Solution of the relativistic Dirac-Woods-Saxon problem

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The Dirac equation is written for the special case of a spinor in a relativistic potential with the even and odd components related by a constraint, and solved exactly with the even component chosen to be the Woods-Saxon potential. The corresponding radial wave functions for the two-component spinor are obtained in terms of the hypergeometric function, and the energy spectrum of the bound states is obtained as a solution to a given equation with boundary constraints in which the nonrelativistic limit reproduces the usual Woods-Saxon potential.

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The Woods-Saxon potential plays an essential role in microscopic fields since it can be used to describe the interaction of a neutron with a heavy nucleus. Although the nonrelativistic Schrödinger equation with this potential has been solved for S states [1] and the single-particle motion in atomic nuclei has been explained quite well, the relativistic effects for a particle under the action of this potential are more important, especially for a strong-coupling system. The relativistic Coulomb and oscillator potential problems, including their bound-state spectra and wave functions, have already been established for a long time [2–9], and their nonrelativistic limits reproduce the usual Schrödinger-Coulomb and Schrödinger-oscillator solutions, respectively. Recently, Kennedy proposed a two-component approach to solving the one-dimensional Dirac equation, and obtained the scattering and bound-state solutions for the Woods-Saxon potential [10]. However, we are still not aware of any solutions of the three-dimensional Dirac equation with the Woods-Saxon potential, which may be more important in the field of nuclear structure. Fortunately, Alhaidari has just put forward an approach to the three-dimensional Dirac equation [11], and solved a class of shape-invariant potentials that includes Dirac-Rosen-Mörse, Dirac-Eckart, Dirac-Pöschl-Teller, and Dirac-Scarf potentials [12], and presented the relativistic bound-state spectra and spinor wave functions. Following the procedure used in [11], we make an attempt, in this paper, to solve the three-dimensional Woods-Saxon potential problem. Suppose a spinor particle populates a relativistic potential with the even and odd components $V(r)$ and $W(r)$, and the even component is chosen to be the spherically symmetric Woods-Saxon potential. We solve the Dirac equation by employing the same strategy as that used by Alhaidari in the Dirac-Mörse problem, namely, by adding a radial term to the odd part of the Dirac operator, and applying a unitary transformation to the Dirac equation such that the resulting second-order differential equation becomes Schrödinger-like, and solvable.

For simplicity, atomic units ($m=e=\hbar=1$) are used, and the corresponding speed of light $c=\alpha^{-1}$ is taken. It is well known that the Hamiltonian for a charged Dirac particle in a four-component electromagnetic potential $(A_0,\vec{A})$ can be written as

$$H=\begin{pmatrix} 1+\alpha A_0 & -i\alpha\vec{\sigma}\cdot\vec{\nabla}+\alpha\vec{\sigma}\cdot\vec{A} \\ -i\alpha\vec{\sigma}\cdot\vec{\nabla}+\alpha\vec{\sigma}\cdot\vec{A} & -1+\alpha A_0 \end{pmatrix}, \qquad (1)$$

where α is the fine structure constant and $\vec{\sigma}$ are the three 2×2 Pauli spin matrices. Replacing the two off-diagonal terms $\alpha\vec{\sigma}\cdot\vec{A}$ in the Hamiltonian of Eq. (1) by $\pm i\alpha\vec{\sigma}\cdot\vec{A}$ and writing $(A_0,\vec{A})$ as $(\alpha V(r),\hat{r}W(r))$, we obtain the following two-component radial Dirac equation:

$$\begin{pmatrix} 1+\alpha^2V(r) & \alpha\left[\dfrac{\kappa}{r}+W(r)-\dfrac{d}{dr}\right] \\ \alpha\left[\dfrac{\kappa}{r}+W(r)+\dfrac{d}{dr}\right] & -1+\alpha^2V(r) \end{pmatrix}=\varepsilon\begin{pmatrix} g(r) \\ f(r) \end{pmatrix}, \qquad (2)$$

where $V(r)$ and $W(r)$ are the even and odd components of the relativistic potential, κ is the coupling parameter defined as $\kappa=\pm(j+\frac{1}{2})$ for $l=j\pm\frac{1}{2}$, and ε is the relativistic energy. This equation gives two coupled first-order differential equations for the two radial spinor components. By eliminating the lower component we obtain a second-order differential equation for the upper. Unfortunately, the resulting equation may turn out to be not Schrödinger-like, i.e., it may contain first-order derivatives. However, a general local unitary transformation may be applied to eliminate the first-order derivative as follows:

$$r=q(x)$$

and

$$\begin{pmatrix} g(r) \\ f(r) \end{pmatrix}=\begin{pmatrix} \cos[\rho(x)] & \sin[\rho(x)] \\ -\sin[\rho(x)] & \cos[\rho(x)] \end{pmatrix}\begin{pmatrix} \phi(x) \\ \theta(x) \end{pmatrix}. \qquad (3)$$

