Equivalent linear two-body method for Bose-Einstein condensates in time-dependent harmonic traps

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(Received 4 March 2002; published 11 November 2002)

The recently developed time-independent effective linear two-body method [J. Phys. B: At., Mol. Opt. Phys. **33**, 55 (2000)] has been generalized for time-dependent traps. The method is used to describe the dynamics of trapped Bose-Einstein condensates beyond the Thomas-Fermi regime. The calculated aspect ratios after ballistic expansion are found to be in good agreement with experimental data obtained recently by Görlitz *et al.* [Phys. Rev. Lett. **87**, 130402 (2001)].

DOI: 10.1103/PhysRevA.66.053602 PACS number(s): 03.75.Fi, 05.30.Jp

I. INTRODUCTION

The newly created Bose-Einstein condensates (BEC) of weakly interacting alkali-metal atoms [1] stimulated a number of theoretical investigations (see recent review [2]). According to the Hohenberg theorem [3], the BEC is impossible in (one-dimensional) 1D and 2D homogeneous Bose gases. But BEC can occur in inhomogeneous systems, for example, in atomic traps [4]. The theoretical aspects of the BEC in highly elongated shaped traps (quasi-one-dimensional regime) have been reported in many papers [5–13]. The Gross-Pitaevskii (GP) equation [14] is widely used to describe the experimental results for BEC. In Ref. [5] it was found that the GP predictions for nonlinear dynamics (the aspect ratio after ballistic expansion) are in good agreement with those observed in a recent experiment [15]. We note that, *a priori*, it was not obvious that the GP equation gives the correct description of the nonlinear dynamics of the quasi-1D BEC.

Recently, an alternative method of equivalent linear two-body (ELTB) equations for many-body systems has been developed based on the variational principle [16–19]. It was shown that the ELTB method gives a good result for the ground state of Bose-condensed atoms in harmonic traps. The purpose of this work is to generalize the ELTB method [16–19] for the dynamics of trapped Bose-condensed gases. A recently developed approximation [5] provides the possibility of avoiding extensive numerical integration of the time-dependent ELTB equation. As an example of its application, this approximation is used to describe the ballistic expansion of the BEC after the cigar-shaped trap is switched off. The calculated aspect ratios are found to be in good agreement with the GP calculations and with the recent experimental results [15].

In Sec. II we derive the ELTB method for the time-independent trap. The accuracy of the ELTB method is confirmed by numerical computations. Section III considers the large-N limit. In Sec. IV, we developed an analytical formula for the lower bounds to the ELTB ground-state energy. Section V develops the analytical approximation for the time-dependent ELTB equation. We conclude the paper in Sec. VI with a brief summary.

II. TIME-INDEPENDENT TRAP

For the stationary N-body system, our method for obtaining the ELTB equation consists of following two steps.

The first step is to give the N-body wave function $\psi(\mathbf{r}_1,\mathbf{r}_2,\ldots)$ a particular functional form,

$$\psi(\mathbf{r}_1,\mathbf{r}_2,\ldots)\approx\tilde{\psi}(\zeta_1,\zeta_2,\zeta_3), \tag{1}$$

where ζ_1, ζ_2, and ζ_3 are known functions. We limit ζ's to three variables in order to obtain the ELTB equation, since a relative motion in the two-body problem depends on one vector described by three component variables. We note that approximation (1) allows us to study systems that are not spherically symmetric. The second step is to derive an equation for $\tilde{\psi}(\zeta_1,\zeta_2,\zeta_3)$ by requiring that $\tilde{\psi}$ must satisfy a variational principle

$$\delta\langle\tilde{\psi}|H|\tilde{\psi}\rangle=0 \tag{2}$$

with a subsidiary condition $\langle\tilde{\psi}|\tilde{\psi}\rangle=1$. H is the Hamiltonian. This leads to a linear two-body equation from which both eigenvalues and eigenfunctions can be obtained.

To fix collective coordinates ζ_1, ζ_2, and ζ_3, we note that the hyper radius

$$\rho^2=\sum_i^N (x_i^2+y_i^2+z_i^2) \tag{3}$$

for an isotropic case [16] and also collective variables

$$x^2=\sum_i^N x_i^2,\quad y^2=\sum_i^N y_i^2,\quad z^2=\sum_i^N z_i^2, \tag{4}$$

for an anisotropic case [17–19], yield good results for the dilute BEC of atoms in harmonic traps for both positive and negative scattering length. This success motivates us to introduce more general collective variables:

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$$x^{p_1}=\sum_i^N x_i^{p_1}, \quad y^{p_2}=\sum_i^N y_i^{p_2}, \quad z^q=\sum_i^N z_i^q, \tag{5}$$

where optimal values of p_1, p_2, and q, restricted to even numbers, are to be chosen to minimize the energy.

