

Conditions for compatibility of quantum-state assignments

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Suppose N parties describe the state of a quantum system by N possibly different density operators. These N state assignments represent the beliefs of the parties about the system. We examine conditions for determining whether the N state assignments are compatible. We distinguish two kinds of procedures for assessing compatibility, the first based on the compatibility of the *prior* beliefs on which the N state assignments are based and the second based on the compatibility of *predictive* measurement probabilities they define. The first procedure leads to a compatibility criterion proposed by Brun, Finkelstein, and Mermin [BFM, Phys. Rev. A **65**, 032315 (2002)]. The second procedure leads to a hierarchy of measurement-based compatibility criteria which is fundamentally different from the corresponding classical situation. Quantum mechanically none of the measurement-based compatibility criteria is equivalent to the BFM criterion.

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I. INTRODUCTION

There are good reasons [1] to view a quantum state not as representing a true state of affairs, but as a state of knowledge or, more provocatively, a *state of belief*. This view corresponds to the Bayesian approach to probability theory, according to which probabilities are an agent's necessarily subjective *degrees of belief* about a set of alternatives. Different scientists can have different beliefs about the same physical system, resulting in different quantum-state assignments. This can arise for a variety of reasons. For instance, one of the scientists might have no access or only partial access to another's measurement results. In general, N scientists, or *parties*, can assign N different states, pure or mixed, to a given system.

In this paper, we are not concerned with how to justify a particular state assignment. Instead, we start from a given set of N states, representing the beliefs of N parties, and ask for conditions for determining that the N quantum states are *compatible* (or, conversely, that they are contradictory). The conditions we derive can be viewed as criteria for the mutual compatibility of N quantum-state assignments.

There are two distinct procedures for assessing the compatibility of quantum states (or, conversely, for uncovering contradictions among the states). The first procedure examines the firm beliefs on which the parties base their state assignments and asks whether these beliefs are compatible. This procedure leads to the compatibility criterion proposed by Brun, Finkelstein, and Mermin (BFM) [2]. The second procedure examines measurement probabilities predicted by the parties' state assignments and asks whether these probabilities are compatible. We show that this second procedure leads to a hierarchy of measurement-based compatibility criteria. The two-party version of one of our criteria is equivalent

to a compatibility criterion proposed by Peierls [3,4]; thus this criterion provides an N -party generalization of Peierls's criterion, which we call *post-Peierls* (PP) compatibility.

The BFM compatibility criterion is based on the compatibility of the beliefs of the N parties. We show that the BFM criterion is not equivalent to any of the measurement-based compatibility criteria. This means that there is generally no way to use measurements to confirm the compatibility of states that are BFM compatible or to reveal the incompatibility of states that are BFM incompatible.

The compatibility criteria we derive can be specialized to classical probabilities by considering density operators all of which are diagonal in a common eigenbasis—i.e., commuting density operators—and by restricting the allowed measurement operators to be diagonal in the same basis. There are interesting differences between the hierarchy of measurement-based criteria in the classical and quantum cases. Moreover, the classical version of BFM compatibility is equivalent to the classical version of PP compatibility, in contrast to the quantum case.

The paper is organized as follows. In Sec. II, we review briefly the concepts of Dutch-book consistency and strong Dutch-book consistency, which provide the foundations for Bayesian probability theory, and define the notions of contradictory beliefs and contradictory probability assignments. (See the Appendix for an explanation of the term “Dutch book.”) In Sec. III, we derive the BFM compatibility criterion for the quantum states of N parties from the requirement that there be a state assignment that does not contradict the belief of any party. Section IV introduces our hierarchy of measurement-based compatibility criteria and highlights the surprising differences between the classical and quantum cases. We consider how the measurement-based compatibility criteria change when one generalizes from measurements described by one-dimensional orthogonal projectors to positive operator valued measurements (POVMs). Section V focuses on PP compatibility, the only one of the compatibility criteria for which—as far as we can tell—it is not possible to

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formulate a simple universal mathematical condition applicable to all sets of quantum states in all Hilbert-space dimensions; we consider nontrivial examples of PP compatibility for three states of a three-dimensional system. Section VI closes the paper with a summary and discussion.

II. CONSISTENT PROBABILITIES AND CONTRADICTIONARY BELIEFS

The notion of consistent beliefs has its roots in the approach to Bayesian or subjective probability via consistent betting behavior. Bayesian probabilities quantify one's degree of belief in the various alternatives from among a set of possibilities [5]. Bayesian probabilities are given an operational definition in terms of betting behavior. Suppose that A is willing to place a bet at odds of $(1-p)/p$ to 1 on the occurrence of some event. This means that A is willing to pay in a *stake* px , with the promise of receiving a *payoff* x if the event occurs and nothing otherwise. That A considers this to be a *fair* bet—she is willing to accept the bet at these odds no matter what the payoff, positive or negative—*defines* p to be the probability that A assigns to the event.

Suppose now that A makes probability assignments to a set of events. We say that A 's probability assignments are *consistent* (or Dutch-book consistent) if there exists no set of bets which she regards as fair, but in which she loses for every outcome that she believes to be possible. Notice that this is a purely internal consistency criterion; it refers only to A 's subjective beliefs.

The so-called Dutch-book argument [6,7] shows that consistency alone implies that A 's probability assignment must satisfy the usual probability axioms: (i) $p \geq 0$, (ii) $p(E) = 1$ if A believes that event E is certain to occur, (iii) $p(E \vee F) = p(E) + p(F)$ if E and F are mutually exclusive, and (iv) $p(E \wedge F) = p(E|F)p(F)$ (Bayes's rule). In the Appendix we first review the formulation of Bayesian probabilities in terms of betting behavior and then give the Dutch-book derivation of the standard probability rules.

Consistency enforces the probability axioms, but it does not dictate particular probability assignments, leaving these to whatever way A chooses to translate what she knows or believes into probabilities. The only exception is in the case of certainty, where consistency requires that all probabilities be 0 or 1. Indeed, a consequence of the probability axioms—it also follows directly from the Dutch-book argument—is that $p = 0$ for any outcome that A believes to be impossible. We call the belief that an outcome is impossible a *firm belief*.

Surprisingly, consistency does not imply that $p > 0$ for any outcome that A believes to be possible. In other words, if A assigns probability zero to an outcome, one cannot infer from consistency alone that A believes the event to be impossible. To make this inference, we need a slightly stronger version of consistency: we say that A 's probability assignment is *strongly* (Dutch-book) *consistent* [8,9] if there exists no bet that she regards as fair in which there is at least one losing outcome but no winning outcomes among those outcomes she deems possible. If A 's probability assignment is strongly consistent, then an outcome has zero probability if

and only if she believes it to be impossible (see, the Appendix).

Dutch-book consistency has to do with the beliefs and probability assignments of a single party. Our concern in this paper is compatibility among many parties. We say that the beliefs of N parties about a set of alternatives are *compatible* if there is at least one alternative that all parties judge to be possible. Conversely, the beliefs of N parties are *contradictory* (or incompatible) if they are not compatible in the sense just described, i.e., if every alternative is deemed impossible by at least one party. Examining the alternatives to determine which applies is thus guaranteed to contradict at least one party. Contradictory beliefs cannot be reconciled unless at least one party abandons a firm prior belief.

It is easy to imagine situations that give rise to compatible or to contradictory beliefs. Take a die as an example. Suppose A has seen the North face of the die, which shows 4 dots. She, therefore, believes that it is impossible for the top face of the die to show either 3 or 4 dots, but that it is possible for it to show 1, 2, 5, or 6 dots. Suppose B has seen the East face of the die, which shows 1 dot. He believes that the top face can show 2, 3, 4, or 5 dots, but not 1 or 6. These beliefs are compatible since both parties believe that the top face can show 2 or 5 dots.