The stated requirement gives the following constraint:

$$\frac{dq}{dx}\left[-\alpha^2V+\cos(2\rho)+\alpha\sin(2\rho)(W+\kappa/q)+\alpha\frac{d\rho/dx}{dq/dx}+\varepsilon\right]=\text{const}\equiv\eta\neq 0. \qquad (4)$$

This transformation and the resulting constraint are the relativistic analog of a point canonical transformation in nonrelativistic quantum mechanics. Here, we consider the case of a global unitary transformation defined by $q(x)=x$ and $d\rho/dx=0$. Substituting these in the constraint Eq. (4) yields

$$W(r)=\frac{\alpha}{S}V(r)-\frac{\kappa}{r},$$
$$\eta=C+\varepsilon, \tag{5}$$

where $S=\sin(2\rho)$ and $C=\cos(2\rho)$. This maps the radial Dirac equation (2) into the following:

$$\begin{pmatrix} C+2\alpha^2V & \alpha\left[-\frac{S}{\alpha}+\frac{\alpha C}{S}V-\frac{d}{dr}\right] \\ \alpha\left[-\frac{S}{\alpha}+\frac{\alpha C}{S}V+\frac{d}{dr}\right] & -C \end{pmatrix} \times\begin{pmatrix}\phi(r)\\ \theta(r)\end{pmatrix}=\varepsilon\begin{pmatrix}\phi(r)\\ \theta(r)\end{pmatrix}, \tag{6}$$

which in turn gives an equation for the lower spinor component in terms of the upper:

$$\theta(r)=\frac{\alpha}{C+\varepsilon}\left[-\frac{S}{\alpha}+\frac{\alpha C}{S}V+\frac{d}{dr}\right]\phi(r), \tag{7}$$

while the differential equation for the upper component reads

$$\left[-\frac{d^2}{dr^2}+\frac{\alpha^2}{T^2}V^2+2\varepsilon V-\frac{\alpha}{T}\frac{dV}{dr}-\frac{\varepsilon^2-1}{\alpha^2}\right]\phi(r)=0, \tag{8}$$

where $T=S/C=\tan(2\rho)$.

Having given this brief review of the method proposed by Alhaidari [11,12], let us consider the Dirac-Woods-Saxon potential problem. In Eq. (8), the independent even potential component $V(r)$ is chosen to be the spherically symmetric Woods-Saxon potential:

$$V(r)=-\frac{V_0}{1+e^{(r-R)/b}}\quad (V_0>0) \tag{9}$$

with $b\ll R$ (R is the nuclear radius, and b is the range of the potential). Consequently, we obtain the following second-order differential equation for the upper spinor component:

$$\left\{-\frac{d^2}{dr^2}-\frac{\lambda}{b^2}\frac{-\lambda+e^{(r-R)/b}}{[1+e^{(r-R)/b}]^2}-\frac{2\varepsilon V_0}{1+e^{(r-R)/b}}-\frac{\varepsilon^2-1}{\alpha^2}\right\}\phi(r)=0, \tag{10}$$

where $\lambda=\alpha b V_0/T$ is a dimensionless parameter. In order to reproduce the nonrelativistic Woods-Saxon problem, we take the nonrelativistic limit ($\alpha\to 0,\varepsilon\approx 1+\alpha^2E$), where E is the nonrelativistic energy. The above equation is then reduced to the following wave equation:

$$\left\{-\frac{d^2}{dr^2}-\frac{\lambda}{b^2}\frac{-\lambda+e^{(r-R)/b}}{[1+e^{(r-R)/b}]^2}-\frac{2V_0}{1+e^{(r-R)/b}}-2E\right\}\phi(r)=0, \tag{11}$$

which is in the form of the Woods-Saxon problem only for $\lambda=-1$. In that case, Eq. (11) becomes the Schrödinger equation for the potential $V(r)=-V_0^{\rm NR}[1+e^{(r-R)/b}]^{-1}$, where $V_0^{\rm NR}=V_0-1/2b^2$. The energy spectrum for the S-wave nonrelativistic Woods-Saxon problem is already known (see, for example, Ref. [1]). In order to obtain the relativistic energy spectrum easily and directly, we rewrite the relativistic Eq. (10), for $\lambda=-1$, as

$$\left[-\frac{d^2}{dr^2}-2\frac{\varepsilon V_0-1/2b^2}{1+e^{(r-R)/b}}-\frac{\varepsilon^2-1}{\alpha^2}\right]\phi(r)=0. \tag{12}$$