We consider N identical bosonic atoms confined in a harmonic trap with the following Hamiltonian:

$$H=-\frac{\hbar^2}{2m}\sum_{i=1}^N \Delta_i+\frac{m}{2}\sum_{i=1}^N [\omega_\perp^2(x_i^2+y_i^2)+\omega_z^2 z_i^2] + \sum_{i<j} V_{\rm int}(\vec{r}_i-\vec{r}_j). \tag{6}$$

We use the Fermi pseudopotential approximation for $V_{\rm int}$,

$$V_{\rm int}(\vec{r}_i-\vec{r}_j)=\frac{4\pi\hbar^2 a}{m}\delta(\vec{r}_1-\vec{r}_j), \tag{7}$$

where a is the scattering length. For the eigenfunction ψ of H, we assume the following form:

$$\psi(\mathbf{r}_1,\ldots,\mathbf{r}_N)\approx\tilde{\psi}(r,z), \tag{8}$$

where $r^p=\Sigma_{i=1}^N(x_i^p+y_i^p)$ and $z^q=\Sigma_{i=1}^N z_i^q$. The ELTB method leads to the equation for $\tilde{\psi}$,

$$\left[H_0+N^{1-2/p}\alpha_\perp r^2+N^{1-2/q}\alpha_z z^2+N^{2/p+1/q+1}\frac{\gamma}{r^2 z}\right]\tilde{\psi}=E\tilde{\psi}, \tag{9}$$

where

$$H_0=\frac{\hbar^2}{2m_\perp N^{1-2/p}}\left(\frac{\partial^2}{\partial r^2}+\frac{2N-1}{r}\frac{\partial}{\partial r}\right)-\frac{\hbar^2}{2m_z N^{1-2/q}} \times\left(\frac{\partial^2}{\partial z^2}+\frac{N-1}{z}\frac{\partial}{\partial z}\right), \tag{10}$$

$$m_\perp=\frac{m\Gamma(1/p)(2/p)^{2-2/p}\tilde{\gamma}(2N/p,2-2/p,0)}{2\Gamma(2-1/p)},$$

$$m_z=\frac{m\Gamma(1/q)(1/q)^{2-2/q}\tilde{\gamma}(N/q,2-2/q,0)}{2\Gamma(2-1/q)}, \tag{11}$$

$$\alpha_\perp=\frac{m\Gamma(3/p)\omega_\perp^2}{\Gamma(1/p)(2/p)^{2/p}\tilde{\gamma}(2N/p,2/p,0)},$$

$$\alpha_z=\frac{m\Gamma(3/q)\omega_z^2}{2\Gamma(1/q)(1/q)^{2/q}\tilde{\gamma}(N/q,2/q,0)}, \tag{12}$$

and

$$\gamma=\frac{\pi\hbar^2 a(N-1)\tilde{\gamma}(2N/p,0,-2/p)\tilde{\gamma}(N/q,0,-1/q)}{4m(1/p)^{2-2/p}(1/q)^{1-1/q}\Gamma^2(1/p)\Gamma(1/q)2^{1/q}} \tag{13}$$

with

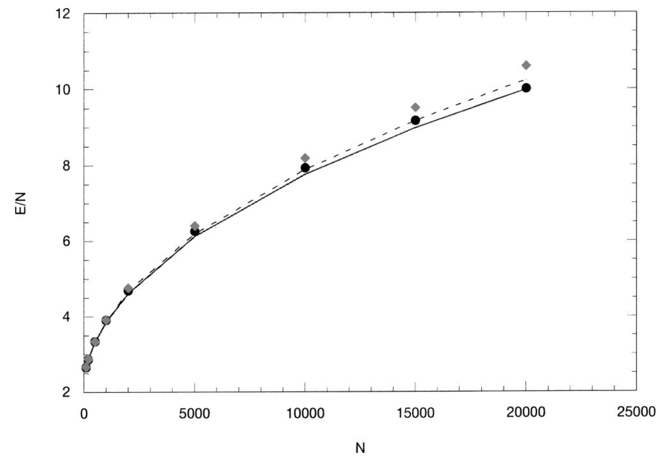

FIG. 1. Ground-state energy per particles, E/N, of ^{87}Rb atoms in a trap with $\lambda=\sqrt{8}$, in units of $\hbar\omega_\perp$, as a function of the number of particles in the trap. Solid circles, diamonds, dashed line, and solid line represent the results of theoretical calculations using the ELTB method, the $p=2$, $q=2$ approximation, the variational Monte-Carlo method [21], and the GP equation [20], respectively.

