleportation via thermally entangled states of a two-qubit Heisenberg *XX* chain

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Recently, entanglement teleportation has been investigated by Lee and Kim [Phys. Rev. Lett. **84**, 4236 (2000)]. In this paper we study entanglement teleportation via two separate thermally entangled states of a two-qubit Heisenberg *XX* chain. We established the condition under which the parameters of the model have to satisfy in order to teleport entanglement. The necessary minimum amount of thermal entanglement for some fixed strength of exchange coupling is a function of the magnetic field and the temperature.

The linearity of quantum mechanics allows building of superposition states of a composite system S_{AB} that cannot be written as products of states of each subsystem (S_A and S_B). Such states are called entangled. States that are not entangled are referred to as separable states. An entangled composite system gives rise to nonlocal correlation between its subsystems, which does not exist classically. This nonlocal property enables the uses of local quantum operations and classical communication to teleport an unknown quantum state via a shared pair of entangled particles [1]. In the standard teleportation protocol, these local quantum operations consist of Bell measurements and Pauli rotations. It is shown in Ref. [2] that standard teleportation with an arbitrary entangled mixed state resource χ_{AB} is equivalent to a generalized depolarizng channel $\Lambda(\chi_{AB})$ with probabilites given by the maximally entangled components of the resource. Quantum teleportation of single-body quantum state via single quantum channel shared between two parties has been studied by a number of authors (see references in [3]). In a recent paper [3], Lee and Kim considered teleportation of an entangled two-body pure spin-$\frac{1}{2}$ state via two independent, equally entangled, noisy quantum channels represented by Werner states [4]. In their two-qubit teleportation protocol, the joint measurement is decomposable into two independent Bell measurements and the unitary operation into two local one-qubit Pauli rotations. They found that quantum entanglement of the spin-$\frac{1}{2}$ state is lost during the teleportation even when the channel has nonzero quantum entanglement, and in order to teleport quantum entanglement the quantum channel should possess a critical value of minimum entanglement.

Recently, the presence of entanglement in condensed-matter systems at finite temperatures has been investigated by a number of authors (see, e.g., Ref. [5], and references therein). The state of a typical condensed-matter system at thermal equilibrium (temperature T) is $\chi = e^{-\beta H}/Z$, where H is the Hamiltonian, $Z = \mathrm{tr}\, e^{-\beta H}$ is the partition function, and $\beta = 1/kT$, where k is the Boltzmann's constant. The entanglement associated with the thermal state χ is referred to as the thermal entanglement [6]. In Ref. [7], quantum teleportation using the thermally entangled state of a two-qubit Heisenberg *XX* chain as a quantum channel is considered. In this paper, we investigate Lee and Kim's two-qubit teleportation protocol using two independent thermally entangled states of two-qubit Heisenberg *XX* chain. In contrast to [3], we con-

sider as input a two-qubit Werner state [4]. We find that quantum entanglement of the Werner state is lost during the teleportation even when the channel has nonzero thermal entanglement, in accordance with Ref. [3]. In order to teleport quantum entanglement, the parameters of the model have to satisfy Eq. (8).

The Hamiltonian H for a two-qubit Heisenberg *XX* chain in an external magnetic field B_m along the z axis is

$$H = \frac{1}{2} J(\sigma_A^1 \otimes \sigma_B^1 + \sigma_A^2 \otimes \sigma_B^2) + \frac{1}{2} B_m(\sigma_A^3 \otimes \sigma_B^0 + \sigma_A^0 \otimes \sigma_B^3),$$

(1)

where σ_α^0 is the identity matrix and $\sigma_\alpha^i (i=1,2,3)$ are the Pauli matrices at site $\alpha = A, B$. J is real coupling constant for the spin interaction. The chain is said to be antiferromagnetic for $J > 0$ and ferromagnetic for $J < 0$. The eigenvalues and eigenvectors of H are given by $H|00\rangle = B_m|00\rangle$, $H|\Psi^\pm\rangle = \pm J|\Psi^\pm\rangle$, and $H|11\rangle = -B_m|11\rangle$, where $|\Psi^\pm\rangle = (1/\sqrt{2}) \times (|01\rangle \pm |10\rangle)$. For the system in equilibrium at temperature T, the density operator is

$$\chi_{AB} = \frac{1}{Z} [e^{-\beta B_m}|00\rangle\langle 00| + e^{-\beta J}|\Psi^+\rangle\langle\Psi^+| + e^{\beta J}|\Psi^-\rangle$$

$$\times \langle\Psi^-| + e^{\beta B_m}|11\rangle\langle 11|],$$

(2)