Now suppose that C asserts that the South face of the die shows 5 dots. He believes that it is impossible for the top face of the die to show 2 or 5 dots, thus contradicting the beliefs of A and B . This situation could arise if C 's assertion was based on a mistaken observation from across the room; the beliefs of the three parties could be reconciled by C 's observing the South face again, finding that it shows 3 dots, and thereby giving up his firm belief that the top face could not show 2 or 5 dots. Another possibility is that the die was tossed after the initial observations by A and B ; the beliefs of the three parties could be reconciled if A and B realized that the die had been tossed after their observations, which would cause them to abandon their prior firm beliefs.

Under the assumption of strong consistency, where the firm belief that an alternative is impossible is equivalent to assigning zero probability, the conditions for contradictory and compatible beliefs can be reexpressed in terms of probabilities. The beliefs (or probability assignments) of N parties are compatible if and only if there is at least one alternative to which all parties assign nonzero probability, i.e., there exists a probability assignment that does not contradict the firm beliefs of any of the parties. This is the classical version of BFM compatibility. The beliefs (or probability assignments) of the N parties are contradictory (or incompatible) if and only if every alternative is assigned zero probability by at least one party.

For ordinary consistency, the existence of one or more alternatives to which all parties assign nonzero probability is sufficient for compatible beliefs, but it is not necessary, because a party can assign zero probability to alternatives the party believes are possible. For ordinary consistency, probabilities do not carry enough information about firm beliefs to allow compatibility to be determined from the parties' probability assignments. Since we are interested in the compatibility of density operators, we need strong consistency so

that probabilities for measurement outcomes generated by the density operators allow one to determine the firm beliefs of the parties. Therefore, we assume strong consistency throughout the remainder of the paper, except where explicitly noted.

There is another stronger kind of compatibility for classical probabilities. Suppose, for example, that in the case of the die, A and B come together, combine their observations, and, thereafter, agree that the top face can show 2 or 5 dots, but not 1, 3, 4, or 6 dots. They generally would not assign the same probabilities to 2 and 5 dots, but they do have the same firm beliefs, a situation we capture by saying that their new beliefs are in concord. Generally, we say that the beliefs (or probability assignments) of N parties are *concordant* if their firm beliefs coincide, i.e., if they assign zero probability to the same alternatives. Concordant probability assignments have the same support. It is not reasonable to demand that parties have concordant probabilities, but they arise naturally when parties with compatible beliefs share those beliefs.

With this background, we turn now to BFM compatibility for quantum state assignments.

III. BFM COMPATIBILITY

The system under consideration is described by a D -dimensional Hilbert space \mathcal{H} . We label the N parties by an index $\alpha=A, B, C, \dots$; their state assignments are denoted by $\hat{\rho}_\alpha$. For a projective measurement in an arbitrary orthonormal basis $\{|k\rangle, k=1, \dots, D\}$, i.e., a measurement described by *orthogonal one-dimensional projectors* [we call such a measurement an (ODOP) for short], the probability assigned by party α to the outcome k is given by $p_k^{(\alpha)} = \text{tr}(\hat{\rho}_\alpha \hat{\Pi}_k) = \langle k | \hat{\rho}_\alpha | k \rangle$, where $\hat{\Pi}_k = |k\rangle\langle k|$. The case of classical probabilities is included automatically as the situation in which all the $\hat{\rho}_\alpha$ are diagonal in the same orthonormal basis $\{|k\rangle\}$, and the only allowed measurement is a measurement in this basis.

Under the assumption of strong consistency, each party assigns zero probability to precisely those outcomes he believes cannot occur; i.e., for each α , $\langle \psi | \hat{\rho}_\alpha | \psi \rangle = 0$ if and only if party α believes that the outcome corresponding to $|\psi\rangle$ is impossible in any measurement containing $|\psi\rangle$. Therefore, each party assigns a density operator $\hat{\rho}_\alpha$ whose null subspace $\mathcal{N}(\hat{\rho}_\alpha)$ consists of all those vectors corresponding to outcomes he believes cannot occur. The support of a density operator is the orthocomplement of the null subspace. All vectors not in the null subspace have a component in the support and thus have nonzero probability, so the party believes that the outcomes corresponding to all such vectors can occur.

A density operator $\hat{\rho}$ contradicts party α 's firm beliefs if $\langle \psi | \hat{\rho} | \psi \rangle > 0$ for some $|\psi\rangle$ that α believes to be impossible. Thus $\hat{\rho}$ does not contradict α 's beliefs if and only if $\mathcal{N}(\hat{\rho}_\alpha) \subseteq \mathcal{N}(\hat{\rho})$. What we want to know is the circumstances under which there is a density operator that does not contradict the firm beliefs of any of the parties, i.e., a density operator $\hat{\rho}$ such that $\mathcal{N}(\hat{\rho}_\alpha) \subseteq \mathcal{N}(\hat{\rho})$ for all α .

Since $\mathcal{N}(\hat{\rho})$ is a subspace, it follows that $\mathcal{M} \equiv \text{span}(\mathcal{N}(\hat{\rho}_A), \mathcal{N}(\hat{\rho}_B), \dots) \subseteq \mathcal{N}(\hat{\rho}) \subset \mathcal{H}$. Such a $\hat{\rho}$ exists if and only if \mathcal{M} is not the entirety of \mathcal{H} , which is equivalent to saying that the orthocomplement of \mathcal{M} contains at least one nonzero vector. Since the orthocomplement of \mathcal{M} is the intersection of the supports of the $\hat{\rho}_\alpha$, we have the result that there exists a $\hat{\rho}$ that does not contradict any party's prior belief if and only if the intersection of the supports of all the $\hat{\rho}_\alpha$ contains at least one nonzero vector. This is the criterion for *BFM compatibility* of state assignments. In the classical case the BFM criterion reduces to the condition that at least one of the common eigenvectors has nonzero eigenvalue for all parties.

What we have shown is that BFM compatibility is equivalent to the existence of a density operator that does not contradict the firm beliefs of any party. Just as in the classical case, the assumption of strong consistency, as opposed to ordinary Dutch-book consistency, is essential for this conclusion. The reason is that *any* set of consistent (but not necessarily strongly consistent) state assignments can arise from a set of noncontradictory beliefs. Let $\hat{\rho}_A, \hat{\rho}_B, \dots$ be N arbitrary states. These are consistent state assignments for N parties all of whom believe that *any* outcome is possible, since consistency alone allows a party to believe that a vector in his null subspace corresponds to a possible outcome. Obviously, there is a posterior state $\hat{\rho}$ that does not contradict the firm beliefs of any party; indeed, any posterior state $\hat{\rho}$ will do. Merely consistent state assignments do not reveal enough about the parties' prior beliefs to rule out the existence of a noncontradictory posterior state assignment.

Suppose parties with BFM compatible state assignments share their beliefs, each adopting the firm beliefs of all the others. BFM compatibility guarantees that there are density operators that are consistent with the firm beliefs of all the parties. The parties will generally not end up assigning the same density operator, but they will assign density operators that incorporate the same firm beliefs and thus have the same support. We say such density operators are *concordant* in the same sense as for probability assignments.

Our derivation of the BFM criterion is different from the one given by Brun, Finkelstein, and Mermin [2]. They show that their criterion follows if one assumes that each of the state assignments $\hat{\rho}_A, \hat{\rho}_B, \dots$ "incorporates some subset of a valid body of currently relevant information about the system, all of which could, in principle, be known by a particularly well-informed Zeno." Their formulation suggests that each of the N -state assignments should be consistent with some real state of affairs captured in Zeno's state $\hat{\rho}$. This impression is reinforced shortly, thereafter, in their paper, where one of the explicit assumptions leading to the BFM criterion is that "if anybody describes a system with a density matrix $\hat{\rho}$, then nobody can find (the system) to be in a pure state in the null space of $\hat{\rho}$." In contrast, our derivation is couched wholly in terms of the beliefs of the parties and does not appeal to a real state of affairs. It is, therefore, preferable in a Bayesian approach to quantum mechanics.