$$\tilde{\gamma}(z,a,b)=z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)}. \tag{14}$$

Equation (9) simplifies if we introduce the new function $u(r,z)$,

$$\tilde{\psi}(r,z)=\frac{u(r,z)}{r^{(2N-1)/2}z^{(N-1)/2}}. \tag{15}$$

In terms of $u(r,z)$ Eq. (9) reads

$$\left[-\frac{\hbar^2}{2m_\perp N^{1-2/p}}\frac{\partial^2}{\partial r^2}-\frac{\hbar^2}{2m_z N^{1-2/q}}\frac{\partial^2}{\partial z^2}+V_{\rm eff}(r,z)\right] \times u(r,z)=Eu(r,z). \tag{16}$$

The effective potential $V_{\rm eff}(r,z)$ is given by formula

$$V_{\rm eff}(r,z)=\frac{\hbar^2(2N-1)(2N-3)}{8m_\perp N^{1-2/p}r^2}+\frac{\hbar^2(N-1)(N-3)}{8m_z N^{1-2/q}z^2} +\alpha_\perp N^{1-2/p}r^2+\alpha_z N^{1-2/q}z^2+\gamma\frac{N^{2/p+1/q}}{r^2 z}. \tag{17}$$

To study the validity of the ELTB method, we consider an example of the ground-state of ^{87}Rb atoms in a harmonic trap, as investigated in Ref. [20] with the S-wave triplet-spin scattering length $a=100a_B$, where a_B is the Bohr radius, the axial frequency $\omega_z/2\pi=220$ Hz, and asymmetry parameter $\lambda=\omega_z/\omega_\perp=\sqrt{8}$. The calculated energies per particle E/N are compared with those obtained from the solutions of the GP equation [20] and with the variational Monte Carlo (VMC) calculations [21] in Fig. 1. These comparisons show that the optimal choice of the parameters p and q greatly improves the results with $p=q=2$ [17–19]. For $100\leqslant N \leqslant 20\,000$, the difference between our results and those of the solution of the GP equation [20] are about 2%, and the difference between our results and the calculations [21] are less than 1%.

Reference [22] proves that the GP mean-field theory describes correctly the energy and particle density of a dilute 3D Bose gas in a trap to the leading order in the small parameter $\overline{\rho}a^3$ (where $\overline{\rho}$ is the mean density and a is the scattering length) when N is large but a is small with fixed Na.

We note also that for the case of lower dimensions $d<3$, it is known that the quantum-mechanical two-body t-matrix vanishes [23] at low energies. Therefore the replacement of the two-body interaction by the t matrix, as done in deriving the GP mean-field theory, is not correct in general for the $d<3$ case [24].

The ELTB can be applied to $d<3$ cases. To illustrate this let us consider the McGuire-Yang (MY) 1D N-body problem [25] with the Hamiltonian

$$H=-\frac{1}{2}\sum_{i=1}^{N}\frac{d^2}{dx_i^2}+c\sum_{i<j}\delta(x_i-x_j). \tag{18}$$

For the case of $c<0$, there is one bound state for a system of N bosons with the wave function

$$\psi(x)=\exp\left[\frac{c}{2}\sum_{i<j}|x_i-x_j|\right] \tag{19}$$

and the energy of this state is given by

$$E=-c^2\frac{N(N^2-1)}{24}. \tag{20}$$

The MY N-body problem provides a unique possibility of checking the validity of various approximations made for the Schrödinger equation describing N one-dimensional particles interacting via short-range potentials. For this case we seek the ELTB wave function in the form of $\psi(x_1,x_2,\ldots,x_N)\approx\tilde{\psi}(\rho)$, where $\rho^p=\Sigma_{i=1}^N|x_i|^p$. Using Eq. (2) we obtain in the leading order of $N\to\infty$,

$$E=-c^2\frac{N^3}{2^{3+2/p}\Gamma(1/p)\Gamma(2-1/p)}. \tag{21}$$

Choosing an optimal value of p, which minimizes the energy, leads to

$$\frac{E}{c^2N^3}=-0.041\,217\,2. \tag{22}$$

On the other hand, for large N, we have from Eq. (20)

$$\frac{E}{c^2N^3}=-\frac{1}{24}=-0.041\,666\,7. \tag{23}$$

The relative error for the binding energy between Eqs. (22) and (23) is about 1%. Therefore, we have demonstrated that the ELTB method is a very good approximation for the MY N-body problem for large N.

