Bohr's correspondence principle: The cases for which it is exact

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Two-dimensional central potentials leading to the identical classical and quantum motions are derived and their properties are discussed. Some of zero-energy states in the potentials are shown to cancel the quantum correction $Q=(-\hbar^2/2m)\Delta R/R$ to the classical Hamilton-Jacobi equation. The Bohr's correspondence principle is thus fulfilled exactly without taking the limits of high quantum numbers, of $\hbar\to 0$, or of the like. In this exact limit of $Q=0$, classical trajectories are found and classified. Interestingly, many of them are represented by closed curves. Applications of the found potentials in many areas of physics are briefly commented.

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I. INTRODUCTION

The quantum-classical correspondence has been a subject of intensive studies from the very beginning of quantum description in physics. If, as it is commonly believed, quantum mechanics were correct, it would agree with classical mechanics in the appropriate limit. The idea, named *the correspondence principle*, was proposed by N.Bohr in the early days of quantum mechanics.

Latest development in laser technique has made possible an experimental exploration of many fundamental quantum problems. The quantum-classical border is, no doubt, one of them. Quite strong activity in this field includes experiments on studying quantum-classical regime for high quantum number states of Rydberg atoms [1] and also for circular states of the hydrogen atom [2]. Worth mentioning are experiments on the observation of environment-induced decoherence [3], cold-atom collisions [4] or single-atom trajectories in cavity in real time [5].

Recent research, including test of Wigner function [6], an analysis of quantum-classical transition in nonlinear dynamical systems [7] or discussion of the problem, how long classical and quantum evolutions stay close [8], also contribute to better understanding of quantum-classical correspondence.

In this context, we should emphasize that no commonly accepted definition of the correspondence principle exists. To one, it means taking $\hbar\to 0$, though the limit is not well-defined mathematically unless some additional conditions are specified. Moreover, the limit does not commute [9] with another one, $t\to\infty$, which is important for the systems, with chaotic classical counterparts. To other, it is correctly formulated for the limit of large quantum numbers [10,11]. This expectation was critically reviewed in the recent comments [12,13] for the potentials of the form $-C_n/r^n$ with $n>2$.

An important approach to the quantum-classical correspondence is also given via expectation values for the position and momentum, as in Ehrenfest's theorem [14]. Yet, this approach is known to be of a restricted validity, since the evolution of a wave packet is governed by Ehrenfest's equation for short times only [15,16].

To have a full classical description emerging from quantum mechanics, it would be necessary to formulate quantum systems in phase space. One can try to do this [17] with the help of Wigner-transform formalism [18]. Though very useful, the approach is also of a limited validity since in most cases the Wigner function is not positively determined. The modification, due to Husimi [19], removes the last disadvantage, but contrary to the Wigner function, now the marginal distributions in coordinates and momenta are not exact. Nevertheless, the phase-space approach to the quantum-classical correspondence, appeared to be also very useful in studying effects of interactions with a stochastic environment [20], the so-called decoherence [21], since it is easily generalized to the case of density matrix.

In conclusion, we can say that there is no commonly accepted definition of the correspondence principle and maintaining that classical mechanics is contained in quantum mechanics, or that the latter is an extension of the former, are too far-reaching simplifications. Therefore we should perhaps acknowledge after Ref. [22] that: "*In its general context, the correspondence principle requires that any two valid physical theories which have an overlap in their domains of validity must, to relevant accuracy, yield the same predictions for physical observations*."

There can be situations when approaching the classical limit is not possible at all. Simple examples are provided by the Rydberg's constant $R_\infty = me^4/8ch^3\varepsilon_0^2$ and famous Einstein's formula for the photoelectric effect $E=h\nu-P$. Contemporary examples [22,23] are given by some classically chaotic models.

In contrast, there are interesting relations between the classical and quantum theories valid even for low quantum numbers and relatively small values of the classical action, known as the correspondence identities [24].

On the other hand, one can ask whether there are any potentials, states and energies, when without taking the limits of $\hbar\to 0$, of large quantum numbers, of small de Broglie wave length , or of the like, the quantum and classical descriptions yield *exactly* the same predictions. This way of thinking was originated by Rosen [25] who found few examples of such potentials and states. Their number [26–30] was greatly multiplied since and general formulas for such potentials were also found [31–33].