IV. MEASUREMENT-BASED COMPATIBILITY

A. Compatibility conditions

We turn our attention now to compatibility conditions based on the compatibility of measurement probabilities. We focus first on ODOP measurements, i.e., those described by complete sets of one-dimensional orthogonal projectors $\{\hat{\Pi}_k, k=1, \dots, D\}$, the probability party α assigns to outcome k being $p_k^{(\alpha)} = \text{tr}(\hat{\rho}_\alpha \hat{\Pi}_k) = \langle k | \hat{\rho}_\alpha | k \rangle$. Most importantly, we assume that all parties agree on this description of the measurement. In Sec. IV C below we generalize the compatibility conditions to the context of measurements described by POVMs.

Our hierarchy of measurement-based compatibility conditions can be stated very simply as whether the parties' measurement probabilities are compatible or concordant and whether this holds for all measurements or for at least one measurement. In mathematical language the compatibility conditions are the following:

$$\forall \{\hat{\Pi}_{jj}\} \forall k ((\forall \alpha: p_k^{(\alpha)} > 0) \vee (\forall \alpha: p_k^{(\alpha)} = 0)) \quad (\text{ES}), \quad (1)$$

$$\forall \{\hat{\Pi}_{jj}\} \exists k \forall \alpha: p_k^{(\alpha)} > 0 \quad (\text{PP}), \quad (2)$$

$$\exists \{\hat{\Pi}_{jj}\} \forall k ((\forall \alpha: p_k^{(\alpha)} > 0) \vee (\forall \alpha: p_k^{(\alpha)} = 0)) \quad (W), \quad (3)$$

$$\exists \{\hat{\Pi}_{jj}\} \exists k \forall \alpha: p_k^{(\alpha)} > 0 \quad (W'). \quad (4)$$

Condition (1), called ES for “equal support,” says that for all measurements and any outcome $\hat{\Pi}_k$, either all parties assign nonzero probability to $\hat{\Pi}_k$, or they all assign zero probability to $\hat{\Pi}_k$. In other words, the parties' probabilities for all measurements are concordant. It is trivial to see that ES is equivalent to all the density operators $\hat{\rho}_\alpha$ having the same support, i.e., being concordant as defined in the preceding section. As a consequence, ES implies BFM. ES is a very strong compatibility condition, which is violated in many practical situations, but which arises naturally when parties with BFM compatible beliefs combine their beliefs.

Unlike ES and BFM, there are fundamental differences between the classical and quantum versions of the remaining three conditions. Condition (2), called PP for “post-Peierls,” is implied by ES. It says that for all measurements, there is at least one outcome to which all parties assign nonzero probability; i.e., all measurements have compatible probabilities. It is often useful to think in terms of the conditions for violating PP compatibility: PP is violated if there exists a measurement such that at least one party assigns zero probability to every outcome; for this measurement the measurement probabilities are contradictory, the outcome, whatever it is, guaranteed to contradict one or more parties. The two-party version of PP is the original compatibility condition of Peierls [3,4]; it is equivalent to $\hat{\rho}_A \hat{\rho}_B \neq 0$. As far as we can tell, the conditions for multiparty PP compatibility cannot be put in a simple universal mathematical form, unlike the other compatibility criteria. In Sec. V, we consider nontrivial examples of three-party PP compatibility in three dimensions.

Condition (3), called W for “weak,” says that there is at least one measurement such that for any outcome $\hat{\Pi}_k$, either all parties assign nonzero probability to $\hat{\Pi}_k$, or they all assign probability zero to $\hat{\Pi}_k$. In other words, there exists a measurement whose measurement probabilities are concordant.

Finally, condition (4), called W' , is implied by W . It states that there is at least one measurement and at least one outcome for that measurement to which all parties assign nonzero probability; i.e., there exists a measurement whose measurement probabilities are compatible.

Summarizing the implications we have identified up till now, we have

$$\text{ES} \Rightarrow \text{BFM}, \quad \text{ES} \Rightarrow \text{PP}, \quad W \Rightarrow W'. \quad (5)$$

The latter two relations involve only the logical structure of the compatibility conditions; all the relations hold for both the classical and quantum cases.

In the case of classical probabilities, there is only one allowed measurement—a measurement in the basis that diagonalizes all the $\hat{\rho}_\alpha$ —so it is clear that ES is equivalent to W and PP is equivalent to W' . It is equally clear that PP and W' are equivalent to BFM. Summarizing the implications for the classical case, we have

$$W \Leftrightarrow \text{ES} \Rightarrow \text{BFM} \Leftrightarrow \text{PP} \Leftrightarrow W' \quad (\text{Classical}). \quad (6)$$

This chain reflects the two kinds of compatibility for classical probabilities: W and ES correspond to the parties having concordant probabilities, whereas BFM, PP, and W' correspond to the parties having compatible probabilities.

B. Relations among quantum compatibility conditions

In the quantum case, the relations (6) change in an interesting way to

$$\text{ES} \Rightarrow \text{BFM} \Rightarrow \text{PP} \Rightarrow W' \Leftrightarrow W \quad (\text{Quantum}). \quad (7)$$

Here, unlike the classical case, BFM is stronger than PP, and PP is stronger than W' , but the most striking difference is that W , the strongest condition classically, is the weakest condition quantum mechanically. As a matter of fact, W is satisfied by *any* set of state assignments, as we show below. The different structure of quantum implications in Eq. (7) is due to the far greater freedom quantum mechanics allows for measurements.

We now prove the relations (7). The first implication, $\text{ES} \Rightarrow \text{BFM}$, is trivial: equal support implies that the supports have at least one state in common. It is also clear that the reverse implication does not hold.

To see that BFM implies PP, consider an arbitrary measurement $\{\hat{\Pi}_j = |j\rangle\langle j|\}$. BFM compatibility is equivalent to saying that $\mathcal{M} \equiv \text{span}(\mathcal{N}(\hat{\rho}_A), \mathcal{N}(\hat{\rho}_B), \dots)$ is not the entire Hilbert-space \mathcal{H} . Since the vectors $\{|j\rangle\}$ are an orthonormal basis, at least one outcome $|k\rangle$ lies outside \mathcal{M} and thus has a nonzero projection onto the orthocomplement of \mathcal{M} , which is the intersection of the supports of the $\hat{\rho}_\alpha$. For this out-

come we have $\langle k|\hat{\rho}_\alpha|k\rangle = \text{tr}(\hat{\rho}_\alpha\hat{\Pi}_k) > 0$ for all α . To see that PP does not imply BFM, consider two nonorthogonal pure states. There is no one-shot measurement that can distinguish the two states reliably, so the two states satisfy PP, but since the intersection of the supports of the two states contains only the zero vector, the two states violate BFM.

A simple example shows that W' does not imply PP: two orthogonal pure states violate PP, but they satisfy W' , as can be seen by considering a measurement in a basis that includes a vector that lies in the two-dimensional subspace spanned by the two states, but is not equal to either of them.

To show that W' is equivalent to W and is implied by PP, we prove the stronger result that *any* set of N states $\hat{\rho}_A, \hat{\rho}_B, \dots$ satisfies W , which shows that W follows from any of the other conditions. We construct a measurement $\{\hat{\Pi}_j\}$ each of whose projectors has nonzero overlap with the supports of all the $\hat{\rho}_\alpha$. Let $|\phi_\alpha\rangle$ be an eigenvector of $\hat{\rho}_\alpha$ with nonzero eigenvalue λ_α . We need to find an orthonormal basis $\{|k\rangle\}$ such that $0 < |\langle k|\phi_\alpha\rangle|^2$ for all k and α , since this implies $0 < \lambda_\alpha |\langle k|\phi_\alpha\rangle|^2 \leq \langle k|\hat{\rho}_\alpha|k\rangle = \text{tr}(\hat{\Pi}_k\hat{\rho}_\alpha)$ for all k and α , where $\hat{\Pi}_k = |k\rangle\langle k|$. Letting S be the set of all state vectors that are orthogonal to at least one $|\phi_\alpha\rangle$, we see that what we need to do is to construct an orthonormal basis none of whose basis vectors is in S .