where the partition function $Z = 2\cosh\beta B_m + 2\cosh\beta J$, the Boltzmann's constant $k \equiv 1$ from hereon, and $\beta = 1/T$. To quantify the amount of entanglement associated with χ_{AB}, we consider the measure of entanglement [3], $E(\chi_{AB}) = \max\{-2\Sigma_m \lambda_m^-, 0\}$, where λ_m^- is a negative eigenvalue of $\chi_{AB}^{T_B}$, the partial transposition of χ_{AB}. The density operator χ_{AB} is entangled if and only if $\chi_{AB}^{T_B}$ has any negative eigenvalues [8,9]. After some straightforward algebra, the amount of thermal entanglement is

$$E(\chi_{AB}) = \max\left\{ \frac{\sqrt{\cosh^2\beta B_m + \cosh^2\beta J - 2} - \cosh\beta B_m}{\cosh\beta B_m + \cosh\beta J}, 0 \right\}.$$

(3)

The amount of thermal entanglement is invariant under the substitutions $B_m \to -B_m$ and $J \to -J$. The latter indicates that the entanglement is the same for the antiferromagnetic and ferromagnetic cases. We thus restrict our considerations

to $B_m>0$ and $J>0$. Notice that the critical temperature $T_{critical}\approx 1.13459J$, beyond which the thermal entanglement is zero, is independent of the magnetic field B_m. This is in agreement with Ref. [10], where the concurrence [11,12] has been adopted as a measure of entanglement.

Now we look at Lee and Kim's two-qubit teleportation protocol, using two copies of the above two-qubit thermal state, $\chi_{A_1B_1}\otimes\chi_{A_2B_2}$, as resource. We consider as input two qubits in the Werner state [4], $\rho_W=\frac{1}{4}(\sigma^0\otimes\sigma^0-[(2\Phi+1)/3]\Sigma_{i=1}^3\sigma^i\otimes\sigma^i)$ $(-1\leqslant\Phi\leqslant1)$. The amount of entanglement associated with ρ_W is given by $E(\rho_W)=\max\{\Phi,0\}$. When $\Phi=1$, $\rho_W=|\Psi^-\rangle\langle\Psi^-|$ is a maximally entangled pure state. When $0<\Phi<1$, ρ_W is an entangled mixed state. Lastly, when $-1\leqslant\Phi\leqslant0$, ρ_W is a separable mixed state. Since our concern is entanglement teleportation,

we focus on $0<\Phi\leqslant1$. The output state is then given by [2]

$$\tilde\rho_W=\Lambda(\chi_{A_1B_1}\otimes\chi_{A_2B_2})\rho_W$$
$$=\sum_{j=0}^3\sum_{k=0}^3 \mathrm{tr}[(E^j\otimes E^k)(\chi_{A_1B_1}\otimes\chi_{A_2B_2})]$$
$$\times(\sigma_{A_1}^j\otimes\sigma_{A_2}^k)\rho_W(\sigma_{A_1}^j\otimes\sigma_{A_2}^k),\quad(4)$$

where $E^0=|\Psi^-\rangle\langle\Psi^-|$, $E^1=|\Phi^-\rangle\langle\Phi^-|$, $E^2=|\Phi^+\rangle\langle\Phi^+|$, $E^3=|\Psi^+\rangle\langle\Psi^+|$, and $|\Phi^\pm\rangle=(1/\sqrt2)(|00\rangle+|11\rangle)$.

To characterize the quality of the teleported state $\tilde\rho_W$, it is often quite useful to look at the fidelity between ρ_W and $\tilde\rho_W$, defined by [13]

$$F(\rho_W,\tilde\rho_W)=\{\mathrm{tr}[\sqrt{(\rho_W)^{1/2}\tilde\rho_W(\rho_W)^{1/2}}]\}^2$$
$$=\frac{1}{36(\cosh\beta B_m+\cosh\beta J)^2}\{2\sqrt{(1-\Phi)[(1-\Phi)\cosh^2\beta B_m+2(2+\Phi)\cosh\beta B_m\cosh\beta J+(1-\Phi)\cosh^2\beta J]}$$
$$+\sqrt{(1-\Phi)[(2+\Phi)\cosh^2\beta B_m+2(1-\Phi)\cosh\beta B_m\cosh\beta J+(1-\Phi)\cosh^2\beta J+(1+2\Phi)]}$$
$$+\sqrt3\sqrt{(1+\Phi)[(2+\Phi)\cosh^2\beta B_m+2(1-\Phi)\cosh\beta B_m\cosh\beta J+3(1+\Phi)\cosh^2\beta J-(1+2\Phi)]}\}^2.\quad(5)$$

The concept of fidelity has been an useful indicator of the teleportation performance of a quantum channel when the input state is a pure state (see, e.g., Ref. [7] and references therein). However, it fails in our context where we consider the mixed Werner state. In particular, we observe that, in the infinite temperature limit, $\beta\to0$, when there is zero thermal entanglement in the channels, we have

$$F(\rho_W,\tilde\rho_W)\to\frac14[(2-\Phi)+\sqrt{3(1-\Phi^2)}],\quad(6)$$

which increases as $\Phi\to0$.