In this paper, we want to extend our recently obtained results [34] for two-dimensional (2D) central potentials hav-

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ing the property that some of their states have vanishing quantum correction (the Bohm's potential) to the classical Hamilton-Jacobi equation. In this way, the velocity fields, and hence also the trajectories, are identical in both the classical and quantum descriptions.

The paper is organized as follows: Sec. II presents a systematic search for the 2D potentials with the above property, previously [34] found basing mostly on a "good guessing method." Thanks to that, some new potentials can be proposed here. Section III gives a review of trajectories that are possible in the limit of vanishing the quantum or of the Bohm potential Q. Then Sec. IV presents the special wave functions with the above property. The paper concludes with Sec. V.

II. SPECIAL POTENTIALS

We consider a special class of potentials, derivable from the stationary Schrödinger equation, from which, we have

$$V=E+\frac{\hbar^2}{2m}\frac{\Delta\psi}{\psi}. \tag{1}$$

Now, using the wave function in polar form $\psi=R\exp[(i/\hbar)S]$, with real functions R and S, we get

$$V=E-\left\{\frac{(\nabla S)^2}{2m}-\frac{i\hbar}{2m}\left[\frac{2}{R}\nabla R\cdot\nabla S+\Delta S\right]-\frac{\hbar^2}{2m}\frac{\Delta R}{R}\right\}. \tag{2}$$

The vanishing of the square brackets in Eq. (2), i.e., the continuity equation, guarantees conservation of the probability flux and reality of potentials. What is remaining is the classical Hamilton-Jacobi equation supplemented with the quantum correction, called the inner or Bohm's potential $Q:=(-\hbar^2/2m)(\Delta R/R)$. We look for special potentials with the property that for some of their states the amplitude R obeys the Laplace equation. Thus, the quantum correction is zero and the classical limit of quantum mechanics is attained exactly, without taking large quantum numbers, or the limit of $\hbar\to 0$, or of the like. Finally, if V in Eq. (2) is to be a central potential, then $(\nabla S)^2$ must be a function, say f, of a distance only.

In this way, one has to search for the solutions of the three coupled partial differential equations,

$$\nabla\cdot(R^2\nabla S)=0, \tag{3}$$

$$\Delta R=0, \tag{4}$$

$$V=E-\frac{1}{2m}f, \tag{5}$$

for the three unknown functions R, S, and V, where

$$(\nabla S)^2=f. \tag{6}$$

In what follows we restrict ourselves to the two-dimensional (2D) central potentials. The method that can be useful in this case, in tackling the problem of integration of Eqs. (3), (4), and (5), is based on the observation that for any solutions of Eq. (4), the function S must be such that $(\nabla S)^2$ in Eq. (6) is independent of an angle. In polar coordinates: $\rho=\sqrt{x^2+y^2}$ and $\theta=\arctan(y/x)$, Eq. (6) takes the form of $(\partial S/\partial x)^2+(\partial S/\partial y)^2=(\partial S/\partial\rho)^2+\rho^{-2}(\partial S/\partial\theta)^2=f(\rho)$. We are now able to distinguish the following cases.

First, let us start with

$$S(\rho,\theta)=A\theta+\int\sqrt{f(\rho)-A^2/\rho^2}\,d\rho. \tag{7}$$

This is a complete integral of Eq. (6) with the *explicit* form for the function $f(\rho)$ being for now unknown. It can be determined after $S(\rho,\theta)$ is substituted into Eq. (3). Then, we have

$$4\left[\frac{A}{\rho}\sqrt{f-\frac{A^2}{\rho^2}}\frac{\partial R}{\partial\theta}+\rho\left(f-\frac{A^2}{\rho^2}\right)\frac{\partial R}{\partial\rho}\right]+R\left(\rho\frac{df}{d\rho}+2f\right)=0. \tag{8}$$

The requirement for an independence of $f(\rho)$ on the angle greatly restricts the number of acceptable solutions of Eqs. (3) and (4). One possible solution of Eq. (8) is obviously given by

$$f=\frac{A^2}{\rho^2}. \tag{9}$$

Another solution of Eq. (8) is found if

$$R(\rho)=\ln(d\rho^c), \tag{10}$$

which, of course, obeys the 2D Laplace equation (4). Again, we obtain the solution (9), but additionally we can also get

$$f(\rho)=\frac{A^2}{\rho^2}+\frac{C_1^2}{\rho^2\ln^4(d\rho^c)}, \tag{11}$$

where henceforth all undefined symbols stand for real constants.