To do the construction, we begin by defining the distance between two state vectors:

$$d(|\psi\rangle, |\chi\rangle) \equiv \cos^{-1} |\langle \psi|\chi\rangle|, \quad (8)$$

which allows us to define the distance between an arbitrary state vector $|\psi\rangle$ and the set S by

$$d(|\psi\rangle, S) \equiv \min_{|\chi\rangle \in S} d(|\psi\rangle, |\chi\rangle). \quad (9)$$

A state vector $|\psi\rangle$ is in S if and only if $d(|\psi\rangle, S) = 0$. Our construction relies on the fact that, arbitrarily close to any vector $|\chi\rangle \in S$, there exists a vector $|\psi\rangle$ that is a finite distance away from S ; i.e., any ϵ -ball around $|\chi\rangle \in S$ contains a vector $|\psi\rangle$ such that $d(|\psi\rangle, S) > 0$.

Now choose an orthonormal basis $\{|k\rangle\}$ such that $d(|1\rangle, S) > 0$. Assume that $d(|k\rangle, S) = 0$ for at least one of the basis vectors—otherwise we have the desired basis—and let $|m\rangle$ be the first such basis vector in the list, i.e., $d(|m\rangle, S) = 0$ and $d(|k\rangle, S) > 0$ for $k < m$. We now show that the basis can be rotated in such a way that $d(|k\rangle, S) > 0$ for $k \leq m$. Define

$$\epsilon = \frac{1}{2} \min_{k < m} d(|k\rangle, S). \quad (10)$$

Let $|m'\rangle$ be a state such that $d(|m'\rangle, |m\rangle) = d < \epsilon$ and $d(|m'\rangle, S) = \delta > 0$. Then there exists a unitary operator \hat{U} such that $|m'\rangle = \hat{U}|m\rangle$ and $d(|k\rangle, \hat{U}|k\rangle) < \epsilon$ for all $k < m$. To see this, let $|m'\rangle = |m\rangle e^{i\mu} \cos d + |m_\perp\rangle \sin d$, with $\langle m_\perp|m\rangle = 0$. We can use

$$\hat{U} \equiv |m'\rangle\langle m| + |m'_\perp\rangle\langle m_\perp| + 1 - |m\rangle\langle m| - |m_\perp\rangle\langle m_\perp|,$$

where $|m'_\perp\rangle \equiv -|m\rangle e^{i\mu} \sin d + |m_\perp\rangle \cos d$, for which it follows that $\cos d \langle k|\hat{U}|k\rangle = |\langle k|\hat{U}|k\rangle| = |1 - |\langle k|m_\perp\rangle|^2 (1 - \cos d)| \geq \cos d$. Now define $|k'\rangle = \hat{U}|k\rangle$ for all k . Then $d(|k'\rangle, S) > \epsilon$ for $k < m$, and, therefore, $d(|k'\rangle, S) > 0$ for $k \leq m$. By repeating this procedure, one arrives at a basis with the property that each basis state is a finite distance from S , as required.

C. Generalized measurements

In this section we investigate how the compatibility criteria change if generalized measurements, described by POVMs, are included in the allowed measurements. A POVM is a collection of positive operators $\{\hat{E}_b\}$ satisfying $\sum_b \hat{E}_b = \hat{1}$; the probability assigned by party α to outcome b is $p_b^{(\alpha)} = \text{tr}(\hat{\rho}_\alpha \hat{E}_b)$.

It is clear that BFM is not affected by generalizing to POVMs, since it is phrased in terms of firm beliefs, not in terms of measurements. For the measurement-based criteria, it is logically possible that states that are W or W' incompatible relative to ODOPs can be made compatible by including additional measurements; indeed, the uninformative measurement with a single outcome does make all states W and W' compatible. Since all states are already W and W' compatible under ODOPs, however, allowing POVMs makes no difference to W and W' . It is also possible that states that are ES or PP compatible relative to ODOPs can be made incompatible by including additional measurements. It is clear, however, that density operators with the same support satisfy ES compatibility with POVMs included among the measurements; thus allowing POVMs makes no difference to ES compatibility.

The only compatibility criterion that is affected by generalizing to POVMs is PP. We distinguish the two kinds of post-Peirler compatibility by using PP-ODOP to denote compatibility relative to ODOPs and PP-POVM to denote compatibility relative to POVMs. Clearly PP-POVM implies PP-ODOP. To investigate PP-POVM, it is easiest to focus on the conditions for violating PP-POVM: PP-POVM is violated if there exists a measurement, described by a POVM $\{\hat{E}_b\}$, such that at least one party assigns zero probability to every outcome b .

Given any POVM, we can write the POVM elements \hat{E}_b in terms of their eigendecompositions, thus obtaining a finer-grained POVM consisting of rank-one operators. If a POVM $\{\hat{E}_b\}$ shows that a set of density operators violates PP-POVM compatibility, then the underlying rank-one POVM reveals the same incompatibility. Thus, in investigating PP-POVM, we can restrict attention to rank-one POVMs. Moreover, since a rank-one POVM can be extended to an ODOP in a higher-dimensional Hilbert space (the Neumark extension) [10], the question of the PP-POVM compatibility of a set of states is equivalent to the question of whether the states are PP-ODOP compatible when they are embedded in a Hilbert space of arbitrary dimension.

The condition for two-party PP-ODOP compatibility, $\hat{\rho}_A \hat{\rho}_B \neq 0$, is independent of the dimension in which the two states are embedded, so it is also equivalent to PP-POVM compatibility [4]. The case of two nonorthogonal pure states, which are PP-POVM compatible, but not BFM compatible, establishes that PP-POVM is not equivalent to BFM.

States of a two-state system (qubit) illustrate the difference between PP relative to ODOPs and POVMs. Since density operators of full rank give nonzero probabilities for all measurement outcomes, they can be added to or removed from a set of density operators without affecting the PP compatibility of the other density operators in the set. For a qubit this means that we only need to consider the situation in which the parties assign pure states $\hat{\rho}_\alpha = \frac{1}{2}(\hat{I} + \mathbf{n}_\alpha \cdot \hat{\boldsymbol{\sigma}})$, $\alpha = 1, \dots, N$, where \mathbf{n}_α is the (unit) Bloch vector for party α 's pure state. Two pure states are PP compatible, relative to either ODOPs or POVMs, if and only if they are not orthogonal. Three or more distinct pure states in two dimensions are PP-ODOP compatible if and only if no two of the states are orthogonal. PP-POVM compatibility is more complicated. The states are incompatible if and only if there is a POVM such that each outcome has zero probability for at least one of the states. Such a POVM must consist of rank-one positive operators, each of which is orthogonal to one of the pure states $\hat{\rho}_\alpha$, i.e., $\hat{E}_\alpha = q_\alpha(\hat{I} - \mathbf{n}_\alpha \cdot \hat{\boldsymbol{\sigma}})$, where $0 \leq q_\alpha \leq 1/2$. Requiring the POVM elements to sum to \hat{I} implies that the q_α 's are a normalized probability distribution and that the Bloch vectors average to zero:

$$0 = \sum_{\alpha} q_{\alpha} \mathbf{n}_{\alpha}. \quad (11)$$

The result is that a set of pure states in two dimensions is PP-POVM compatible if and only if the convex set generated by the Bloch vectors does not contain zero or, equivalently, the convex set generated by the states $\hat{\rho}_\alpha$ does not contain the maximally mixed state $\hat{I}/2$. These results for a two-state system establish that PP-ODOP is not equivalent to PP-POVM.

We are left with the following chain of implications:

$$\text{ES} \Rightarrow \text{BFM} \Rightarrow \text{PP-POVM} \Rightarrow \text{PP-ODOP} \Rightarrow W' \Leftrightarrow W \text{ (Quantum)}. \quad (12)$$

It is interesting to compare these relations to what happens to the classical relations (6) when one generalizes to the coarse-grained measurements that are POVMs diagonal in the common eigenbasis of the density operators. ES still corresponds to concordant probabilities, BFM and PP still correspond to compatible probabilities, but the uninformative measurement makes all probabilities compatible under W and W' . Thus we have the following classical implications when we allow coarse-grained measurements:

$$\text{ES} \Rightarrow \text{BFM} \Leftrightarrow \text{PP} \Rightarrow W' \Leftrightarrow W \text{ (Classical)}. \quad (13)$$

When we generalize to coarse-grained classical measurements, W migrates from the strongest to the weakest compatibility condition.