Comparing this with the nonrelativistic Eq. (11) for $\lambda=-1$, we arrive at the following parameter correspondence map:

$$b\to b,\quad R\to R,$$
$$V^{\rm NR}\to\varepsilon V_0-1/2b^2, \tag{13}$$
$$E\to(\varepsilon^2-1)/2\alpha^2.$$

Using this parameter map we obtain immediately the relativistic energy spectrum

$$\arg\Gamma(2i\gamma)-2\arg\Gamma(\mu+i\gamma)-\tan^{-1}\frac{\gamma}{\mu}+\frac{\gamma R}{b}=(n+1/2)\pi,\quad n=0,\pm1,\pm2,..., \tag{14}$$

and the upper spinor component $\phi(r)$

$$\phi_n(r)=e^{i\gamma(r-R)/b}[1+e^{(r-R)/b}]^{-\mu-i\gamma}F(\mu+i\gamma,\mu+i\gamma+1,2\mu+1;[1+e^{(r-R)/b}]^{-1})=0, \tag{15}$$

where

$$\mu=b\sqrt{(1-\varepsilon^2)/\alpha^2}$$

and

$$\gamma=\sqrt{(\varepsilon^2-1)b^2/\alpha^2+2b^2\varepsilon V_0-1}.$$

The lower spinor component can also be derived from Eq. (7) as

$$\theta_n(r)=\frac{\alpha}{C+\varepsilon_n}\left\{\left[-\frac{S}{\alpha}-\frac{\mu}{b}+\left(\frac{\mu+i\gamma}{b}-\frac{V_0\alpha}{T}\right)\times\frac{1}{1+e^{(r-R)/b}}\right]\phi_n(r)-\frac{1}{b}\left(\mu-\frac{-1+\nu^2}{2\mu+1}+i\gamma\right)\times e^{(1+i\gamma)(r-R)/b}[1+e^{(r-R)/b}]^{-\mu-2-i\gamma}\times F\left(\mu+i\gamma+1,\mu+i\gamma+2,2\mu+2;\frac{1}{1+e^{(r-R)/b}}\right)\right\}. \tag{16}$$

Moreover, we can also give the solution of the relativistic Dirac-Woods-Saxon problem with a general λ as the deformation parameter. In order to do that, a new variable $x=[1+e^{(r-R)/b}]^{-1}$ is introduced. Now, the equation of the upper spinor component has been transformed into a second-order differential equation as follows:

$$x^2(1-x)^2\frac{d^2\phi}{dx^2}+x(1-x)(1-2x)\frac{d\phi}{dx}-[\mu^2+\lambda(\lambda+1)x^2-(\lambda+\nu^2)x]\phi=0, \tag{17}$$

where the parameters $\mu^2=b^2(1-\varepsilon)^2/\alpha^2$, $\nu^2=2b^2\varepsilon V_0$, and $\lambda=\alpha b V_0/T$. Bound-state solutions of Eq. (17) satisfy the boundary conditions $\phi(0)=0\,(r\to\infty)$ and $\phi(1)=0\,(r\to 0)$. Using the trial function

$$\phi(x)=x^{\mu}(1-x)^{\delta}\psi(x) \tag{18}$$

with

$$\delta=\sqrt{\mu^2-\nu^2+\lambda^2}, \tag{19}$$

we find that Eq. (17) can be turned into the hypergeometric differential equation

$$x(1-x)\psi''(x)+[2\mu+1-(2\mu+2\delta+2)x]\psi'(x)-(\mu^2+\delta^2-\lambda^2+2\mu\delta+\mu+\delta-\lambda)\psi(x)=0. \tag{20}$$

Taking into account the boundary condition $\phi(0)=0$, we get

$$\phi(x)=x^{\mu}(1-x)^{\delta}F(\mu+\lambda+1+\delta,\mu-\lambda+\delta,2\mu+1;x). \tag{21}$$