Equation (8) can also be solved exactly if

$$R(\rho,\theta)=e^{\alpha\theta}F(\rho), \tag{12}$$

where $R(\rho,\theta)$ is to obey Eq. (4) which now simplifies to

$$\rho^2\frac{d^2F}{d\rho^2}+\rho\frac{dF}{d\rho}+\alpha^2F=0. \tag{13}$$

This is Euler's type equation and its solution for $\alpha^2>0$, has the form

$$F(\rho)=C_2\sin[\ln(\beta\rho^\alpha)]. \tag{14}$$

Obviously, for $\alpha^2=0$ we would get the solution (10). This time, instead of Eq. (8), we have to solve the equation

$$4\left[\frac{\alpha A}{\rho}\sqrt{f-\frac{A^2}{\rho^2}}F+\rho\left(f-\frac{A^2}{\rho^2}\right)\frac{dF}{d\rho}\right]+F\left(\rho\frac{df}{d\rho}+2f\right)=0, \tag{15}$$

with the function $F(\rho)$ given in Eq. (14). This nonlinear equation can be reduced to a linear one, as

$$f(\rho)=\frac{A^2}{\rho^2}+U^2(\rho). \tag{16}$$

Omitting details of further calculations, we shall write down the final result,

$$U(\rho)=\frac{\xi-A\ln(\beta\rho^{\alpha})+(A/2)\sin[2\ln(\beta\rho^{\alpha})]}{\rho\sin^2[\ln(\beta\rho^{\alpha})]}. \tag{17}$$

Equations (5), (16), and (17) give, for particular choices of real constants ξ, A, α, β, the new class of singular central 2D potentials leading to identical classical and quantum trajectories.

Another possible choice for the phase S is

$$S(\rho,\theta)=C_a\rho^a\sin(a\theta+D_a). \tag{18}$$

For this choice of $S(\rho,\theta)$, the function f in Eq. (6) will also be dependent on ρ only, as it should be. Substituting the function into Eq. (6), we have

$$f(\rho)=a^2C_a^2\rho^{2(a-1)} \quad (a\neq 0,1). \tag{19}$$

Furthermore, one can verify easily that Eqs. (3) and (4) are fulfilled if

$$R(\rho,\theta)=C_a\rho^a\cos(a\theta+D_a). \tag{20}$$

The parameter a is a real positive or negative constant, not necessarily integer, and thus we extend again the class of potentials derived previously [34].

For the sake of further discussion let us gather the potentials altogether as

$$V(\rho)=E-\frac{1}{2m}\frac{A^2}{\rho^2}, \tag{21}$$

$$V(\rho)=E-\frac{1}{2m}\left[\frac{A^2}{\rho^2}+\frac{C_1^2}{\rho^2\ln^4(d\rho^c)}\right], \tag{22}$$

$$V(\rho)=E-\frac{1}{2m}\left[\frac{A^2}{\rho^2}+U^2(\rho)\right], \tag{23}$$

$$V_a(\rho)=E-\frac{1}{2m}a^2C_a^2\rho^{2(a-1)} \quad (a\neq 0,1). \tag{24}$$

From their construction the potentials have eigenstates belonging to the continuous spectrum. When substituted to the stationary Schrödinger equation they correspond to the threshold energy $E=0$. Studying of the physical systems in between the bound-state regime and the continuum has attracted much attention recently [35–39]. It was, among other reasons, motivated by an interest in cold atoms and their interactions. The potentials found here are also widely used in other physical applications. For instance, $V_{1/2}(\rho)$, derived from Eq. (24), is a 2D Kepler problem [40], and $V_2(\rho)$ represents an inverted 2D oscillator, the simplest model of an unstable system in quantum mechanics [39,41,42]. Worth of mentioning are also application of the potentials $-\rho^{\pm 3}$, $-\rho^{\pm 4}, -\rho^{\pm 5}$ in the collision theory [43] and low-dimensional quantum dots [44]. The potential singled out in Eq. (21) was also a subject of many studies, among them, as a convenient model for an analysis of anomalous symmetry breaking in quantum mechanics [45].