V. THREE-PARTY POST-PEIERLS COMPATIBILITY IN THREE DIMENSIONS

PP seems to be the only one of our compatibility criteria for which there is no simple, general mathematical condition for deciding whether a given set of density operators is compatible. For two parties, however, it is easy to determine whether the two density operators are PP compatible: Mermin [4] showed that PP is satisfied if and only if the states are not orthogonal, i.e., $\text{tr}(\hat{\rho}_A \hat{\rho}_B) \neq 0$ or, equivalently, $\hat{\rho}_A \hat{\rho}_B \neq 0$. This condition follows from the fact that two density operators are not PP compatible if and only if there is a measurement that can distinguish them reliably, and there is such a measurement if and only if the two density operators are orthogonal. This two-party PP compatibility condition is the same for ODOPs and POVMs, as can be shown directly [4] or from the fact that as an ODOP condition, it is independent of the dimension of the Hilbert space in which the two states are embedded. Notice that if any two parties assign PP incompatible states, then the states of all parties are PP-ODOP and PP-POVM incompatible.

In three or more Hilbert-space dimensions, the general condition for N states to be PP compatible, relative to ODOPs or to POVMs, is highly nontrivial. We report results in this section for the first interesting situation, three parties assigning states in three Hilbert-space dimensions. As noted above, full-rank density operators are irrelevant to questions of PP compatibility, so we can assume that all the density operators are either rank one or rank two. There are four cases to consider, depending on how many of the states are pure. We consider the three cases where one or more of the states is mixed in Sec. V A and deal with the case of three pure states in Sec. V B.

A. Mixed and pure states

Throughout this section, we investigate the conditions for constructing POVMs (or ODOPs) that show that the density operators are PP incompatible. Such a POVM (or ODOP) must be made up of the operators that give zero probability for each density operator in turn. In doing this construction, we adopt the following conventions. If $\hat{\rho}_\alpha$ is a rank-two density operator, we let \mathcal{S}_α denote the two-dimensional subspace that is the support of $\hat{\rho}_\alpha$, with \hat{S}_α being the projector onto \mathcal{S}_α , and we let $\hat{\Pi}_\alpha = |e_\alpha\rangle\langle e_\alpha|$ be the one-dimensional projection operator that projects orthogonal to \mathcal{S}_α . A POVM element that has zero probability given $\hat{\rho}_\alpha$ must have the form $\hat{E}_\alpha = r_\alpha \hat{\Pi}_\alpha$, $0 \leq r_\alpha \leq 1$; if the POVM is to be an ODOP, we need $r_\alpha = 1$ or $r_\alpha = 0$. If $\hat{\rho}_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$ is a rank-one density operator, we let

$$\begin{aligned} \hat{E}_\alpha &= r_{\alpha,1} \hat{\Pi}_{\alpha,1} + r_{\alpha,2} \hat{\Pi}_{\alpha,2} = r_{\alpha,1} |e_{\alpha,1}\rangle\langle e_{\alpha,1}| + r_{\alpha,2} |e_{\alpha,2}\rangle\langle e_{\alpha,2}|, \\ 0 &\leq r_{\alpha,1}, r_{\alpha,2} \leq 1, \end{aligned} \quad (14)$$

denote the general POVM element orthogonal to $\hat{\rho}_\alpha$; its eigenvectors $|e_{\alpha,1}\rangle$ and $|e_{\alpha,2}\rangle$ are orthogonal to $|\psi_\alpha\rangle$. A POVM element that has zero probability given $\hat{\rho}_\alpha$ must have

the form (14). The operators $\hat{E}_{\alpha,j} = r_{\alpha,j} \hat{\Pi}_{\alpha,j}$, $j=1,2$, are rank-one POVM elements that give a fine graining of \hat{E}_α . If the POVM is to be an ODOP, we must have $r_{\alpha,j}=1$ or $r_{\alpha,j}=0$.

The first (and easiest) case is that of three rank-two density operators $\hat{\rho}_\alpha$. A POVM each of whose outcomes contradicts at least one of the parties must consist of positive multiples of the projectors $\hat{\Pi}_\alpha$. The only way three such POVM elements can sum to $\hat{1}$ is if they are orthogonal projectors (an ODOP). Thus we have that three rank-two density operators in three dimensions are PP-POVM incompatible if and only if the vectors orthogonal to their supports are mutually orthogonal. Since the measurement that reveals incompatibility is an ODOP, there is no difference between PP-POVM and PP-ODOP for this case.

A straightforward way to generalize from two parties is to say that $N>2$ density operators are *pairwise PP compatible* if $\hat{\rho}_\alpha \hat{\rho}_\beta \neq 0$ for all pairs α, β . Though PP-POVM or PP-ODOP clearly implies pairwise PP, the converse does not hold, as is plain from the three states

$$\begin{aligned}\hat{\rho}_1 &= \frac{1}{2}(|e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|), \\ \hat{\rho}_2 &= \frac{1}{2}(|e_1\rangle\langle e_1| + |e_3\rangle\langle e_3|), \\ \hat{\rho}_3 &= \frac{1}{2}(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|),\end{aligned}\quad (15)$$

which are pairwise PP compatible even though they are PP incompatible when considered together.

The next case is that of one pure state $\hat{\rho}_1 = |\psi_1\rangle\langle\psi_1|$ and two rank-two density operators, $\hat{\rho}_2$ and $\hat{\rho}_3$. If $\hat{\rho}_2$ and $\hat{\rho}_3$ have the same support, we are back in the situation of pairwise PP compatibility, and the three states are PP incompatible if and only if $|\psi_1\rangle$ is orthogonal to the common support of $\hat{\rho}_2$ and $\hat{\rho}_3$. Thus assume that $\hat{\rho}_2$ and $\hat{\rho}_3$ do not have the same support. Then $|e_2\rangle$ and $|e_3\rangle$ span a two-dimensional subspace \mathcal{R} ; denote the projector onto \mathcal{R} by \hat{R} . Let $|\chi\rangle$ be the unique (up to a phase) pure state that lies in the intersection of \mathcal{S}_2 and \mathcal{S}_3 ; $|\chi\rangle$ is orthogonal to \mathcal{R} . In addition, let $|\phi_\alpha\rangle$, $\alpha=2,3$, be the unique (up to a phase) pure state in \mathcal{S}_α that is orthogonal to $|\chi\rangle$; $|\phi_2\rangle$ and $|\phi_3\rangle$ lie in \mathcal{R} .

With this setup, we can turn to formulating the conditions for the existence of a POVM that shows the density operators are incompatible. Such a POVM must consist of the POVM elements \hat{E}_α defined above. Since only \hat{E}_1 has support outside \mathcal{R} , the only way the POVM elements can sum to $\hat{1}$ is to have $|\chi\rangle$ be an eigenvector of \hat{E}_1 with eigenvalue 1, i.e., $|e_{1,1}\rangle = |\chi\rangle$ and $r_{1,1}=1$. Consequently, $|\psi_1\rangle$ and $|e_{1,2}\rangle$ are orthogonal vectors in \mathcal{R} , and $\hat{E}_{1,2} = r_{1,2}|e_{1,2}\rangle\langle e_{1,2}|$ is a rank-one POVM element that acts only in \mathcal{R} . The only remaining requirement is that $\hat{R} = \hat{E}_{1,2} + \hat{E}_2 + \hat{E}_3$. This means that we are back to the question of constructing a POVM in two

dimensions, here the two-dimensional subspace \mathcal{R} . What we have shown is that the PP incompatibility of the original three states is equivalent to the PP incompatibility of the three pure states $|\psi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$, all of which lie in the two-dimensional subspace \mathcal{R} .