In Ref. [19], it was shown that in the case of large N the ground-state wave function of N bosons confined in a harmonic anisotropic trap can be written in a separable form as

$$\Psi(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N)=\eta(x,y,z)\chi(\Omega), \tag{24}$$

where $x^2=\Sigma_i^N x_i^2$, $y^2=\Sigma_i^N y_i^2$, $z^2=\Sigma_i^N z_i^2$, and Ω is a set of $(3N-3)$ angular variables.

Equation (24) may explain why the ELTB results are expected to be valid and are so close to the GP results for 3D dilute systems ($\overline{\rho}a^3\ll 1$). However Eq. (24) is valid also for nondilute systems, while the GP mean-field theory is proven to be applicable for dilute systems. Therefore, we may expect that the ELTB approach will not be quantitatively equivalent to the solution of the GP equation for these cases. In our future work, we hope also to investigate the large gas parameter regimes [26].

Here we note that if scattering length a is larger than the van der Waals length r_0 [27], there is a regime when the Bose system is dilute, but with respect to r_0, $\overline{\rho}r_0^3\ll 1$ [28]. For these systems the three-body contributions, given by the Efimov effect [29], can become the dominant term of the energy functional [28].

III. LARGE-N LIMIT

To consider the large-N limit, we rescale variables r and z in Eq. (15),

$$r=N^{1/p}\tilde{r},\quad z=N^{1/q}\tilde{z}, \tag{25}$$

and rewrite Eq. (16) as

$$\begin{aligned}&-\frac{\hbar^2}{2m_\perp N^2}\frac{\partial^2}{\partial\tilde{r}^2}-\frac{\hbar^2}{2m_zN^2}\frac{\partial^2}{\partial\tilde{z}^2}+V_{\text{eff}}(\tilde{r},\tilde{z}))u(\tilde{r},\tilde{z})\\&\quad=\frac{E}{N}u(\tilde{r},\tilde{z}).\end{aligned} \tag{26}$$

In the large-N limit, $\tilde{\gamma}$ in Eqs. (11)–(13) is of the order of unity and the expression for $V_{\text{eff}}(\tilde{r},\tilde{z})$ simplifies to

$$\begin{aligned}V_{\text{eff}}(\tilde{r},\tilde{z})&=\frac{\hbar^2}{2mm'_\perp\tilde{r}^2}+\frac{\hbar^2}{8mm'_z\tilde{z}^2}+m\omega_\perp^2\alpha'_\perp\tilde{r}^2+m\omega_z^2\alpha'_z\tilde{z}^2\\&\quad+\frac{\hbar^2aN}{m}\frac{\gamma'}{\tilde{r}^2\tilde{z}},\end{aligned} \tag{27}$$

where

$$m'_\perp=\frac{\Gamma(1/p)(2/p)^{2-2/p}}{2\Gamma(2-1/p)},\quad m'_z=\frac{\Gamma(1/q)(1/q)^{2-2/q}}{\Gamma(2-1/q)}, \tag{28}$$

and

$$\alpha'_\perp=\frac{\Gamma(3/p)}{\Gamma(1/p)(2/p)^{2/p}},\quad \alpha'_z=\frac{\Gamma(3/q)}{2\Gamma(1/q)(1/q)^{2/q}}, \tag{29}$$

$$\gamma' = \frac{\pi p^{2-2/p} q^{1-1/q}}{4\Gamma(1/p)^2\Gamma(1/q)2^{1/q}}. \tag{30}$$

Quantum fluctuations are unimportant in the limit $N\to\infty$, and the most significant contribution to the ground-state energy is given by

$$E = N V_{\rm eff}(r_0, z_0), \tag{31}$$

where r_0 and z_0 are to be obtained from

$$\frac{\partial V_{\rm eff}(r_0,z_0)}{\partial r_0} = \frac{\partial V_{\rm eff}(r_0,z_0)}{\partial z_0} = 0. \tag{32}$$

Obviously Eq. (31) fails if the effective potential does not possess a minimum.

Instead of variables $\tilde{r}$ and $\tilde{z}$, we introduce the new quantities

$$r_t = \tilde{r}/a_\perp, \quad z_t = \tilde{z}/a_\perp, \tag{33}$$

where $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$.

