Theoretical studies of the long-range Coulomb potential effect on photoionization by strong lasers

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Using the second-order Coulomb-corrected Volkov function as a continuum state, we derive quantum mechanically analytical formulas for the photoionization rate of hydrogen atoms irradiated by a linearly polarized electric field in the tunneling regime. From the analytical formula is directly drawn the important conclusion that the role of the first-order Coulomb correction is to reduce the ionization potential. As a result, the photoionization rate is enhanced compared with that in the absence of the Coulomb correction. In addition, the second-order correction modifies the Keldysh parameter, decreases the binding energy, and increases the photoionization rates relative to those of the first-order Coulomb correction. We estimate the effects of the respective Coulomb corrections on the resonance structure of the photoionization rate, Keldysh parameter, and ponderomotive energy.

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I. INTRODUCTION

In recent years, owing to the rapid development of laser technology, much attention has been paid to photoionization and photodissociation processes of atoms and molecules. In order to understand and/or extend our knowledge to complicated atomic and molecular systems, it is necessary to clarify the mechanism of photoionization processes of hydrogen or hydrogenlike atoms in more detail. Even for the hydrogen atom, it is not possible to claim that we know the detailed mechanism of the photoionization process.

The most important and well-known theories for description of the photoionization rate of hydrogenlike atoms were proposed by Keldysh, Faisal, and Reiss (KFR) [1–3]. In the KFR theory, due to the fact that it presumes photoionization from a short-range potential, whereas the real potential exerts a long-range Coulomb force between the residual core and the ionizing electron, one would not expect good agreement between experiments and theoretical predictions. However, at relatively high intensities and high orders, where external electromagnetic field effects on the electron become dominant, good agreement can be expected. The principal cause of the problem mentioned above is that Keldysh used the Gordon-Volkov function [4,5] as the final state of the photoionized electron. This normal Volkov function is an exact solution of the quantum-mechanical equations of motion for a free electron in a plane-wave electromagnetic field. However, in the presence of an atomic potential among particles, the Volkov function is not necessarily an exact description of the motion of the photoionizing electron. In order to incorporate appropriately the effect of the atomic potential into the Volkov function, there have been many efforts so far [6–16] and the revised Volkov function is usually called the "Coulomb-Volkov function."

Therefore, it is essential to construct a theory to bridge the large discrepancy between the results based on the normal Volkov function and on the Coulomb-corrected Volkov function.

For this purpose, Reiss and Krainov [16] improved the free-electron Volkov function and obtained a first-order Coulomb-corrected Volkov function for an electron irradiated by a circularly polarized electric field. In addition, they demonstrated numerically that the time-dependent phase shift included in the Coulomb-Volkov function leads to a greatly enhanced photoionization rate compared to that calculated by using the free-electron Volkov function.

Later, based on the development of Reiss and Krainov, Bauer [17] introduced one more unitary transformation for the Hamiltonian and the Coulomb-Volkov function and succeeded in deriving a second-order Coulomb-corrected Volkov function for an electron irradiated by a circularly polarized electric field. However, in his paper, the kind of effect introduced into the photoionization rate formulas by such an improvement of the Volkov function was not clarified.

Based on their ideas, we derive analytical photoionization rate formulas for hydrogen atoms irradiated by a linearly polarized electric field in the presence of a Coulomb interaction.

Our strategy is as follows. We express the final ionized state using the Coulomb-Volkov function instead of the normal Volkov function that was utilized by Keldysh. As in our previous paper [18], we avoid using the saddle-point method for the integration of $L(\vec{p})$ of Eq. (15) in Ref. [1]. Instead, we use the residue theorem for its evaluation. The dependence of the preexponential factors on $\vec{p}$ (the momentum vector of the electron), ignored by Keldysh in the expression

$$2\sqrt{\pi a_0^3}\frac{I_0}{eFa_0}\frac{\hbar\omega}{(1-u_s^2)^{1/2}} \times\exp\left[\frac{i}{\hbar\omega}\int_0^{u_s}\left\{I_0+\frac{1}{2m}\left(\vec{p}+\frac{e\vec{F}}{\omega}v\right)^2\right\}\frac{dv}{(1-v^2)^{1/2}}\right], \tag{1.1}$$

will be incorporated in our derivation. In addition, we change the summation of $S(\gamma,x)$ in Eq. (18) of Ref. [1] into an integration with respect to n. It will be shown that these modifications allow us to obtain insightful analytical expressions for the photoionization rate in the simultaneous presence of an electric field and a Coulomb potential.