Our conclusion is the following. The three states $|\psi_1\rangle$, ρ_2 , and ρ_3 , with $\mathcal{S}_2 \neq \mathcal{S}_3$, are PP-POVM incompatible if and only if $\langle\psi_1|\chi\rangle=0$ and the convex set generated by $|\psi_1\rangle\langle\psi_1|$, $|\phi_2\rangle\langle\phi_2|$, and $|\phi_3\rangle\langle\phi_3|$ contains $\hat{R}/2$. Similarly, the three states are PP-ODOP incompatible if and only if $\langle\psi_1|\chi\rangle=0$ and two of the states $|\psi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$ are orthogonal.

The third case is that of two pure states, $\hat{\rho}_1 = |\psi_1\rangle\langle\psi_1|$ and $\hat{\rho}_2 = |\psi_2\rangle\langle\psi_2|$, and one rank-two density operator $\hat{\rho}_3$. We have not been able to determine the conditions for PP-POVM compatibility in this case, so we restrict ourselves to PP-ODOP compatibility. One projector in an ODOP that reveals the incompatibility of these states must be $\hat{\Pi}_3 = |e_3\rangle\langle e_3|$. The other two elements of the ODOP, $\hat{\Pi}_1 = |e_1\rangle\langle e_1|$ and $\hat{\Pi}_2 = |e_2\rangle\langle e_2|$ must operate in \mathcal{S}_3 . The states they contradict must be orthogonal to them. Thus $|\psi_1\rangle$ must lie in the subspace spanned by $|e_3\rangle$ and $|e_2\rangle$, and its projection onto \mathcal{S}_3 , i.e., $\hat{S}_3|\psi_1\rangle$, must be proportional to $|e_2\rangle$. Similarly, $|\psi_2\rangle$ must lie in the subspace spanned by $|e_3\rangle$ and $|e_1\rangle$, and its projection onto \mathcal{S}_3 , i.e., $\hat{S}_3|\psi_2\rangle$, must be proportional to $|e_1\rangle$. Our conclusion is that the three states are PP-ODOP incompatible if and only if $\langle\psi_1|\hat{S}_3|\psi_2\rangle=0$.

B. Three pure states

The final case is that of three pure states in three dimensions. Again we have not been able to prove the conditions for PP-POVM compatibility, although we have numerical evidence that PP-POVM is equivalent PP-ODOP for this case. We restrict our attention in this subsection to ODOPs.

We can assume that the states are all different, since if two are the same, we are back in the two-party case. Moreover, since not being pairwise PP compatible implies not being PP compatible, the interesting case is where the three states are pairwise PP compatible, i.e., no pair is orthogonal. Thus we address the following question: under what circumstances is there an ODOP whose outcome will definitely contradict one of three distinct, nonorthogonal pure states? The criterion we derive is interesting in its own right, independent of compatibility considerations.

Let $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$ be the three distinct, normalized, pairwise PP compatible, pure states, i.e.,

$$\begin{aligned}0 &< |\langle\psi_1|\psi_2\rangle| < 1, \\ 0 &< |\langle\psi_2|\psi_3\rangle| < 1, \\ 0 &< |\langle\psi_3|\psi_1\rangle| < 1.\end{aligned}\quad (16)$$

The vectors $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$ violate PP-ODOP if and only if there exist angles θ_k , $0 < \theta_k < \pi/2$, $k=1,2,3$, such that

$$a \equiv |\langle\psi_1|\psi_2\rangle|^2 = (\sin\theta_1 \cos\theta_2)^2, \quad (17)$$

$$b \equiv |\langle \psi_2 | \psi_3 \rangle|^2 = (\sin \theta_2 \cos \theta_3)^2,$$

$$c \equiv |\langle \psi_3 | \psi_1 \rangle|^2 = (\sin \theta_3 \cos \theta_1)^2.$$

These conditions can be seen as follows. If the vectors violate PP-ODOP, there exists an orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$ for the space spanned by $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$ such that

$$|\psi_1\rangle = e^{i\chi_1}(\cos \theta_1|2\rangle + e^{i\phi_1}\sin \theta_1|3\rangle), \quad (18)$$

$$|\psi_2\rangle = e^{i\chi_2}(\cos \theta_2|3\rangle + e^{i\phi_2}\sin \theta_2|1\rangle),$$

$$|\psi_3\rangle = e^{i\chi_3}(\cos \theta_3|1\rangle + e^{i\phi_3}\sin \theta_3|2\rangle),$$

where $0 \leq \chi_k < 2\pi$, $0 \leq \phi_k < 2\pi$, $0 < \theta_k < \pi/2$ ($k=1,2,3$). Taking the inner products, we see that

$$\langle \psi_1 | \psi_2 \rangle = e^{i(\chi_2 - \chi_1)} e^{i\phi_1} \sin \theta_1 \cos \theta_2, \quad (19)$$

$$\langle \psi_2 | \psi_3 \rangle = e^{i(\chi_3 - \chi_2)} e^{i\phi_2} \sin \theta_2 \cos \theta_3,$$

$$\langle \psi_3 | \psi_1 \rangle = e^{i(\chi_1 - \chi_3)} e^{i\phi_3} \sin \theta_3 \cos \theta_1.$$

The conditions (17) follow immediately.

Conversely, if the conditions (17) are satisfied, then it is clear that we can find angles χ_k and ϕ_k such that the inner products $\langle \psi_i | \psi_j \rangle$ are given by Eqs. (19). Since the pairwise inner products specify the vectors up to a unitary transformation, there exists an orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$ such that $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$ have the form (18). A measurement in this basis shows that the vectors are PP-ODOP incompatible.

To find a simpler criterion, define $x_k = \sin^2 \theta_k$ for $k=1,2,3$. The Eqs. (17) are then equivalent to

$$\begin{aligned} x_1(1-x_2) &= a, \\ x_2(1-x_3) &= b, \\ x_3(1-x_1) &= c. \end{aligned} \quad (20)$$

Solving these equations for, e.g., x_2 , we obtain

$$\begin{aligned} x_2 &= \frac{1-a+b-c \pm \sqrt{(1-a+b-c)^2 - 4b(1-a)(1-c)}}{2(1-c)} \\ &= \frac{1-a+b-c \pm \sqrt{(a+b+c-1)^2 - 4abc}}{2(1-c)}, \end{aligned} \quad (21)$$

the expressions for x_3 and x_1 follow from cyclic permutations of a , b , and c . Equations (17) are equivalent to the existence of solutions that satisfy $0 < x_k < 1$ for $k=1,2,3$. The first equality in Eq. (21) shows that, if there are two real solutions, both have the same sign. The existence of a solution $0 < x_2 < 1$ is thus equivalent to the following three conditions:

$$1-a+b-c > 0, \quad (22)$$

$$(a+b+c-1)^2 > 4abc, \quad (23)$$

$$1-a+b-c - \sqrt{(a+b+c-1)^2 - 4abc} < 2(1-c). \quad (24)$$

The third of these conditions is equivalent to

$$-(1-b+a-c) < \sqrt{(a+b+c-1)^2 - 4abc}. \quad (25)$$

This is implied by the condition $1-b+a-c > 0$, which is a cyclic permutation of the inequality (22). The full set of conditions is therefore

$$1-a+b-c > 0,$$

$$1-b+c-a > 0,$$

$$1-c+a-b > 0,$$

$$(a+b+c-1)^2 > 4abc. \quad (26)$$

An equivalent form is

$$|a-b| < 1-c, \quad (27)$$

$$a+b < 1+c,$$

$$\frac{(a+b-1)^2}{c} + \frac{(a-b)^2}{(1-c)} > 1.$$

For fixed c , with $0 < c < 1$, it is straightforward to show that the ellipse in the a - b plane defined by the last inequality has the following properties: it is centered at the point $a=b=1/2$, and its principal axes, of length $\sqrt{c}/2$ and $\sqrt{(1-c)}/2$, form angles of 45° with the a and b axes. The ellipse has exactly one point of intersection with the a axis at $a=1-c$ and exactly one point of intersection with the b axis at $b=1-c$. The ellipse and the associated region of PP-ODOP incompatibility are shown in Fig. 1.

