# Theory of excess noise in unstable resonator lasers

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We theoretically investigate the quantum dynamics of an unstable resonator laser. Compared to a stable cavity laser of equal gain and loss it exhibits a $K$-fold enhanced linewidth. This excess noise factor $K$ is a measure of the nonorthogonality of the resonator eigenmodes and amounts to an enlargement of the quantum vacuum fluctuations. Using a quantum treatment starting from first principles based on the nonorthogonal eigenmodes, we put previous theoretical predictions onto a more firm ground. While we find a position-dependent enhancement of the spontaneous emission rate into an empty mode of only $\sqrt{K}$, the constructive quantum interference of the spontaneous emission with a single oscillating mode lets the Petermann excess noise factor $K$ reappear in the phase diffusion (laser linewidth). Hence locally enhanced spontaneous emission as well as noise enhanced by interference (amplified spontaneous emission) play an equal role in the origin of excess noise.

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## INTRODUCTION

Since its first discovery more than 30 years ago the phenomenon of excess noise in unstable resonators has led to substantial theoretical confusion and controversies. The effect was first predicted by Petermann [1] as a simple enhancement factor $K$ of the minimal laser linewidth compared to the Schawlow-Townes formula [2,3]. Later on the effect was connected to the nonorthogonality of the resonator modes by Siegman [4]. Experimentally the existence of excess noise and the validity of this rule has been extensively tested by various groups [5–8] finding excellent agreement. The effect not only depends on the reflectivity and size of the mirrors but as well on other parameters as the Fresnel number or the mirror curvature and shape [8–11]. From the beginning it was unclear if the effect should be attributed to an enhanced atomic spontaneous emission rate to be simply plugged into the Schawlow-Townes formula or to an amplification of the spontaneously emitted light field via the laser gain [12]. Since a full quantum-mechanical derivation starting from first principles was hampered by the nonorthogonality of the unstable resonator eigenmodes, different heuristic or numerical integration approaches were developed.

The theoretical modeling of excess noise progressed along several lines. In one approach one avoids the problem of the mode nonorthogonality by embedding the whole system into a larger closed volume (''modes of the universe approach'') and use the corresponding modes for numerical simulations. Bardroff and co-worker here found a close connection between the amount of excess noise and the difference of the spatial distribution of gain and loss [13]. Up to small corrections they could reproduce Siegman's predictions and extend the model to a nonlinear gain medium [14]. Alternatively, Poizat and co-workers showed that central properties of excess noise can be mimicked by a toy input-output model containing only three orthogonal modes but with suitably coupled noise terms [15] which can be generalized to a larger set of modes [16]. However, in both cases the connection and interpretation in terms of the nonorthogonal resonator modes of an unstable resonator geometry is not so straightforward.

In a renewed effort we based our considerations on the infinite set of nonorthogonal matched and adjoint quasimodes [17], which were the basis of the original predictions by Petermann [1] and Siegman [4]. As one example the longitudinal multimode dynamics of very lossy Fabry-Perot resonator with overlapping lines has been studied in great detail [18].

As a first step to get a deeper understanding of excess noise in unstable lasers we developed a quantum theory based on the nonorthogonal modes [17]. In this case, photon creation and annihilation in a certain mode are no longer Hermitian conjugate processes and can show an asymmetry. The main results of this approach have also been confirmed by a more detailed work of Dalton and co-worker [19]. As a first test we tried to identify the microscopic origin of excess noise and considered a single inverted atom inside an unstable resonator as a quantum detector for the local amount of quantum noise [20]. Indeed we found an enhancement of the spontaneous atomic decay into the resonator modes approximately, which for realistic cases was only proportional to the square root of the excess noise factor $\sqrt{K}$. The full $K$ factor appeared only in the limit of a tiny high-$Q$ cavity, where only a single mode dominates the decay. Interestingly, we discovered that one only gets a spatial redistribution of the quantum noise in the cavity and not a global enhancement. Averaging over the volume leads to a total cancelation of the $K$ factor in the total emission rate. Again by external active mode selection we could obtain an excess noise enhancement of spontaneous emission in perfect agreement with other predictions of van der Lee *et al.* [21,22].