On making the substitution (33), Eqs. (27) and (30) become

$$V_{\rm eff}(r_t, z_t) = \hbar\omega_\perp\left[\frac{1}{2m_1' r_t^2} + \frac{1}{8m_z' z_t^2} + \alpha_\perp' r_t^2 + \lambda^2\alpha_z' z_t^2 + N(a/a_\perp)\frac{\gamma'}{r_t^2 z_t}\right], \tag{34}$$

with

$$\alpha_\perp' r_t^2 - (a/a_\perp)N\frac{\gamma'}{r_t^2 z_t} = \frac{1}{2m_\perp r_t^2},$$

$$2\lambda^2\alpha_z' z_t^2 - (a/a_\perp)N\frac{\gamma'}{r_t^2 z_t} = \frac{1}{4m_z' z_t^2}, \tag{35}$$

and $\lambda = \omega_z/\omega_\perp$.

In the case of large $N(a/a_\perp)$, solutions of Eqs. (35)

$$r_t = [N(a/a_\perp)\gamma'\sqrt{2\lambda^2\alpha_z'/\alpha_\perp'^3}]^{1/5},$$

$$z_t = \left[N(a/a_\perp)\gamma'\left(\frac{\alpha_\perp'^3}{2\lambda^2\alpha_z'}\right)^2\right]^{1/5}\frac{1}{\alpha_\perp'} \tag{36}$$

give for the ground-state energy

$$\frac{E}{N\hbar\omega_\perp(N\lambda a/a_\perp)^{2/5}} = \frac{5}{2^{4/5}}(\alpha_\perp'^2\gamma'^2\alpha_z')^{1/5}. \tag{37}$$

Optimal values of $p=4$ and $q=4$ minimize the energy, Eq. (37), and we have

$$\frac{E}{N\hbar\omega_\perp(N\lambda a/a_\perp)^{2/5}} = 1.081\,99. \tag{38}$$

For the case of large N, one can obtain an essentially exact expression for the ground-state energy by neglecting the kinetic-energy term in the Ginzburg-Pitaevskii-Gross equation (the Thomas-Fermi approximation) [30,31] as

$$\frac{E}{N\hbar\omega_\perp(N\lambda a/a_\perp)^{2/5}} = \frac{5}{7}\left[\left(\frac{15}{8}\right)^2 2\right]^{1/5} = 1.055\,06. \tag{39}$$

Comparing Eq. (38) with Eq. (39), we can see that for the case of large N the ELTB method $(p=4, q=4)$ is a very good approximation, with a relative error of about 2.5% for the binding energy (note that the $p=2$, $q=2$ case gives about 8.2% error for the binding energy).

IV. LOWER BOUNDS

In this section we formulate a lower-bound method for the solution of the ELTB equation (9). Following Ref. [5], we introduce auxiliary Hamiltonians

$$H_\perp = -\frac{\hbar^2}{2m_\perp N^{1-2/p}}\left(\frac{\partial^2}{\partial r^2} + \frac{2N-1}{r}\frac{\partial}{\partial r}\right) + N^{1-2/p}\alpha_\perp r^2,$$

$$H_z = -\frac{\hbar^2}{2m_z N^{1-2/q}}\left(\frac{\partial^2}{\partial z^2} + \frac{N-1}{z}\frac{\partial}{\partial z}\right) + N^{1-2/q}\alpha_z z^2, \tag{40}$$

and

$$\tilde{H}_\perp = \hbar N\sqrt{2\alpha_\perp\gamma_\perp/m_\perp} + N^{1-2/p}\alpha_\perp(1-\gamma_\perp)r^2,$$

$$\tilde{H}_z = \frac{\hbar N}{2}\sqrt{2\alpha_z\gamma_z/m_z} + N^{1-2/q}\alpha_z(1-\gamma_z)z^2, \tag{41}$$

where $\gamma_\perp$ and γ_z are parameters, restricted by $0\leqslant\gamma_\perp<1$ and $0\leqslant\gamma_z<1$, respectively. Using these auxiliary Hamiltonians we write the ELTB energy functional as

$$E = \langle\tilde{\psi}|(H_\perp + H_z - \tilde{H}_\perp - \tilde{H}_z)|\tilde{\psi}\rangle + \langle\tilde{\psi}|\left(\tilde{H}_\perp + \tilde{H}_z + N^{2/p+1/q+1}\frac{\gamma}{r^2 z}\right)|\tilde{\psi}\rangle. \tag{42}$$