In order to describe the behavior of the formula (21) in the vicinity of $x=1$, we transform $\phi(x)$ into the following form:

$$\begin{aligned}\phi(x)=&x^{\mu}(1-x)^{\delta}\frac{\Gamma(2\mu+1)\Gamma(-2\delta)}{\Gamma(\mu-\lambda-\delta)\Gamma(\mu+\lambda+1-\delta)}\\&\times F(\mu+\lambda+1+\delta,\mu-\lambda+\delta,1+2\delta;1-x)+x^{\mu}\\&\times(1-x)^{-\delta}\frac{\Gamma(2\mu+1)\Gamma(2\delta)}{\Gamma(\mu+\lambda+1+\delta)\Gamma(\mu-\lambda+\delta)}\\&\times F(\mu-\lambda-\delta,\mu+\lambda+1-\delta,1-2\delta;1-x).\end{aligned} \tag{22}$$

Near the origin $(x\to 1, r=0)$ the wave function behaves as

$$\phi(x)\sim(1-x)^{\delta}\frac{\Gamma(2\mu+1)\Gamma(-2\delta)}{\Gamma(\mu-\lambda-\delta)\Gamma(\mu+\lambda+1-\delta)}+(1-x)^{-\delta}\frac{\Gamma(2\mu+1)\Gamma(2\delta)}{\Gamma(\mu+\lambda+1+\delta)\Gamma(\mu-\lambda+\delta)}. \tag{23}$$

To discuss formula (23) we note that $\mu^2-\nu^2+\lambda^2<0$, so that according to Eq. (19) δ turns out to be imaginary:

$$\delta=i\gamma,\quad \gamma=\sqrt{\nu^2-\mu^2-\lambda^2}. \tag{24}$$

We then write

$$\begin{aligned}\phi(x)\sim&\frac{\Gamma(2\mu+1)\Gamma(-2i\gamma)}{\Gamma(\mu-\lambda-i\gamma)\Gamma(\mu+\lambda+1-i\gamma)}\Bigg[(1-x)^{i\gamma}\\&+\frac{\Gamma(2i\gamma)\Gamma(\mu-\lambda-i\gamma)\Gamma(\mu+\lambda+1-i\gamma)}{\Gamma(-2i\gamma)\Gamma(\mu-\lambda+i\gamma)\Gamma(\mu+\lambda+1+i\gamma)}\\&\times(1-x)^{-i\gamma}\Bigg].\end{aligned} \tag{25}$$

In the neighborhood of $r=0$, $1-x=e^{-R/b}$ approximately, so we have

$$\phi(x)\sim\frac{\Gamma(2\mu+1)\Gamma(-2i\gamma)}{\Gamma(\mu-\lambda-i\gamma)\Gamma(\mu+\lambda+1-i\gamma)}\Bigg[e^{-i\gamma R/b}+\frac{\Gamma(2i\gamma)\Gamma(\mu-\lambda-i\gamma)\Gamma(\mu+\lambda+1-i\gamma)}{\Gamma(-2i\gamma)\Gamma(\mu-\lambda+i\gamma)\Gamma(\mu+\lambda+1+i\gamma)}e^{i\gamma R/b}\Bigg]. \tag{26}$$

The boundary condition $\phi(1)=0$ $(r\to 0)$ leads to

$$\frac{\Gamma(2i\gamma)\Gamma(\mu-\lambda-i\gamma)\Gamma(\mu+\lambda+1-i\gamma)}{\Gamma(-2i\gamma)\Gamma(\mu-\lambda+i\gamma)\Gamma(\mu+\lambda+1+i\gamma)}e^{i\gamma R/b}+e^{-i\gamma R/b}=0, \tag{27}$$

or

$$\frac{\Gamma(2i\gamma)\Gamma(\mu-\lambda-i\gamma)\Gamma(\mu+\lambda+1-i\gamma)}{\Gamma(-2i\gamma)\Gamma(\mu-\lambda+i\gamma)\Gamma(\mu+\lambda+1+i\gamma)}e^{2i\gamma R/b}=-1. \tag{28}$$