As a next process we studied another purely quantum noise driven process, namely, parametric down conversion. Again we found excess noise effects as a locally enhanced pair photon production at the expense of a reduction of the photon correlations [23,24]. For this purpose we generalized known phase-space methods based on the Wigner, the Glauber or the positive-$P$ representation to the case of the nonorthogonal quasimodes. Fortunately, using these methods we are now in the situation to come back to the roots of the problem and study the dynamics of an unstable resonator laser starting from first principles. Here the central question still is to identify the physical mechanism yielding the excess linewidth in unstable resonator lasers.

We will proceed as follows. As a basic model we consider the interaction of the unstable cavity field with a number of pumped two-level atoms. Following a standard laser model (see, e.g., Refs. [25,26]) we first derive coupled stochastic differential equations for the field and atomic variables. Based on our previous findings (cf. Ref. [20]) we can assume a total spontaneous decay rate of the atoms, which is dominated by the transverse nonresonator modes and hence shows no significant excess noise enhancement. Adiabatically eliminating the atomic degrees of freedom, we end up with stochastic differential equations for the field amplitudes similar to the Van der Pol oscillator. Averaging over the atomic position we can obtain simple expressions for the oscillation threshold or the phase diffusion coefficient, which will be the basis for our discussion of the essential physical effects.

## FIELD QUANTIZATION

Let us first review some aspects of the field quantization in terms of nonorthogonal quasimodes [20]. For the free electromagnetic field confined to a volume with partially absorbing boundaries one can find a complete set of quasimodes $\{u_{\mathbf{n}}(\mathbf{x})\}$. They are known as matched modes and defined as self-reproducing field configurations after one full round trip. The multiple index $\mathbf{n}$ includes all longitudinal, transverse, and polarization degrees of freedom. Within the paraxial approximation this corresponds to eigenfunctions of Huygens' integral operator, i.e., $L(u_{\mathbf{n}})=\gamma_{\mathbf{n}}u_{\mathbf{n}}$ (see, e.g., Ref. [28]). The eigenvalues $\gamma_{\mathbf{n}}$ determine the possible frequencies $\omega_{\mathbf{n}}$ and loss rates $\kappa_{\mathbf{n}}$. An analytically solvable example is a one-dimensional symmetric unstable resonator with a Gaussian reflectivity profile [20]. In general these modes are not necessarily orthogonal, but are biorthogonal to a second set of adjoint modes $\{v_{\mathbf{n}}(\mathbf{x})\}$, such that

$$\int_V d\mathbf{x}\; v_{\mathbf{n}}^*(\mathbf{x})u_{\mathbf{m}}(\mathbf{x})=\delta_{\mathbf{nm}}. \tag{1}$$

In fact the adjoint modes correspond to quasimodes traveling in the opposite direction. Whereas the matched modes can be normalized to unity, Eq. (1) implies a normalization constant $K_{\mathbf{n}}\geqslant 1$ for the adjoint modes, called the Petermann excess noise factor [1]. The connection between the excess noise factor in lasers and the norm of the adjoint modes was found by Siegman [4] ten years after the prediction of a $K$-enhanced laser noise [1].

These normalization properties can be rewritten as

$$\int_V d\mathbf{x}\;\; u_{\mathbf{n}}^*(\mathbf{x})u_{\mathbf{m}}(\mathbf{x})=A_{\mathbf{nm}} \quad \text{with} \quad A_{\mathbf{nn}}=1, \tag{2}$$

$$\int_V d\mathbf{x}\;\; v_{\mathbf{n}}^*(\mathbf{x})v_{\mathbf{m}}(\mathbf{x})=B_{\mathbf{nm}} \quad \text{with} \quad B_{\mathbf{nn}}=K_{\mathbf{n}}, \tag{3}$$

where the integral extends over the resonator volume, where $A$ and $B$ are just their inverse. For a stable geometry the adjoint modes are identical to the matched modes and the overlap matrices $A,B$ are the identity matrices. A general property of the quasimodes is their completeness. Hence, every field distribution may be expanded in terms of each set of quasimodes. Accordingly two sets of field operators may be defined associated with an expansion in the matched modes $\{a_{\mathbf{n}},a_{\mathbf{n}}^\dagger\}$ or in the adjoint modes $\{b_{\mathbf{n}}=A_{\mathbf{nm}}a_{\mathbf{m}},b_{\mathbf{n}}^\dagger=A_{\mathbf{mn}}a_{\mathbf{m}}^\dagger\}$. One finds commutation relations reflecting the normalization properties, i.e.,