III. TRAJECTORIES

For the potentials derived above, the quantum correction Q to the classical Hamilton-Jacobi equation is exactly zero. Obviously, this is the case only for some states in the potentials. As a result of that, the notion of trajectory is identical both in the classical and quantum descriptions. The orbits can be found from the known relation

$$\dot{\boldsymbol{\rho}}=\frac{1}{m}\nabla S=\frac{i\hbar}{2m}\frac{\psi\nabla\psi^*-\psi^*\nabla\psi}{|\psi|^2}, \tag{25}$$

which, for the special states of our 2D potentials, is exactly integrable for all cases discussed here. For instance, for the potential (23), we have, after using Eqs. (7), (16), (17), and (25), the solution

$$\theta=\frac{-1}{2\alpha}\ln\left|C\left\{\xi-A\ln(\beta\rho^{\alpha})+\frac{A}{2}\sin[2\ln(\beta\rho^{\alpha})]\right\}\right|, \tag{26}$$

where θ and ρ stand for the polar coordinates defined in Sec. II. Much more interesting are potentials specified in Eq. (24). In this case, we can find for the orbits, using Eqs. (18) and (25), that

$$\rho_a(\theta)=\left[\frac{L}{aC_a\cos(a\theta+D_a)}\right]^{1/a}. \tag{27}$$

The symbol $L=A=m(x\dot{y}-\dot{x}y)$ represents quantity of the dimension of angular momentum which is, of course, a constant of the motion.

Very interesting feature of Eq. (27) is that closed orbits are possible for some values of the parameter a. To observe this, it is convenient to take for arbitrary constants $L/aC_a=1$ and $D_a=0$. Then, it is not difficult to prove the existence of closed orbits for negative values of a. By all means, when $a=-1,-2,-3,\ldots$, we will get respectively, the circle $(x-1/2)^2+y^2=1/4$, the Bernoulli lemniscate $(x^2+y^2)^2=x^2-y^2$, and then n-leaved roses $(n=-3,-4,-5,\ldots)$. Other sequence of closed orbits can be generated when $a=-1/k$ with $k=2,3,4,\ldots$. The simplest $(k=2)$ member of the family is cardioid $(x^2+y^2-x)^2=x^2+y^2$ and the next curves are some deformations of the limacon of Pascal. The missing two conics can also be derived, and indeed for $a=2$ we can find from Eq. (27) the hyperbola $x^2-y^2=1$, and for $a=1/2$ the parabola $y^2=4(1-x)$.

In this connection, we should mention the nineteenth-century theorem by Bertrand [46], that asserts that the only central power-law potentials having closed *bounded* orbits are the Coulomb potential $V(r)=-\alpha/r$, $\alpha>0$ and the isotropic harmonic oscillator $V(r)=\beta r^2$, $\beta>0$. Our results show that in the limit of vanishing Bohm's potential, the trajectories for a large class of 2D central potentials are given by closed curves, some of them being special curves studied long ago by mathematicians.

The trajectories for positive values of a are not very interesting. Some of them have been discussed elsewhere [34], together with those for the potentials (21) and (22).

IV. CLASSICAL WAVE FUNCTIONS

All the states we have derived in the paper belong to the continuous spectrum and are not square-integrable functions. Since the practically important potentials are those given in Eq. (24), we shall restrict ourselves mostly to the discussion of wave functions represented by the amplitudes (20) and phases (18), i.e., for the power-law potentials of Eq. (24). Eigenstates for one of them, $V(\rho)\sim-\rho^2$, are best known and have been intensively studied both in the 1D [47–49] and 2D [39,41,42] spaces. It follows from a recent remarkable observation by Kobayashi and Shimbori [39] that all the wave functions for arbitrary values of the parameter a, can be, in suitable coordinates, represented by the same functions of the inverted 2D oscillator potential. Details can be found in Ref. [39], and therefore we shall report here only necessary formulas.