Omission of $(H_\perp + H_z - \tilde{H}_\perp - \tilde{H}_z)$ yields a lower bound for the ground-state energy. Projecting $|\tilde{\psi}\rangle$ on the complete basis states $|n\rangle$, obtained from

$$h|n\rangle = \epsilon_n|n\rangle,$$

where

$$h = H_0 + N^{1-2/p}\alpha_\perp\gamma_\perp r^2 + N^{1-2/q}\alpha_z\gamma_z z^2,$$

we get

$$\langle\tilde{\psi}|h_i|\tilde{\psi}\rangle = \sum_n \epsilon_n|\langle\tilde{\psi}|n\rangle\langle n|\tilde{\psi}\rangle| \geqslant \epsilon_1 = N\hbar\left(\sqrt{2\alpha_\perp\gamma_\perp/m_\perp} + \tfrac{1}{2}\sqrt{2\alpha_z\gamma_z/m_z}\right). \tag{43}$$

Therefore a set of optimal values of parameters $\gamma_\perp$ and γ_z, which maximizes our lower bound, will yield an optimal lower-bound value for the ground-state energy given by

TABLE I. Calculated results for the lower bound $E_{\rm low}/N$, Eq. (36), and for the ground-state energy per particle, E/N, ELTB Eq. (9), in units of $\hbar\omega_\perp$, for the same case as in Fig 1. Δ is defined as $\Delta=(E-E_{\rm low})/E$.

N	$E_{\rm low}/N$	E/N	Δ
100	2.660 93	2.662 86	7.2×10^{-4}
200	2.866 33	2.867 97	5.7×10^{-4}
500	3.342 59	3.343 78	3.6×10^{-4}
1000	3.913 11	3.913 92	2.1×10^{-4}
2000	4.696 83	4.697 42	1.3×10^{-4}
5000	6.255 03	6.255 39	5.8×10^{-5}
10000	7.940 76	7.941 01	3.1×10^{-5}
15000	9.172 08	9.172 27	2.1×10^{-5}
20000	10.187 7	10.187 9	2.0×10^{-5}

$$\frac{E}{N}=\max_{\gamma_\perp,\gamma_z}\left[\hbar\left(\sqrt{2\alpha_\perp\gamma_\perp/m_\perp}+\tfrac{1}{2}\sqrt{2\alpha_z\gamma_z/m_z}\right)+\frac{5}{2^{4/5}}[\gamma^2\alpha_\perp^2(1-\gamma_\perp)^2\alpha_z(1-\gamma_z)]^{1/5}\right]. \quad (44)$$

Using this approximation we calculate the energy per particle, E/N, for the same case as in Fig. 1. The calculated results are compared with those obtained from the numerical solutions of the ELTB equation in Table I. These comparisons show that the analytical approximation, Eq. (44), gives excellent results. The difference between E/N, Eq. (44) and numerical solutions of the ELTB equation is less than 0.07% for $100\leqslant N\leqslant5000$ and less than 0.006% for $N>5000$.

V. TIME-DEPENDENT TRAP

In this section, we consider N identical bosonic atoms confined in a time-dependent harmonic trap with the Hamiltonian

$$H(t)=-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\Delta_i+\frac{m}{2}\sum_{i=1}^{N}[\omega_\perp^2(t)(x_i^2+y_i^2)+\omega_z^2(t)z_i^2]+\frac{4\pi\hbar^2 a}{m}\sum_{i<j}\delta(\vec{r}_i-\vec{r}_j).$$

To obtain the wave function, we apply the variational principle

$$\delta A=0, \quad (45)$$

where the action integral A is given by

$$A=\int_{t_0}^{t_1}\langle\tilde{\Psi}|\left[i\hbar\frac{\partial}{\partial t}-H(t)\right]|\tilde{\Psi}\rangle dt, \quad (46)$$

and $\tilde{\Psi}(r,z,t)$ is the trial wave function.

This generalizes the time-independent ELTB equation [16–19] for time-dependent traps and leads to the equation

$$i\hbar\frac{\partial\tilde{\Psi}}{\partial t}=\left[H_0+N^{1-2/p}\alpha_\perp(t)r^2+N^{1-2/q}\alpha_z(t)z^2+N^{2/p+1/q+1}\frac{\gamma}{r^2 z}\right]\tilde{\Psi}, \quad (47)$$

where

$$\alpha_\perp(t)=\frac{m\Gamma(3/p)\omega_\perp^2(t)}{\Gamma(1/p)(2/p)^{2/p}\tilde{\gamma}(2N/p,2/p,0)},$$

$$\alpha_z(t)=\frac{m\Gamma(3/q)\omega_z^2(t)}{2\Gamma(1/q)(1/q)^{2/q}\tilde{\gamma}(N/q,2/q,0)}, \quad (48)$$

with the initial condition $\tilde{\Psi}(r,z,0)=\tilde{\psi}(r,z)$, where $\tilde{\psi}(r,z)$ is a ground-state solution of the time-independent ELTB equation (9) with $\alpha_\perp(0)=\alpha_\perp$, $\alpha_z(0)=\alpha_z$.