$$[a_{\mathbf{n}},a_{\mathbf{m}}^\dagger]=B_{\mathbf{nm}}, \tag{4}$$

$$[b_{\mathbf{n}},b_{\mathbf{m}}^\dagger]=A_{\mathbf{nm}}, \tag{5}$$

$$[a_{\mathbf{n}},b_{\mathbf{m}}^\dagger]=\delta_{\mathbf{nm}}. \tag{6}$$

Note that matched mode operators invoke the adjoint overlap matrix. Including field losses and within the Markov approximation the field dynamics may be described by the following master equation [20]:

$$\dot{\rho}=\frac{-i}{\hbar}(H_{eff}\rho-\rho H_{eff}^\dagger)+\sum_{\mathbf{nm}} A_{\mathbf{nm}}(\tilde{\kappa}_{\mathbf{m}}+\tilde{\kappa}_{\mathbf{n}}^*)a_{\mathbf{m}}\rho a_{\mathbf{n}}^\dagger, \tag{7}$$

with

$$H_{eff}=\hbar\sum_{\mathbf{n}}\; \tilde{\omega}_{\mathbf{n}}b_{\mathbf{n}}^\dagger a_{\mathbf{n}}. \tag{8}$$

Here we have introduced complex frequencies $\tilde{\omega}_{\mathbf{n}}=\omega_{\mathbf{n}}-i\kappa_{\mathbf{n}}$ and complex loss rates $\tilde{\kappa}_{\mathbf{n}}=\kappa_{\mathbf{n}}+i\omega_{\mathbf{n}}$, respectively. This effective Hamiltonian may now be used to define generalized Fock states for each mode $n$,

$$|N_n\rangle=\frac{b_n^{\dagger N}}{\sqrt{N!}}|0\rangle, \tag{9}$$

or generalized coherent states,

$$\|\vec{\alpha}\rangle=\exp\{\vec{b}^\dagger\cdot\vec{\alpha}\}\|0\rangle, \tag{10}$$

using the vectorial notation $\vec{b}^\dagger\cdot\vec{\alpha}\equiv\Sigma_{\mathbf{n}}b_{\mathbf{n}}^\dagger\alpha_{\mathbf{n}}$. The operators $b_{\mathbf{n}}^\dagger$ and $a_{\mathbf{n}}$ here correspond to photon creation and annihilation in these modes.

## LASER MODEL

As a next step we couple the electromagnetic field to $N$ two-level atoms of frequency $\omega_A$ described by the Hamiltonians $H_A^{(j)}=\omega_a\sigma_z^{(j)}/2$. The interaction is given by

$$H_{Int}=-i\hbar\sum_{\mathbf{n},j}\;(g_{\mathbf{n}}^{(j)}\sigma_+^{(j)}a_{\mathbf{n}}-\tilde{g}_{\mathbf{n}}^{(j)}b_{\mathbf{n}}^\dagger\sigma_-^{(j)}), \tag{11}$$

and invokes the matched and adjoint mode coupling

$$g_{\mathbf{n}}^{(j)}=\sqrt{\frac{\omega_{\mathbf{n}}}{2\hbar\epsilon_0}}u_n(\mathbf{x}_j)d,\quad \tilde{g}_{\mathbf{n}}^{(j)}=\sqrt{\frac{\omega_{\mathbf{n}}}{2\hbar\epsilon_0}}v_{\mathbf{n}}^*(\mathbf{x}_j)d, \tag{12}$$

where $d$ denotes the atomic dipole moment and $\mathbf{x}_j$ the atomic position. At the end of our calculation we will average over all possible positions in the mode accounting for a homogeneous gain medium. As usual [25–27], we introduce collective atomic operators as $S_-=\Sigma_j g_{\mathbf{n}}^{(j)}\sigma_-^{(j)}$ and neglect the direct interaction between the atomic dipoles. In this model the two atomic levels describe the lasing transition only. All other transitions of the real laser atoms that contribute quantum noise are formally absorbed in a quantum reservoir of inverted harmonic oscillators as a quantum model for pumping (see Refs. [25,26], and references therein). Using the Glauber $P$ representation one may now derive stochastic differential equations for the atomic variables $\{s_-,s_+,s_z\}$ corresponding to $\{S_-,S_+,S_z\}$ and the field variables $\alpha_{\mathbf{n}}$ corresponding to $a_{\mathbf{n}}$ yielding