The present paper is organized as follows. In Sec. II, we show a derivation of Keldysh-like photoionization rate formulas for hydrogenlike atoms, taking into account the main influence of the long-range Coulomb potential on the Volkov function. As for the initial state, we focus on the $1s$ state of hydrogenlike atoms. In Sec. III, we numerically show the validity of performing the integration over n for $S(\gamma,x)$ and discuss the features of our formulas. Section IV is devoted to the concluding remarks.

II. THEORY

Let us consider a hydrogenlike atom in the $1s$ state which is excited directly to the continuum state. We assume that the initial $1s$ state is not perturbed by the laser field.

First of all, we have to define the wave function of the continuum state in such a way that it involves the effect of the long-range Coulomb potential (Coulomb-Volkov function). The Schrödinger equation to be solved for the continuum state $\psi_A(\vec{r},t)$ is expressed as

$$i\hbar\frac{\partial\psi_A(\vec{r},t)}{\partial t}=\left\{\frac{1}{2m}[-i\hbar\vec{\nabla}-e\vec{A}(t)]^2+V(\vec{r})\right\}\psi_A(\vec{r},t), \tag{2.1}$$

where $V(\vec{r})$ is the potential between the nucleus and the electron and $\vec{A}(t)$ is the vector potential. Hereafter we adopt the dipole approximation: $\vec{A}(t)$ is independent of the location $\vec{r}$ of the electron.

Let us transform the wave function in Eq. (2.1), $\psi_A(\vec{r},t)$, to $\Phi(\vec{r},t)$ by the following relation:

$$\psi_A(\vec{r},t)=\exp\left(-\frac{ie^2}{2m\hbar}\int_{-\infty}^{t}\vec{A}^2(\tau)d\tau\right)\times\exp(\vec{\alpha}\cdot\vec{\nabla})\Phi(\vec{r},t), \tag{2.2}$$

where

$$\vec{\alpha}(t)=\frac{e}{m}\int_{-\infty}^{t}\vec{A}(\tau)d\tau. \tag{2.3}$$

Here, it is assumed that the laser field is adiabatically turned on at $t=-\infty$. Note that the phase-factor transformation is not applied [19]. The quantity $\vec{\alpha}(t)$ denotes that the classical electron driven by the laser field $\vec{A}(\tau)$ has a quiver motion of the vector radius.

Using the Kramers-Henneberger transformation [20], it is shown that the wave function $\Phi(\vec{r},t)$ satisfies the following Schrödinger equation:

$$i\hbar\frac{\partial\Phi(\vec{r},t)}{\partial t}=\left\{-\frac{\hbar^2\vec{\nabla}^2}{2m}+V\left(\left|\vec{r}-\vec{\alpha}(t)\right|\right)\right\}\Phi(\vec{r},t). \tag{2.4}$$

Equation (2.4) is the space-translated version of the Schrödinger equation. In the special case of the Coulomb potential $V(\vec{r})=-Ze^2/r$, where Z is the nuclear charge, V in Eq. (2.4) is given by

$$V(|\vec{r}-\vec{\alpha}(t)|)=-\frac{Ze^2}{|\vec{r}-\vec{\alpha}(t)|}. \tag{2.5}$$

Equation (2.4) cannot be solved in simple closed form owing to the presence of the term of Eq. (2.5).

We assume that

$$\alpha_0\gg a_0/Z, \tag{2.6}$$

where α_0 is the radius of the quiver motion of a classical electron in a laser field. The quantity a_0 is the Bohr radius. In the case of a linearly polarized laser field, the maximum quiver radius α_0 can be determined as [21,22]

$$\alpha_0=\frac{eF}{m\omega^2}, \tag{2.7}$$

where F is the laser amplitude and ω is the laser frequency. Because we are interested in the tunneling process, the assumption (2.6) is valid in most cases.