Let us introduce the potentials (24) into the stationary Schrödinger equation, then

$$\Delta\psi+a^2\gamma_a^2\rho^{2(a-1)}\psi=0, \tag{28}$$

where $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$, $\rho=\sqrt{x^2+y^2}$ and $\gamma_a^2=C_a^2/\hbar^2>0$. Thus, ψ represents zero-energy states of the potentials $V(a;\rho)=-(a^2C_a^2/2m)\rho^{2(a-1)}$. Now, using the conformal mapping

$$(x+iy)^a e^{iD_a}=\rho^a e^{i(a\theta+D_a)}=u_a+iv_a, \tag{29}$$

with

$$u_a=\rho^a\cos(a\theta+D_a), \tag{30}$$

$$v_a=\rho^a\sin(a\theta+D_a),$$

we have for the transformation $(x,y)\to(u_a,v_a)$, that

$$\Delta=a^2\rho^{2(a-1)}\Delta_a, \tag{31}$$

where $\Delta_a=\partial^2/\partial u_a^2+\partial^2/\partial v_a^2$. In this way, we have from Eq. (28)

$$(\Delta_a+\gamma_a^2)\psi(u_a,v_a)=0. \tag{32}$$

As already shown in Ref. [41], zero-energy states of the inverted 2D oscillator are infinitely degenerate and they correspond to the complex eigenvalues of the type $\mp i(n_x-n_y)\hbar\omega$, $\omega=2C_a/m$, as $n_x=n_y$. Since the form of Eq. (32) is the same for all values of a, including $a=2$, solutions of this equation can thus be written by the same functions that are known for the inverted 2D oscillator [41]. They have the form [39]

$$\psi_n^{\pm}(v_a)=N_a\exp[\pm i\gamma_a v_a(D_a)]f_n^{\pm}(u_a,v_a)$$
$$\equiv R_n^{(a)}(u_a,v_a)\exp[(i/\hbar)S_n^{(a)}(u_a,v_a)], \tag{33}$$

where $n=\mathbb{N}$, the function $f^{\pm}(u_a,v_a)$ is a polynomial of u_a and v_a, and $R_n^{(a)}$ and $S_n^{(a)}$ are real functions of their arguments. The first four terms of $f_n^{\pm}$ are

$$f_0^{\pm}=1, f_1^{\pm}=4\gamma_a u_a, \tag{34}$$

$$f_2^{\pm}=4(4\gamma_a^2u_a^2+1\pm4i\gamma_a v_a),\quad f_3^{\pm}=16\gamma_a u_a(4\gamma_a^2u_a^2+9 \pm12i\gamma_a v_a). \tag{35}$$

For the solution $\psi_n^{\pm}(u_a)$ we should take $f_n^{\pm}(v_a,u_a)$, where the symbols u_a and v_a are exchanged.

When $n\geqslant1$ the solutions cannot be normalized in terms of Dirac δ functions, and instead, we have to treat them as the eigenfunctions of the conjugate space $\mathcal{S}(\mathbb{R}^2)^{\times}$ in the Gel'fand triplets $\mathcal{S}(\mathbb{R}^2)\subset\mathcal{L}^2(\mathbb{R}^2)\subset\mathcal{S}(\mathbb{R}^2)^{\times}$, where $\mathcal{S}(\mathbb{R}^2)$ and $\mathcal{L}^2(\mathbb{R}^2)$ are, respectively, Schwartz space and Lebesgue space in two dimensions. General properties of such functions have been discussed in a series of elaborated papers [39,41,42,47] and for the excellent introduction to the problem we refer the reader to the papers.

Now, according to Eqs. (18), (20), and (30), the wave functions in our case, can be written as

$$\psi_1^{\pm}(v_a)=N_a\exp[\pm i\gamma_a v_a(D_a)]u_a(D_a). \tag{36}$$

It is thus clear that among the states (33) only those for $n=0$ and $n=1$ lead to the identical classical and quantum motions for the potentials (24). Formally, Eq. (36) is the proper solution as well, if the roles of v_a and u_a are exchanged.

For an illustration, we present in two figures trajectories related to the states given in Eq. (33). In order to prepare the plots, one needs to determine the phases $S_n^{(a)}(u_a,v_a)$ and then to solve the guidance equations for trajectories, i.e., $\dot{\boldsymbol{\rho}}=(1/m)\boldsymbol{\nabla}S_n^{(a)}$.