From this it can be seen that the conditions (27) are equivalent to the following, final set of conditions,

$$a+b+c < 1,$$

$$(a+b+c-1)^2 > 4abc, \quad (28)$$

which are manifestly symmetric in the three squared inner products, a , b , and c . To summarize, the three pure states are PP-ODOP incompatible if and only if their pairwise inner products satisfy the conditions (28).

VI. DISCUSSION

We have shown that the BFM criterion can be viewed as one member in a hierarchy of five compatibility criteria for quantum-state assignments. Parties whose state assignments are BFM compatible can come to agreement about a joint state assignment without any party having to abandon a firm prior belief. By contrast, the four other criteria are based on measurements. They all have distinct roles, and none is equivalent to BFM.

The ES criterion can be applied to a situation where all parties have shared their available information. They may

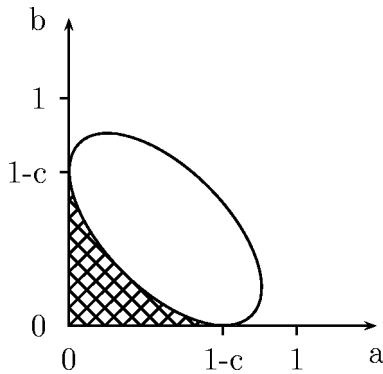


FIG. 1. Three pure states in three dimensions, $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$, are PP-ODOP incompatible if there exists a measurement described by three one-dimensional, orthogonal projectors such that each outcome rules out at least one of the three states. The incompatibility of the states depends only on the three squared inner products. For a fixed value of one squared inner product, $c = |\langle\psi_3|\psi_1\rangle|^2$, the plot shows the region of PP-ODOP incompatibility in terms of the other two squared inner products, $a = |\langle\psi_1|\psi_2\rangle|^2$ and $b = |\langle\psi_2|\psi_3\rangle|^2$. The ellipse is defined in Eq. (27). The region of a and b corresponding to PP-ODOP incompatible states, indicated by cross hatching, lies between the ellipse and the axes.

still assign different states, but they agree on the nullspace. The states assigned by the different parties all have the same support.

The PP criterion rules out the possibility of a measurement that all parties agree will, regardless of outcome, contradict one of their state assignments. In other words, if the states assigned by the parties are not PP compatible, then there exists a measurement that will definitely reveal disagreement among the parties. The PP criterion is in some ways the most interesting: it puts nontrivial constraints on the set of density operators, and it depends on whether all generalized measurements, described by POVMs, are allowed or the permitted measurements are restricted to ODOPs.

The W criterion (which is equivalent to W') shows that there is no strictly necessary constraint on a finite set of density operators to be compatible. Any such set is compatible in the sense that there exists a measurement that allows the parties to come to agreement. In this sense, the BFM criterion is neither sufficient nor necessary.

It turns out that there are important differences between the classical and quantum cases. Whereas the BFM criterion is stronger than the PP criterion quantummechanically, the two are equivalent classically. Curiously, the criterion W , which is the weakest quantummechanically, is the strongest classically, at least for fine-grained (ODOP) measurements.

Finally, we identify *strong* Dutch-book consistency as a necessary assumption in the derivation of the BFM criterion. In particular, we show that N parties who violate strong Dutch-book consistency might come to agreement about a joint state assignment without abandoning any of their firm prior beliefs, even if their prior state assignments are not BFM consistent.

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APPENDIX: BAYESIAN PROBABILITIES AND THE DUTCH-BOOK ARGUMENT

Bayesian probabilities are a measure of one’s degree of belief in or, equivalently, one’s degree of uncertainty about the various alternatives in a set [5]. Bayesian probabilities receive an operational definition in decision theory [11], i.e., the theory of how to decide in the face of uncertainty. The Bayesian approach captures naturally the notion that probabilities represent one’s beliefs about a set of alternatives.

The simplest operational definition of Bayesian probabilities is in terms of betting behavior, which is decision theory in a nutshell. To formulate this definition, let A be a bettor who is willing to place a bet at odds of $(1-p)/p$ to 1 on the occurrence of an event E . These odds mean that A is willing to pay in an amount px —the *stake*—up front, with the promise of receiving an amount x —the *payoff*—if E occurs and nothing otherwise. To say that A considers this a *fair* bet is to say that she is willing to accept the bet at these odds no matter what the payoff; in particular, the payoff can be either positive or negative, meaning that A is willing to accept either side of the bet. This situation is used to *define* probabilities: that A considers it fair to bet on E at odds of $(1-p)/p$ to 1 is the operational definition of A ’s assigning probability p to the occurrence of event E .

The bookmaker who accepts the stakes and makes the payoffs is called the *Dutch bookie*. In a betting situation with A , he has the freedom to set the payoffs for the various outcomes at will. A ’s probability assignment to the outcomes of a betting situation is called *inconsistent* if it forces her to accept bets on which she loses for every outcome that she deems possible. A probability assignment is called *consistent* (or Dutch-book consistent, often called coherent in the literature) if it is not inconsistent in this sense. Remarkably, requiring consistent behavior implies that A must obey the standard probability rules in her probability assignments: (i) $p \geq 0$, (ii) $p(E) = 1$ if A believes that E is certain to occur, (iii) $p(E \vee F) = p(E) + p(F)$ if E and F are mutually exclusive, and (iv) $p(E \wedge F) = p(E|F)p(F)$ (Bayes’s rule). A probability assignment that violates any of these rules is inconsistent in the above sense. This is the so-called *Dutch-book argument* [6,7], which we review below. We stress that it does not invoke expectation values or averages over repeated bets; a bettor who violates the probability rules places bets that, according to her own assessment of what is possible, will result in a sure loss in a single instance of the betting situation.

Consider first the situation, where A assigns probability p_E to E 's occurring and probability $p_{\neg E}$ to E 's not occurring (symbolized by $\neg E$). This means that she will accept a bet on E with payoff x_E (stake $p_E x_E$) and a bet on $\neg E$ with payoff $x_{\neg E}$ (stake $p_{\neg E} x_{\neg E}$). The net amount A receives is

$$G = \begin{cases} x_E(1-p_E) - x_{\neg E}p_{\neg E} & \text{if } E \text{ occurs,} \\ -x_E p_E + x_{\neg E}(1-p_{\neg E}) & \text{if } E \text{ does not occur.} \end{cases} \quad (\text{A1})$$

The Dutch bookie can always choose $x_{\neg E} = 0$, in which case A 's gains become $G = x_E(1-p_E)$ if E occurs and $G = -x_E p_E$ if E does not occur. To avoid all-negative gains requires that $1-p_E$ and p_E have the same sign, which implies $0 \leq p_E \leq 1$, thus giving rule (i).

Suppose now that A believes that E is certain to occur. For the only outcome she deems possible, her gain is $G = x_E(1-p_E) - x_{\neg E}p_{\neg E}$. The Dutch bookie can arrange this gain to have any value—in particular, any negative value by choosing $x_E < 0$ and $x_{\neg E} > 0$ —unless $p_E = 1$ and $p_{\neg E} = 0$. The result is rule (ii): an outcome thought certain to occur must be assigned probability 1 (and an outcome thought certain not to occur must be assigned probability 0).