In the approximation, we obtain in the second-order Coulomb correction

$$V(|\vec{r}-\vec{\alpha}(t)|)\approx-Ze^2\left\{\frac{1}{\alpha_0}+\frac{\vec{r}\cdot\vec{\alpha}(t)}{\alpha_0^3}+O(\alpha_0^{-3})\right\}. \tag{2.8}$$

For a Taylor expansion similar to that in Eq. (2.8), Reiss and Krainov and Bauer assumed that the incident laser is *circularly* polarized. In this case, the absolute value of $\vec{\alpha}(t)$ is constant so that the expansion (2.8) is valid for any time t. However, we are interested in the *linearly* polarized electric field. In this case, the absolute value of $\vec{\alpha}(t)$ varies with time so that the Taylor expansion (2.8) is not necessarily adequate.

Based on a rough estimation, let us discuss the conditions of validity of using the approximation Eq. (2.8) in the case of the linearly polarized electric field. The mean tunneling time t_t is given by

$$t_t=\frac{\sqrt{mI_0}}{\sqrt{2}eF}, \tag{2.9}$$

where I_0 is the ionization potential. Usually, tunneling can take place when t_t is less than half the period of the incident laser. Therefore, until the tunneling is almost over, the phase of the laser field will roughly change from zero to ωt_t, which is of a small magnitude. In this case, $\vec{\alpha}(t)$ will hardly change. Within this restriction, we can apply the approximation of Eq. (2.8).

Substituting Eq. (2.8) into Eq. (2.4), the approximate Schrödinger equation is of the form

$$i\hbar\frac{\partial\Phi(\vec{r},t)}{\partial t}=\left[-\frac{\hbar^2\vec{\nabla}^2}{2m}-Ze^2\left\{\frac{1}{\alpha_0}+\frac{\vec{r}\cdot\vec{\alpha}(t)}{\alpha_0^3}\right\}\right]\Phi(\vec{r},t). \tag{2.10}$$

Let us transform the wave function in Eq. (2.10), $\Phi(\vec{r},t)$, into $\Phi'(\vec{r},t)$ by the following unitary transformation:

$$\Phi'(\vec{r},t)=\exp\{i\beta\vec{A}(t)\cdot\vec{r}\}\Phi(\vec{r},t), \tag{2.11}$$

where

$$\beta=\frac{Ze^3}{\hbar m\omega^2\alpha_0^3}. \tag{2.12}$$

Then, the wave function $\Phi'(\vec{r},t)$ satisfies

$$i\hbar\frac{\partial\Phi'(\vec{r},t)}{\partial t}=\left[\frac{\hbar^2}{2m}\{-i\vec{\nabla}-\beta\vec{A}(t)\}^2-\frac{Ze^2}{\alpha_0}\right]\Phi'(\vec{r},t). \tag{2.13}$$

The wave function $\Phi'(\vec{r},t)$ can be directly obtained from Eq. (2.13):

$$\Phi'(\vec{r},t)=\exp\left[\frac{i}{\hbar}\left\{\vec{p}\cdot\vec{r}-\frac{p^2}{2m}t-\frac{1}{2m}\int_{-\infty}^{t}d\tau[-2\hbar\beta\vec{p}\cdot\vec{A}(\tau)+\hbar^2\beta^2\vec{A}^2(\tau)]+\frac{Ze^2}{\alpha_0}t\right\}\right]. \tag{2.14}$$

The wave function $\Phi(\vec{r},t)$ is easily obtained from the transformation of Eq. (2.11). Therefore, the second-order Coulomb-corrected Volkov function in the velocity gauge $\psi_A(\vec{r},t)$ is obtained from the transformation of Eq. (2.2):

$$\psi_A(\vec{r},t)=\exp\left[\frac{i}{\hbar}\left\{\vec{p}\cdot\vec{r}-\frac{p^2}{2m}t-\frac{1}{2m}\int_{-\infty}^{t}d\tau[-2e\vec{p}\cdot\vec{A}(\tau)+e^2\vec{A}^2(\tau)]+\frac{Ze^2}{\alpha_0}t-\frac{1}{2m}\int_{-\infty}^{t}d\tau[-2\hbar\beta\vec{p}\cdot\vec{A}(\tau)+\hbar^2\beta^2\vec{A}^2(\tau)]-\hbar\beta\vec{A}(t)\cdot[\vec{r}+\vec{\alpha}(t)]\right\}\right]. \tag{2.15}$$