Thus we obtain the quantum condition

$$\exp\{2i[\arg\Gamma(2i\gamma)-\arg\Gamma(\mu-\lambda+i\gamma)-\arg\Gamma(\mu+\lambda+1+i\gamma)+\gamma R/b]\}=-1, \tag{29}$$

that is,

$$\arg\Gamma(2i\gamma)-\arg\Gamma(\mu-\lambda+i\gamma)-\arg\Gamma(\mu+\lambda+1+i\gamma)+\gamma R/b=\left(n+\frac{1}{2}\right)\pi,\quad (n=0,\pm1,\pm2,\ldots) \tag{30}$$

or

$$\begin{aligned}&\arg\Gamma\left(2i\frac{b}{\alpha}\sqrt{\varepsilon_n^2-1+2V_0\varepsilon_n\alpha^2-V_0^2\alpha^4/T^2}\right)-\arg\Gamma\left(\frac{b}{\alpha}\sqrt{1-\varepsilon_n^2}-b\alpha V_0/T+i\frac{b}{\alpha}\sqrt{\varepsilon_n^2-1+2V_0\varepsilon_n\alpha^2-V_0^2\alpha^4/T^2}\right)\\&-\arg\Gamma\left(\frac{b}{\alpha}\sqrt{1-\varepsilon_n^2}+b\alpha V_0/T+1+i\frac{b}{\alpha}\sqrt{\varepsilon_n^2-1+2V_0\varepsilon_n\alpha^2-V_0^2\alpha^4/T^2}\right)+\frac{R}{\alpha}\sqrt{\varepsilon_n^2-1+2V_0\varepsilon_n\alpha^2-V_0^2\alpha^4/T^2}\\&=\left(n+\frac{1}{2}\right)\pi,\end{aligned} \tag{31}$$

and the upper spinor component is given by

$$\phi_n(r)=e^{i\gamma(r-R)/b}[1+e^{(r-R)/b}]^{-\mu-i\gamma}F\left(\mu+\lambda+1+i\gamma,\mu-\lambda+i\gamma,2\mu+1;\frac{1}{1+e^{(r-R)/b}}\right). \tag{32}$$

With Eq. (7), the lower spinor component is also obtained as

$$\theta_n(r)=\frac{\alpha}{C+\varepsilon_n}\left\{\left[-\frac{S}{\alpha}-\frac{\mu}{b}+\left(\frac{\mu+i\gamma}{b}-\frac{V_0\alpha}{T}\right)\frac{1}{1+e^{(r-R)/b}}\right]\phi_n(r)-\frac{1}{b}\left(\mu-\frac{\lambda+\nu^2}{2\mu+1}+i\gamma\right)e^{(1+i\gamma)(r-R)/b}[1+e^{(r-R)/b}]^{-\mu-2-i\gamma}\right.$$
$$\left.\times F\left(\mu+\lambda+2+i\gamma,\mu-\lambda+1+i\gamma,2\mu+2;\frac{1}{1+e^{(r-R)/b}}\right)\right\}. \tag{33}$$

It can be easily seen that Eqs. (30), (32), and (33) give the same results as Eqs. (14)–(16) with $\lambda=-1$.

In conclusion, we have obtained the exact solution of the Dirac equation for the Woods-Saxon potential and presented the explicit form of the spinor wave function. The relativistic bound-state spectrum is also obtained through an equation that may be very useful to describe single-particle motion in nuclei, because spin-orbit coupling is involved automatically there.

[1] S. Flügge, *Practical Quantum Mechanics* (Springer-Verlag, Berlin, 1974).
[2] L. C. Biedenharn, Phys. Rev. **126**, 845 (1962).
[3] B. G. Wybourne, *Classical Groups for Physicists* (Wiley-Interscience, New York, 1974).
[4] B. Goodman and S. R. Ignjatovic, Am. J. Phys. **65**, 214 (1997).
[5] M. Moshinsky and A. Szczepaniak, J. Phys. A **22**, L817 (1989).
[6] J. Bentez *et al.*, Phys. Rev. Lett. **64**, 1643 (1990).
[7] O. L. Lange, J. Phys. A **24**, 667 (1991).
[8] V. M. Villalba, Phys. Rev. A **49**, 586 (1994).
[9] P. Rozmej and R. Arvieu, J. Phys. A **32**, 5367 (1999).
[10] P. Kennedy, J. Phys. A **35**, 689 (2002).
[11] A. D. Alhaidari, Phys. Rev. Lett. **87**, 210405 (2001); **88**, 189901 (2002).
[12] A. D. Alhaidari, J. Phys. A **34**, 9827 (2001).