Returning to the main issue of this paper, we calculate the measure of entanglement for the teleported state $\tilde\rho_W$ to be

$$E(\tilde\rho_W)=\max\left\{\frac{3\Phi\cosh^2\beta J-2(2+\Phi)\cosh\beta B_m\cosh\beta J-(1-\Phi)\cosh^2\beta B_m-(1+2\Phi)}{3(\cosh\beta B_m+\cosh\beta J)^2},0\right\}$$
$$=\max\left\{\left[\frac{(\cosh\beta B_m-\cosh\beta J)^2}{3(\cosh\beta B_m+\cosh\beta J)^2}+\frac{2(\cosh^2\beta J-1)}{3(\cosh\beta J+\cosh\beta B_m)^2}\right]\Phi-\frac{4\cosh\beta B_m\cosh\beta J+\cosh^2\beta B_m+1}{3(\cosh\beta B_m+\cosh\beta J)^2},0\right\}.\quad(7)$$

In the zero temperature limit, $\beta\to\infty$, we have $E(\tilde\rho_W)\to\Phi$ when $B_m<J$, but $E(\tilde\rho_W)\to0$ when $B_m\geqslant J$. This is not difficult to understand when $B_m<J$, since at $T=0$ the quantum channels are in the maximally entangled ground state: $\chi_{A_1B_1}=\chi_{A_2B_2}=|\Psi^-\rangle\langle\Psi^-|$. For $B_m>J$, the channels have zero entanglement and hence are not able to teleport entanglement. However, at $B_m=J$, although the channels have nonzero entanglement, it does not allow them to perform better than the "classical channels" [7]. Since entanglement

is a quantum property, we therefore do not expect the channels to teleport entanglement when $B_m=J$.

For nonzero temperatures, it is clear from Eq. (7) that entanglement is lost in the teleportation, since the coefficient of Φ is less than or equal to one, and the term independent of Φ is nonpositive. The interesting thing to note is that the thermally entangled channels are still able to teleport some quantum entanglement even at nonzero temperatures. In order to transmit nonzero entanglement, we require

$$\cosh \beta J > \frac{2+\Phi}{3\Phi} \cosh \beta B_m$$
$$+ \sqrt{\frac{-2\Phi^2 + 7\Phi + 4}{9\Phi^2} \cosh^2 \beta B_m + \frac{1+2\Phi}{3\Phi}},$$

$$(8)$$

and hence $B_m < J$. The "critical" temperature $T_{critical}^{(m,\Phi)}$, beyond which no quantum entanglement is teleported, is therefore dependent on the magnetic field B_m and on Φ. The right-hand side of Eq. (8) is strictly greater than $\sqrt{2}$. This shows that in order to teleport quantum entanglement, there must be some nonzero critical value of minimum thermal entanglement. This minimum is not a constant but depends on B_m and T for some fixed J, in contrast to Ref. [3]. The right-hand side of Eq. (8) increases as $\Phi \rightarrow 0$. This, together with Eq. (7), indicates that quantum entanglement in a less entangled mixed state is more fragile to teleport. It demands

an even smaller $T_{critical}^{(m,\Phi)}$ for some fixed $B_m < J$, that is, $\exp(\beta J) \gg \exp(\beta B_m)$.

In conclusion, we have established the condition under which the quantum entanglement in a Werner state can be teleported via two separate, thermally entangled two-qubit Heisenberg *XX* chain. We also demonstrated that the fidelity [13] in our case is not a good indicator of the performance of these quantum channels. The two-qubit teleportation together with one-qubit unitary operations are sufficient to implement the universal gates for quantum computation [14]. It is hoped that this paper would contribute to the gathering of phenomenology in this direction.

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[1] C. H. Bennett *et al.*, Phys. Rev. Lett. **70**, 1895 (1993).

[2] G. Bowen and S. Bose, Phys. Rev. Lett. **87**, 267901 (2001).

[3] J. Lee and M. S. Kim, Phys. Rev. Lett. **84**, 4236 (2000).

[4] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).

[5] T. J. Osborne and M. A. Nielsen, Phys. Rev. A (to be published).

[6] M. C. Arnesen *et al.*, Phys. Rev. Lett. **87**, 017901 (2001).

[7] Y. Yeo, e-print quant-ph/0205014.

[8] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[9] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[10] X. Wang, Phys. Rev. A **64**, 012313 (2001).

[11] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).

[12] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[13] R. Jozsa, J. Mod. Opt. **41**, 2315 (1994).

[14] D. Gottesman and I. L. Chuang, Nature (London) **402**, 390 (1999).