Equation (2.15) can further be transformed into the length gauge:

$$\psi_{\vec{p}}(\vec{r},t)=\exp\left[\frac{i}{\hbar}\left\{[\vec{p}-e\vec{A}(t)]\cdot\vec{r}-\frac{p^2}{2m}t-\frac{1}{2m}\int_{-\infty}^{t}d\tau[-2e\vec{p}\cdot\vec{A}(\tau)+e^2\vec{A}^2(\tau)]+\frac{Ze^2}{\alpha_0}t-\frac{1}{2m}\int_{-\infty}^{t}d\tau[-2\hbar\beta\vec{p}\cdot\vec{A}(\tau)+\hbar^2\beta^2\vec{A}^2(\tau)]-\hbar\beta\vec{A}(t)\cdot[\vec{r}+\vec{\alpha}(t)]\right\}\right]. \tag{2.16}$$

In Eq. (2.16), the term containing α_0 is the first-order and those involving β are the second-order Coulomb corrections. Note that, if the second-order Coulomb correction can be neglected ($\beta=0$), the Coulomb-Volkov function utilized in Ref. [1] is recovered. The above mentioned derivation is essentially the same as that reported by Bauer [17]. As was pointed out above, however, Bauer did not derive the photoionization rate in his paper [17]. In the following, we show the derivation of analytical formulas using the second-order-corrected Coulomb-Volkov function defined by Eq. (2.16).

To be specific, we consider a hydrogenlike one-electron atom in the presence of a monochromatic electric field. The rate of photoionization w_0 for direct transition from the ground bound state to the continuum state is given by

$$w_0=\frac{2}{\hbar^2}\lim_{T\to\infty}\mathrm{Re}\int\frac{d^3p}{(2\pi\hbar)^3}\int_{-\infty}^{T}dt\cos(\omega T)\cos(\omega t)V_0^*\left(\vec{p}+\frac{e\vec{F}}{\omega}\sin(\omega T)\right)V_0\left(\vec{p}+\frac{e\vec{F}}{\omega}\sin(\omega t)\right)$$
$$\times\exp\left[\frac{i}{\hbar}\int_T^t d\tau\left\{I_0'+\frac{1}{2m}\left(\vec{p}+\frac{e\vec{F}}{\omega}\sin(\omega\tau)\right)^2+\frac{1}{2m}\left(\frac{2\hbar\beta}{\omega}\vec{p}\cdot\vec{F}\sin(\omega\tau)+\frac{\hbar^2\beta^2\vec{F}^2}{\omega^2}\sin^2(\omega\tau)\right)+2\tilde{B}\cos(2\omega\tau)\right\}\right], \tag{2.17}$$

where

$$I_0'=I_0+\tilde{A}, \tag{2.18}$$

$$\tilde{A}=-\frac{Ze^2}{\alpha_0}, \tag{2.19}$$

$$I_0 = -E_g = \frac{Z^2 e^2}{2a_0} \quad (\text{ionization potential}), \tag{2.20}$$

$$\overline{e} = e + \hbar\beta, \tag{2.21}$$

$$\widetilde{B} = -\frac{\hbar\beta e F^2}{2m\omega^2} = \frac{\widetilde{A}}{2}, \tag{2.22}$$

$$V_0(\vec{p}) = \langle \exp(i\vec{p}\cdot\vec{r}/\hbar) | e(\vec{F}\cdot\vec{r}) | \psi_g(\vec{r}) \rangle, \tag{2.23}$$

and $\vec{F}$ is the maximum amplitude of the electric field including the linear polarization.