We substitute the following Eq. (49) into Eq. (47) [5,32,33]:

$$\tilde{\Psi}(r,z,t)=\frac{\Phi(r/\lambda_\perp(t),z/\lambda_z(t),t)}{\lambda_\perp^N(t)\lambda_z^{N/2}(t)}\times\exp[-i\beta(t)+i(f_\perp(t)r^2+f_z(t)z^2)], \quad (49)$$

where

$$f_\perp(t)=-\frac{\dot{\lambda}_\perp(t)m_\perp N^{1-2/p}}{2\hbar\lambda_\perp(t)},\quad f_z(t)=-\frac{\dot{\lambda}_z(t)m_z N^{1-2/q}}{2\hbar\lambda_z(t)}, \quad (50)$$

and β, $\lambda_\perp$, and λ_z are solutions of the following equations:

$$\hbar\dot{\beta}=\frac{E}{\lambda_\perp^2\lambda_z}+\hbar N\sqrt{2\alpha_\perp(0)\gamma_\perp/m_\perp}\left(\frac{1}{\lambda_\perp^2}-\frac{1}{\lambda_\perp^2\lambda_z}\right)+\frac{\hbar N}{2}\sqrt{2\alpha_z(0)\gamma_z/m_z}\left(\frac{1}{\lambda_z^2}-\frac{1}{\lambda_\perp^2\lambda_z}\right),\quad \beta(0)=0, \quad (51)$$

$$\frac{m_\perp}{2}\ddot{\lambda}_\perp=-\alpha_\perp(t)\lambda_\perp+\frac{\alpha_\perp(0)\gamma_\perp}{\lambda_\perp^3}+\frac{\alpha_\perp(0)(1-\gamma_\perp)}{\lambda_\perp^3\lambda_z},$$

$$\frac{m_z}{2}\ddot{\lambda}_z=-\alpha_z(t)\lambda_z+\frac{\alpha_z(0)\gamma_z}{\lambda_z^3}+\frac{\alpha_z(0)(1-\gamma_z)}{\lambda_\perp^2\lambda_z^2},$$

$$\lambda_\perp(0)=1,\ \dot{\lambda}_\perp(0)=0,\ \lambda_z(0)=1,\ \dot{\lambda}_z(0)=0. \quad (52)$$

The above substitution yields the following time-dependent ELTB equation:

$$i\hbar\frac{\partial\Phi}{\partial t}=\left[\frac{H_\perp-\tilde{H}_\perp}{\lambda_\perp^2}+\frac{H_z-\tilde{H}_z}{\lambda_z^2}+\frac{1}{\lambda_\perp^2\lambda_z}\left(\tilde{H}_\perp+\tilde{H}_z+N^{2/p+1/q+1}\frac{\gamma}{r^2 z}-E\right)\right]\Phi. \quad (53)$$

By neglecting $(H_\perp - \tilde{H}_\perp)$ and $(H_z - \tilde{H}_z)$ in Eq. (53), we obtain a generalization of the approximation of Ref. [5] to the time-dependent ELTB equation

$$\tilde{\Psi}(r,z,t) = \frac{\tilde{\psi}(r/\lambda_\perp(t), z/\lambda_z(t))}{\lambda_\perp^N(t)\lambda_z^{N/2}(t)} \times \exp[-i\beta(t) + i(f_\perp(t)r^2 + f_z(t)z^2)], \quad (54)$$

where all the dynamics is in the evolution of the scaling parameters $\lambda_\perp(t)$ and $\lambda_z(t)$, Eq. (52).

The aspect ratio of the cloud in the approximation, Eq. (54) is given by

$$R(t) = \sqrt{\frac{\overline{x_1^2}(t)}{\overline{z_1^2}(t)}} = \frac{\lambda_\perp(t)}{\lambda_z(t)} R(0). \quad (55)$$