$$\dot{\alpha}_{\mathbf{n}}=-\tilde{\kappa}_{\mathbf{n}}\alpha_{\mathbf{n}}+\tilde{g}_{\mathbf{n}}s_-, \tag{13}$$

$$\dot{s}_-=-\gamma_\perp s_-+s_z\beta+\Gamma_-, \tag{14}$$

$$\dot{s}_z=-\gamma_\parallel s_z-2\sum_{\mathbf{n}}(s_+\beta+\beta^* s_-)+\gamma N+\Gamma_z, \tag{15}$$

with $\beta=\Sigma_{\mathbf{n}}g_{\mathbf{n}}\alpha_{\mathbf{n}}$ and $\tilde{\kappa}_{\mathbf{n}}=\kappa_{\mathbf{n}}+i(\omega_{\mathbf{n}}-\omega_A)$. Here $\gamma_\perp,\gamma_\parallel,\gamma$ describe the effective damping rates of the atoms and $\Gamma_-,\Gamma_+,\Gamma_z$ denote white-noise sources fulfilling the correlations

$$\langle\Gamma_-\Gamma_+\rangle=\gamma N, \tag{16}$$

$$\langle\Gamma_-\Gamma_-\rangle=2s_-\beta, \tag{17}$$

$$\langle\Gamma_-\Gamma_z\rangle=-2\gamma s_-, \tag{18}$$

$$\langle\Gamma_z\Gamma_z\rangle=2(\gamma N-\gamma_\parallel s_z)-4(\beta^* s_-+s_+\beta). \tag{19}$$

Note that the atomic equations contain no excess noise. The rates $\gamma_\perp,\gamma_\parallel,\gamma$ describe the atom-reservoir coupling including resonator modes as well as nonresonator modes. As the latter are usually dominating the decay we neglect any emission enhancement due to the unstable geometry (cf. Ref. [20]). The origin of excess noise, however, is present in the field equations in the terms involving the adjoint coupling $\tilde{g}_{\mathbf{n}}$, which contain the excess noise factor $\tilde{g}_{\mathbf{n}}\sim\sqrt{K_{\mathbf{n}}}$. These equations [Eqs. (13)–(15)] could now be simulated using standard techniques and in principle the linewidth and intensity noise could be obtained numerically.

## RESULTS

In order to get some analytic expressions, we will now make some further assumptions and adiabatically eliminate the atomic variables, which is valid in the limit $\kappa\ll\gamma,\gamma_\perp,\gamma_\parallel$. After some algebra one finds

$$s_z^{ss}=\frac{\gamma N}{\gamma_\parallel R(\beta)}+\frac{\Gamma_z}{\gamma_\parallel}-\frac{2}{\gamma_\perp\gamma_\parallel}(\beta^*\Gamma_-+\beta\Gamma_+), \tag{20}$$

$$s_-^{ss}=\frac{\beta}{\gamma_\perp}s_z^{ss}+\frac{\Gamma_-}{\gamma_\perp}, \tag{21}$$

with the saturation denominator $R(\beta)=1+4|\beta|^2/\gamma_\perp\gamma_\parallel$. This leads to the following equations for the field variables:

$$\dot{\alpha}_{\mathbf{n}}=-\tilde{\kappa}_{\mathbf{n}}\alpha_{\mathbf{n}}+\frac{\gamma N}{\gamma_\perp\gamma_\parallel}\frac{\tilde{g}_{\mathbf{n}}\beta}{R(\beta)}+\Gamma_n, \tag{22}$$

with

$$\Gamma_n=\frac{\tilde{g}_{\mathbf{n}}\beta}{\gamma_\perp R(\beta)}\left\{\left(1+\frac{2|\beta|^2}{\gamma_\perp\gamma_\parallel}\right)\Gamma_--\frac{2\beta^2}{\gamma_\perp}\Gamma_++\frac{\beta}{\gamma_\parallel}\right\}. \tag{23}$$

Note that all noise terms contain the adjoint coupling $\tilde{g}_{\mathbf{n}}$ and hence the excess noise factor, which is, however, also present in the gain term. To account for a homogeneous gain medium we now have to average over the atomic positions. Taking into account the normalization properties of the matched and adjoint modes [Eqs. (1)–(3)] we may write

$$\dot{\alpha}_{\mathbf{n}}=-\tilde{\kappa}_{\mathbf{n}}\alpha_{\mathbf{n}}+\frac{g^2}{\gamma_\perp}\overline{s_z^{ss}}\alpha_{\mathbf{n}}+\overline{\Gamma}_n, \tag{24}$$

where we have introduced the mean coupling strength $g^2=\omega_A d/2\hbar\epsilon_0$, the steady-state inversion

$$\overline{s_z^{ss}}=\gamma N/\gamma_\parallel\left(1+\frac{4g^2 I}{\gamma_\perp\gamma_\parallel}\right) \tag{25}$$

and the total field photon number $I=\Sigma_{\mathbf{nm}}A_{\mathbf{nm}}\alpha_{\mathbf{n}}^*\alpha_{\mathbf{m}}$.