At this point, it should be noted that the essential difference between this work and the previous study [18] is that the effective ionization potential I_0' is smaller than that when the usual Volkov function is used (I_0) by the term $\widetilde{A}(=-Ze^2/\alpha_0)$. The effect of this phase-shift term is of course a decrease of the binding energy. Notice that in the calculation the quiver radius α_0 cannot be too small, because then the effective binding energy I_0' in Eq. (2.17) would be negative, which is unphysical.

Carrying out the integration over t and taking infinity for T yields

$$w_0 = \frac{2\pi}{\hbar} \int \frac{d^3 p}{(2\pi\hbar)^3} |L(\vec{p})|^2 \sum_{n=-\infty}^{\infty} \delta\left(\widetilde{I}_0'' + \frac{p^2}{2m} - n\hbar\omega \right), \tag{2.24}$$

where

$$\widetilde{I}_0'' = I_0' + \frac{E^2 F^2}{4m\omega^2} = I_0 + \frac{E^2 F^2}{4m\omega^2} + \widetilde{A}, \tag{2.25}$$

$$E = \sqrt{e^2 + \hbar^2 \beta^2}, \tag{2.26}$$

and

$$L(\vec{p}) = \frac{16 i e I_0^3 \sqrt{\pi a_0^7}}{\pi\hbar} \times \oint du \frac{\vec{F}[\vec{p} + (\overline{e}\vec{F}/\omega)u]}{\{I_0 + (1/2m)[\vec{p} + (\overline{e}\vec{F}/\omega)u]^2\}^3} \times \exp\left[\frac{i}{\hbar\omega} \int_0^u \frac{dv}{\sqrt{1-v^2}} \left\{ I_0' + \frac{1}{2m}\left(\vec{p} + \frac{\overline{e}\vec{F}}{\omega} v \right)^2 + 2\widetilde{B}(1-v^2) \right\} \right]. \tag{2.27}$$

Notice that the singular point of the integrand in Eq. (2.27) is different from the zero point of the time derivative of the exponent. This is the point most different from Keldysh's theory, where they coincide.

One of the singularity points of $L(\vec{p})$, u_s, reads

$$u_s = \overline{\gamma}\left(-\frac{p}{\sqrt{2mI_0}} \cos\theta + i \sqrt{\frac{p^2 \sin^2\theta}{2mI_0} + 1} \right), \tag{2.28}$$

where

$$\vec{p}\cdot\vec{F} = pF\cos\theta \tag{2.29}$$

and the "modified" Keldysh parameter

$$\overline{\gamma} = \frac{\omega\sqrt{2mI_0}}{\overline{e}F}. \tag{2.30}$$

Note that the second-order Coulomb correction of the Volkov function β is introduced in the Keldysh parameter, while the first-order correction does not affect the adiabatic parameter. In addition, we can see that the modified Keldysh parameter $\overline{\gamma}$ is always smaller than the original Keldysh parameter,

$$\gamma = \frac{\omega\sqrt{2mI_0}}{eF}, \tag{2.31}$$

provided that the laser amplitude F and the frequency ω are the same: the photoionization rate tends toward the tunneling ionization region in the presence of a Coulomb potential.

In principle, contour integration with the residue theorem of Eq. (2.24) can be carried out including the $\vec{p}$ dependence of all the preexponential factors, which leads to the most accurate formula. However, it is sometimes quite cumbersome and difficult. Therefore, in the present work, the simplest expression will be presented. When the Coulomb correction up to second order is included and all the preexponential factors of Eq. (2.24) are $\vec{p}$ independent, we have

$$w_0 = N(\gamma, \overline{\gamma}, \omega, I_0, \widetilde{A}, \widetilde{B}) \times \exp\left[-\frac{2}{\hbar\omega} \left\{ \widetilde{I}_0 \left(\sinh^{-1}\overline{\gamma} - \frac{\overline{\gamma}\sqrt{1+\overline{\gamma}^2}}{1+2\overline{\gamma}^2} \right) + \widetilde{A}\sinh^{-1}\overline{\gamma} + \widetilde{B}(\sinh^{-1}\overline{\gamma} + \overline{\gamma}\sqrt{1+\overline{\gamma}^2}) \right\} \right], \tag{2.32}$$

where

$$\widetilde{I}_0 = I_0 + \frac{\overline{e}^2 F^2}{4m\omega^2} = I_0\left(1 + \frac{1}{2\overline{\gamma}^2} \right). \tag{2.33}$$