Now consider two mutually exclusive events, E and F , and suppose A assigns probabilities p_E , p_F , and $p_{E \vee F}$ to the three outcomes E , F , and $E \vee F$ (E or F). This means A will accept the following three bets: a bet on E with payoff x_E (stake $p_E x_E$); a bet on F with payoff x_F (stake $p_F x_F$); and a bet on $E \vee F$ with payoff $x_{E \vee F}$ (stake $p_{E \vee F} x_{E \vee F}$). The net amount A receives is

$$G = \begin{cases} x_E(1-p_E) - x_F p_F + x_{E \vee F}(1-p_{E \vee F}) & \text{if } E, \text{ but not } F \text{ occurs,} \\ -x_E p_E + x_F(1-p_F) + x_{E \vee F}(1-p_{E \vee F}) & \text{if } F, \text{ but not } E \text{ occurs,} \\ -x_E p_E - x_F p_F - x_{E \vee F} p_{E \vee F} & \text{if neither } E \text{ nor } F \text{ occurs.} \end{cases} \quad (\text{A2})$$

We need not consider the possibility that both E and F occur, since they are mutually exclusive. The Dutch bookie can choose payoffs x_E , x_F , and $x_{E \vee F}$ that lead to $G < 0$ for all three outcomes unless

$$0 = \det \begin{pmatrix} 1-p_E & -p_F & 1-p_{E \vee F} \\ -p_E & 1-p_F & 1-p_{E \vee F} \\ -p_E & -p_F & -p_{E \vee F} \end{pmatrix} = p_E + p_F - p_{E \vee F}. \quad (\text{A3})$$

The probability assignment is thus inconsistent unless rule (iii) is satisfied, i.e., $p_{E \vee F} = p_E + p_F$.

Finally, we consider two events, E and F , which are not necessarily exclusive. Suppose that A assigns probability p_F to the occurrence of F , probability $p_{E \wedge F}$ to the occurrence of $E \wedge F$ (E and F), and conditional probability $p_{E|F}$ to the occurrence of E , given that F has occurred. This means A will accept the following three bets: a bet on F with payoff x_F (stake $p_F x_F$); a bet on $E \wedge F$ with payoff $x_{E \wedge F}$ (stake $p_{E \wedge F} x_{E \wedge F}$); and a conditional bet on E given that F has occurred, the payoff being $x_{E|F}$ (stake $p_{E|F} x_{E|F}$). If F does not occur, the conditional bet is called off, with the stake returned. The net amount A receives is

$$G = \begin{cases} -x_F p_F - x_{E \wedge F} p_{E \wedge F} & \text{if } F \text{ does not occur,} \\ x_F(1-p_F) - x_{E \wedge F} p_{E \wedge F} - x_{E|F} p_{E|F} & \text{if } F, \text{ but not } E \text{ occurs,} \\ x_F(1-p_F) + x_{E \wedge F}(1-p_{E \wedge F}) + x_{E|F}(1-p_{E|F}) & \text{if both } E \text{ and } F \text{ occur.} \end{cases} \quad (\text{A4})$$

The Dutch bookie can choose payoffs x_F , $x_{E \wedge F}$, and $x_{E|F}$ that lead to $G < 0$ for all three outcomes unless

$$0 = \det \begin{pmatrix} -p_F & -p_{E \wedge F} & 0 \\ 1-p_F & -p_{E \wedge F} & -p_{E|F} \\ 1-p_F & 1-p_{E \wedge F} & 1-p_{E|F} \end{pmatrix} = -p_{E|F} p_E + p_{E \wedge F}. \quad (\text{A5})$$

Consistency thus requires that Bayes's rule be satisfied, i.e., $p_{E \wedge F} = p_{E|F} p_F$.

In our experience most physicists find it difficult first to accept and then to embrace the notion that Bayesian probabilities receive their *only* operational significance from decision theory, the simplest realization of which is the Dutch-book argument in which probabilities are *defined* in terms of betting odds for fair bets. In the Dutch-book approach, the structure of probability theory follows solely from the requirement of consistent betting behavior. There is no other input to the theory. It is worth emphasizing, for example, that normalization is not a separate assumption, so trivial that it

requires no justification. Rather it is a consequence of Dutch-book consistency, specifically of rules (ii) and (iii), i.e., $1 = p(E \vee \neg E) = p(E) + p(\neg E)$.

Surprisingly, consistency does not imply the converse of rule (ii); i.e., we cannot conclude from consistency alone that if $p_E = 1$, then A believes that E is certain to occur. To see this, return to Eq. (A2), specializing to $p_E = 1$ and $p_{\neg E} = 0$ (the latter required by normalization):

$$G = \begin{cases} 0 & \text{if } E \text{ occurs,} \\ -x_E + x_{\neg E} & \text{if } E \text{ does not occur.} \end{cases} \quad (\text{A6})$$

We can get no further with consistency because the zero gain for outcome E ensures that A cannot be put in a situation where all gains are negative.

To go further, we need the notion of *strong consistency* [8,9] (or strong Dutch-book consistency, often called strong or strict coherence in the literature): A 's probability assignment is said to be inconsistent in the strong sense if she can be forced to accept bets on which, for outcomes she deems possible, no gain is positive, but some gains are negative (she never wins, but sometimes loses); a probability assignment is strongly consistent if it is not inconsistent in the strong sense. Since in Eq. (A6) the second gain can be made negative, strong consistency implies that A must believe that E is certain to occur. Thus strong consistency requires that $p = 1$ be assigned only to events thought certain to occur (and $p = 0$ be assigned only to events thought certain not to occur).

Dutch-book consistency requires a bettor to follow the standard probability rules. That following the rules is sufficient to avoid inconsistency has been shown by Kemeny [9]. Kemeny reduces the most general betting situation to combinations of conditional bets, as in Eq. (A4), and bets on exclusive alternatives, as in Eq. (A2), and he then shows that the expected gain for each of these kinds of bets is zero for probabilities that satisfy the standard rules. The expected gain for bets on the exclusive alternatives in Eq. (A2) is

$$\begin{aligned} & p_E[x_E(1-p_E) - x_F p_F + x_{E \vee F}(1-p_{E \vee F})] \\ & + p_F[-x_E p_E + x_F(1-p_F) + x_{E \vee F}(1-p_{E \vee F})] \\ & + (1-p_{E \vee F})[-x_E p_E - x_F p_F - x_{E \vee F} p_{E \vee F}] \end{aligned}$$

$$= (p_{E \vee F} - p_E - p_F)[p_E x_E + p_F x_F - (1-p_{E \vee F})x_{E \vee F}]. \quad (\text{A7})$$

A similar result holds for the conditional bets of Eq. (A4):

$$\begin{aligned} & (1-p_F)[-x_F p_F - x_{E \wedge F} p_{E \wedge F}] \\ & + (1-p_{E|F})p_F[x_F(1-p_F) - x_{E \wedge F} p_{E \wedge F} - x_{E|F} p_{E|F}] \\ & + p_{E \wedge F}[x_F(1-p_F) + x_{E \wedge F}(1-p_{E \wedge F}) + x_{E|F}(1-p_{E|F})] \\ & = (p_{E \wedge F} - p_{E|F} p_F)[(1-p_F)x_F + (1-p_{E|F})x_{E|F} \\ & \quad - p_{E \wedge F} x_{E \wedge F}]. \end{aligned} \quad (\text{A8})$$

Since the expected gains are zero for probabilities that satisfy the standard rules, it is impossible to have all-negative gains (or, in the case of strong consistency, for those outcomes the bettor deems possible, to have gains some of which are negative with the rest being zero).

Unlike an ordinary bookie, who tries to balance wins and losses and makes money off the fees charged for handling the bets, a Dutch bookie exploits inconsistencies in a bettor's behavior to win under all circumstances (or never to lose, yet sometimes win in the case of strong consistency). To avoid inconsistency, a bettor simply has to follow the rules of probability theory. The Dutch-book argument is not about a contest between a bettor and a Dutch bookie. It is wholly about the internal consistency of the way a bettor translates beliefs into probability assignments. The Dutch bookie is simply the agent who exposes inconsistencies in the bettor's behavior.

In keeping with the notion that probabilities are subjective, the Dutch-book argument does not dictate a bettor's probability assignments, which are based on whatever the bettor believes or knows about the situation at hand. The only exception occurs in the case where the bettor is certain. Then Dutch-book consistency requires that all her probabilities be 0 or 1. For quantum mechanics, this means that when a bettor is certain about the outcome of some ODOP, she must assign the pure state corresponding to the certain outcome. Only if the bettor is strongly consistent, however, can we conclude that a pure state assignment means that the bettor is certain about the outcome of an ODOP that includes the pure state among its outcomes, and this conclusion is crucial for all the compatibility criteria developed in this paper.

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