Using Eqs. (34) and (35), we can find for few values of n,

$$S_n^{(a)}=\pm\hbar\left\{\gamma_a\rho^a\sin(a\theta+D_a)+p\arctan\left(\frac{q\gamma_a\rho^a\sin(a\theta+D_a)}{s+4\gamma_a^2\rho^{2a}\cos^2(a\theta+D_a)}\right)\right\}, \tag{37}$$

where for $n=0,1$, we have to choose $p=0$; for $n=2$, we have $p=1$, $q=4$, $s=1$; and for $n=3$, we have $p=1$, $q=12$, and $s=9$.

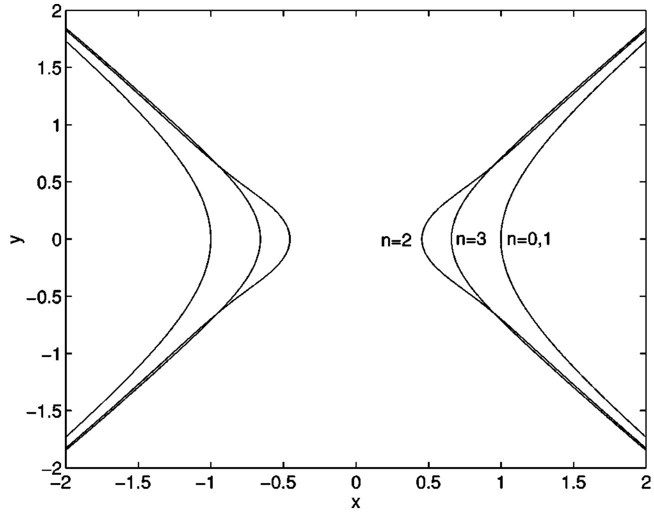

FIG. 1. The trajectories corresponding to the states in Eq. (33) with $n=0,1,2,3$ and $a=2$. Only for $n=0$ and $n=1$, we have the Bohm's potential $Q=0$, and the classical and quantum motions are identical. The curves for $n=2$ and $n=3$ deviate from the hyperbola $x^2-y^2=1$. The spatial coordinates are in dimensional units.

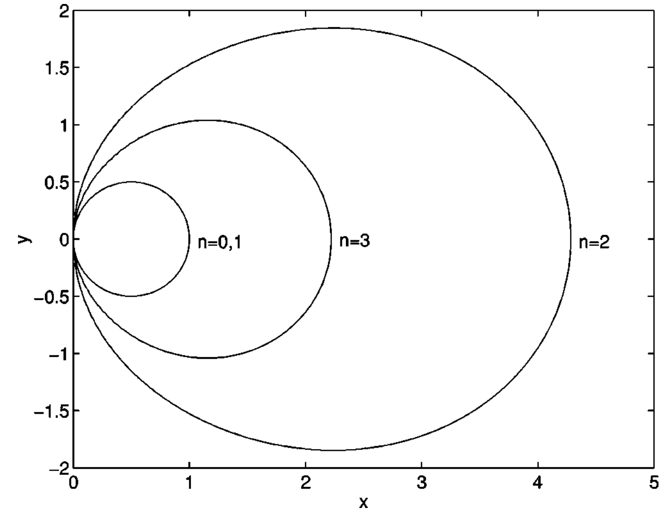

FIG. 2. As in Fig. 1 but for $a=-1$. Now, the curves for $n=2$ and $n=3$ depart from the circle $x^2+y^2=x$.

The guidance equations can easily be integrated formally if one remembers that $m(x\dot{y}-\dot{x}y)\equiv L$ is a conserved quantity for central potentials. Then, with the help of the polar coordinates ρ and θ, equations for trajectories can be reduced to the simple relation $L=(\partial S_n^{(a)}/\partial\theta)$. From this, we have

$$L=\pm\hbar a\gamma_a u_a\left\{1+p\frac{q(s+4\gamma_a^2u_a^2+8\gamma_a^2v_a^2)}{q^2\gamma_a^2v_a^2+(s+4\gamma_a^2u_a^2)^2}\right\}, \quad (38)$$

where u_a and v_a are defined in Eqs. (30). Now, the plots are prepared using $L=\hbar=1$ and $a\gamma_a=1$.