As an example, we consider the application of the above results to the experimental data with ^{23}Na atoms obtained in the Ioffe-Pritchard-type magnetic trap with radial and axial trapping frequencies of $\omega_\perp/(2\pi) = 360$ Hz and $\omega_z/(2\pi) = 3.5$ Hz [15], respectively. In our analysis, we use $a = 2.75$ nm, $t = 4$ ms, and $a/a_\perp = 2.488 \times 10^{-3}$, where $a_\perp = \sqrt{\hbar/m\omega_\perp}$. As in Ref. [5], we consider a sudden and total opening of the trap at $t=0$. For this case, Eqs. (52) become

$$\frac{d^2\lambda_\perp}{d\tau^2} = b_\perp\left(\frac{\gamma_\perp}{\lambda_\perp^3} + \frac{1-\gamma_\perp}{\lambda_\perp^3\lambda_z}\right),$$

$$\frac{d^2\lambda_z}{d\tau^2} = b_z\left(\frac{\gamma_z}{\lambda_z^2} + \frac{1-\gamma_z}{\lambda_\perp\lambda_z}\right)\epsilon^2, \quad (56)$$

where $\tau = \omega_\perp(0)t$ and $\epsilon = \omega_z(0)/\omega_\perp(0) \ll 1$, and

$$b_\perp = \frac{p^2\Gamma(3/p)\Gamma(2-1/p)}{\Gamma^2(1/p)\tilde{\gamma}(2N/p,2/p,0)\tilde{\gamma}(2N/p,2-2/p,0)},$$

$$b_z = \frac{q^2\Gamma(3/q)\Gamma(2-1/q)}{\Gamma^2(1/q)\tilde{\gamma}(N/q,2/q,0)\tilde{\gamma}(N/q,2-2/q,0)}. \quad (57)$$

To zeroth order in ϵ^2, we have $\lambda_z = 1$ and $\lambda_\perp = \sqrt{1 + b_\perp\tau^2}$. For the experimental conditions [15], the terms in ϵ^2 are negligible. Our calculated results for $R(t)$ are compared with those obtained from the solution of the GP equation [5], with the Thomas-Fermi (TF) approximation, and with experimental data [15] in Fig. 2. This comparison shows that the present results, Eqs. (56), are in good agreement with the GP calculations and with the recent experimental results [15].

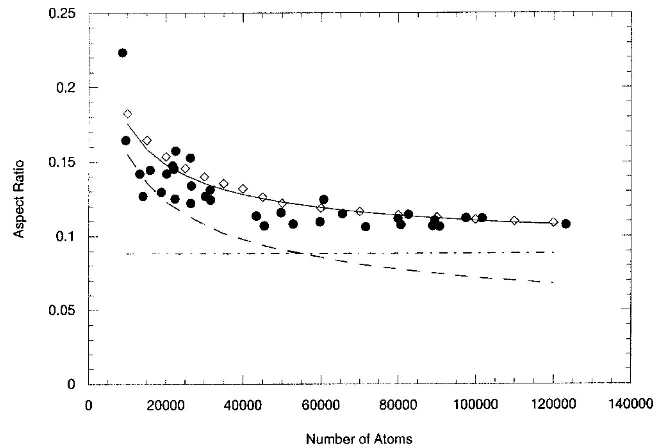

FIG. 2. Aspect ratio R of the ^{23}Na atom cloud after a ballistic expansion of $t=4$ ms, as a function of the number of atoms N, with $\omega_\perp(0) = 2\pi \times 360$ Hz, $\omega_z(0) = 2\pi \times 3.5$ Hz. Experimental data (●) from Ref. [15] are compared with the results of theoretical calculations using Eqs. (48) (——), the GP equations (◇), 1D approximation (— — —), and the TF approximation (—·—·—·—).

We consider 1D approximation, when the radial motion of the atoms becomes frozen and is governed by the ground-state wave function of the radial harmonic oscillator [6,8]. From Fig. 2 one can see that the 1D approximation provides reasonable results with a relative error of less than 10% for the case of $N \leqslant 10^4$.

One can also see that even for the relatively large TF parameter, $Na/a_{ho} \approx 100$, $a_{ho} = \sqrt{h/(m\omega_\perp^{2/3}\omega_z^{1/3})}$, the TF approximation is not valid, the error is larger than 20%.

VI. SUMMARY AND CONCLUSION

In summary, we have generalized the time-independent ELTB method [16–19] to the time-dependent case. As examples of application, we have studied the problem of the ballistic expansion of the condensate after the cigar-shaped traps are switched off. The approximation developed in Ref. [5] provides a possibility of avoiding extensive numerical integrations of the time-dependent ELTB equation.

The calculated aspect ratios after ballistic expansion are found to be in a good agreement with experimental data obtained recently by a group at MIT.

ACKNOWLEDGMENTS

We are grateful to W. Ketterle and J. M. Vogels for providing us with the MIT experimental data. We also thank J. L. DuBois and H. R. Glyde for making their variational Monte Carlo results available to us.

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