Above threshold essentially only the lowest-order mode $n=0$ oscillates due to the rapidly increasing loss with mode order. Neglecting higher-order modes completely the resulting equation is now equivalent to a Van der Pol oscillator. For laser oscillation well above threshold,

$$\frac{g^2\gamma N}{\gamma_\perp\gamma_\parallel}>\kappa_0, \tag{26}$$

we expand the field amplitude into intensity and phase $\alpha=\sqrt{I}\exp(i\Psi)$. The laser linewidth is then determined by the phase diffusion coefficient $D_{\Psi\Psi}$ of the oscillating mode, which is derived from the two-time correlation function of the field phase $\Psi$ [1,27]. Following the standard rules for variable transformations for Langevin equations [26] it can be expressed in the form $D_{\Psi\Psi}=(D_{\alpha\alpha^*}-D_{\alpha\alpha}\cos 2\Psi)/2I$. By help of Eqs. (23) and (24) one then finds

$$D_{\Psi\Psi}\approx\frac{1}{2I}\langle\overline{\Gamma\Gamma^*}\rangle\approx\frac{1}{2I}\frac{g^2\gamma_\parallel}{\gamma_\perp^2}\overline{s_z^{ss}}K. \tag{27}$$

Hence, although the spatially averaged atomic spontaneous emission rate into an empty cavity mode is not influenced by excess noise, we still find the excess noise factor $K$ in the laser linewidth for single-mode operation. Note that the noise correlations of higher modes also scale with $K_n = B_{\mathbf{nn}}$ and one in principle could also expect extra noise in multimode operation. However, if many modes would be excited the excess noise factors destructively interfere after spatial averaging. Calculating the total noise intensity $I_N = \Sigma_{\mathbf{nm}} A_{\mathbf{nm}} \langle \overline{\Gamma_{\mathbf{n}}^* \Gamma_{\mathbf{m}}} \rangle$ for all modes shows no enhancement, since $A$ and $B$ are just inverse. Hence excess noise originates in the projection of spatially redistributed multimode spontaneous emission onto a single oscillating mode. This is facilitated through the mode nonorthogonality. Note that similarly the enhanced coupling strength for the mean field vanishes through averaging over the atomic positions due to the biorthogonality relation Eq. (1). Hence, the semiclassical field equations determining the laser threshold condition do not show any excess noise modification.

## CONCLUSIONS

We have revealed the mechanism connecting the enhanced laser linewidth in unstable resonator lasers and mode nonorthogonality. Starting from first principles we could confirm previous claims that the minimal laser linewidth is enhanced by the excess noise factor, determined by the normalization of the adjoint mode. The mechanism behind this is a spatial redistribution of quantum fluctuations and a projection of this onto a single mode. Following the standard phase diffusion model the atomic noise feeds into the field dynamics and suddenly shows an excess noise enhancement via the enhanced adjoint mode coupling [17,20]. Of course, to derive this result we had to use several approximations, which do, however, not go beyond models of a single laser in a stable resonator geometry. In practice single-mode operation to see the full effect might be harder to achieve in unstable lasers.

We think that the present work resolves some of the controversies and confusion on this subject in the past. We have shown that the effect of excess noise is always present in unstable cavities, but needs mode selection to get visible. Considering a single atom in an unstable cavity the situation is similar [20]. Along the optical axis the ground mode is dominant and an enhancement of the spontaneous emission rate may occur, but averaging over the atomic position cancels out the excess noise enhancement. However, there is not a simple relation between the laser linewidth and the spontaneous emission rate of the single atoms. The one-to-one correspondence fails for nonorthogonal cavity modes, and one finds an enhancement factor for the spontaneous emission rate into the empty resonator modes of only approximately $\sqrt{K}$, whereas the laser linewidth shows a linear $K$ dependence.

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