When $p=0$ $(n=0,1)$, we have straight lines $\pm1=u_a$ on the plane (u_a,v_a) for all potentials in Eq. (24). For this case, the quantum potential $Q=0$, and the classical and quantum trajectories are identical.

In the Cartesian coordinates (x,y) the equation $\pm1=u_a$ represents a circle for $a=-1$, a cardioid for $a=-1/2$ and for example, a hyperbola for $a=2$. Of course, the trajectories are identical both in the classical and quantum theories. In Fig. 1 we present the departure from the hyperbola when the states in Eq. (33) are used with $n=2$ and $n=3$. It is somewhat surprising that some kind of a shape invariance of the trajectories is preserved, even if the quantum correction Q to classical equations of the motion is different from zero. A similar behavior we have observed for the curves related to other values of a. An example for a closed curve is given in Fig. 2, where the trajectories for $n=2$ and $n=3$ depart from the circle.

From among the states (33) only those given in Eq. (36), when substituted to the Schrödinger equation, lead to the classical Hamilton-Jacobi (HJ) equation without any quantum correction. We shall call the states (36) the classical wave functions. The same HJ equation can be obtained if the Bohm's potential $Q=(-\hbar^2/2m)\Delta R/R$ is substracted in the Schrödinger equation and then the wave function $\psi=R\exp[(i/\hbar)S]$ is used as its solution [50]. This shows, what crucial role is played by the Bohm potential Q in attaining an exact classical limit of quantum mechanics. Note, moreover, that vanishing of Q does not depend at all on the intensity of wave function ψ and only on the functional form of its amplitude R.

In this connection, the last but not least comment of this section concerns the wave function given in Eqs. (7), (12), (16), and (17), i.e., for the potential (23). Its peculiar property is that the amplitude R in Eq. (12) is, for real α, undergoing a jump every 2π and thus the full wave function is not invariant under rotation. Independently of that the condition of $Q=0$ is obeyed exactly. This is because R appears both in the numerator and in the denominator of Q. In such a limit, the quantum wave function becomes the classical one and the former itself plays no significant part in the determination of the classical state of a system (trajectories). Besides, the classical S function may be well defined even in nodal regions where $R=0$, and in the quantum case the S function is undefined at nodes. The classical wave function is thus of a purely descriptive nature, and as such, it is not required to have properties of a "decent" quantum wave functions. We have reported the case of potential (23) here to show that even if there is no quantum correction to the HJ equation, the quantum and classical points of view lead to a number of subtle differences in both approaches.

V. CONCLUSIONS

We have discussed a class of the 2D central potentials and their special states leading to the identical classical and quantum motions. Many of them are shown to undergo along closed curves. This greatly extends the number of potentials having closed orbits that could have been considered in the known theorem [46] by Bertrand.

All the special states of our paper belong to the threshold value of $E=0$ of the continuous spectrum and some of them are particular members of infinitely degenerated zero-energy states of power-law potentials. The states have been found important [39] in creation vortices around nodal points of wave functions. Also the potentials considered here have numerous applications in physics. Examples were mentioned in Sec. II.

The most important point of our paper was, however, studying of the classical limit of quantum mechanics without taking particular values for $\hbar$, for quantum numbers, and for other similar quantities. This was achieved by demanding that the quantum correction Q to the classical HJ equation were zero. In consequence, we could contribute to the still not cleanly resolved problem of the crossover from quantum to classical. In spite of what is claimed in some books on quantum mechanics, the passage from classical to quantum description involves much more than the introduction only of probabilistic arguments into classical mechanics. On the other hand, no simple classical limit of quantum mechanics exists in general as well. We can say that there can be quantum systems with no classical analog and classical systems with no quantum analog. Even in the case of vanishing Q the classical and quantum points of view may be different. The reason is that the condition $Q=0$ distinguishes one representation of quantum mechanics, the position representation.

With this restriction in mind, we can conclude after Holland [51] that: ''*The necessary and sufficient condition for the classical limit is embodied in* $Q\rightarrow 0$, *which may be thought of as a Correspondence Principle*.''

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