The definition of $N(\gamma, \overline{\gamma}, \omega, I_0, \widetilde{A}, \widetilde{B})$ is given in Appendix A. However, it should be noted that it sometimes happens that the behaviors of the photoionization rates at the tunneling limit are quite different depending on the various treatments of the preexponential factors (see Sec. III). For comparison, the photoionization rates for different treatments of the preexponential factors in the first-order Coulomb correction $(\beta = 0)$ are presented in Appendix B.

As mentioned above, the main role of the first-order Coulomb correction is to lower the ionization potential [see Eq.

(2.18)]. Here, let us discuss the effect of the second-order Coulomb correction. In fact, by comparing the first-order Coulomb-corrected Volkov function,

$$\psi_{\vec{p}}(\vec{r},t)=\exp\left[\frac{i}{\hbar}\left\{[\vec{p}-e\vec{A}(t)]\cdot\vec{r}-\frac{p^2}{2m}t-\frac{1}{2m}\times\int_{-\infty}^{t}d\tau[-2e\vec{p}\cdot\vec{A}(\tau)+e^2\vec{A}^2]-\tilde{A}t\right\}\right], \tag{2.34}$$

and the second-order corrected function of Eq. (2.16) with a changed order of terms in the exponent,

$$\psi_{\vec{p}}(\vec{r},t)=\exp\left[\frac{i}{\hbar}\left\{[\vec{p}-(e+\hbar\beta)\vec{A}(t)]\cdot\vec{r}-\frac{p^2}{2m}t-\frac{1}{2m}\int_{-\infty}^{t}d\tau[-2(e+\hbar\beta)\vec{p}\cdot\vec{A}(\tau)+(e^2+\hbar^2\beta^2)\vec{A}^2(\tau)]-\tilde{A}t-\hbar\beta\vec{A}(t)\cdot\vec{\alpha}(t)\right\}\right], \tag{2.35}$$

we could easily have obtained some hint as to what kind of photoionization rate formulas should finally be derived.

First, it is apparent that when we include the second-order correction the following transformations have to be performed:

$$e\rightarrow e+\hbar\beta \tag{2.36}$$

and

$$e^2\rightarrow e^2+\hbar^2\beta^2. \tag{2.37}$$

In Eqs. (2.32) and (A6), the electron charge e is included in the Keldysh parameter and the effective ionization potential. However, it is not necessarily apparent which transformation [Eq. (2.36) or (2.37)] would affect the Keldysh parameter or ionization potential without careful derivation. Nevertheless, it can be qualitatively concluded that the second-order correction β lowers the Keldysh parameter and enhances the effective ionization potential.

Secondly, it is noted that the extra term $-\hbar\beta\vec{A}(t)\cdot\vec{\alpha}(t)$ in Eq. (2.35) is not contained in Eq. (2.34). The main role of this term is to add an extra term of $\tilde{B}(\sinh^{-1}\overline{\gamma}+\overline{\gamma}\sqrt{1+\overline{\gamma}^2})$ in the exponents of Eqs. (2.32) and (A6), and to reduce the effective ionization potential [note that $\tilde{B}$ defined by Eq. (2.22) is negative].

Notice that our expression is almost the same as that of Keldysh if $\tilde{A}=\tilde{B}=0$ (no Coulomb effect), although ours is larger than that of Keldysh by a factor of 4. This is due to the fact that we utilized the residue theorem for the evaluation of Eq. (2.27), whereas Keldysh used the saddle-point method for that purpose.

An important conclusion can be drawn from Eq. (B3) with the assumption of Eqs. (B10) and (B11), and that with the assumption $\tilde{A}=0$ (no Coulomb effect). By comparing these, one notices that the preexponential factors and the exponents of the former are always larger than those of the latter [note that $\tilde{A}$ is negative and that $\tilde{I}_0'$ of Eq. (B3) is defined by Eq. (B2)]. Therefore, we can predict that the photoionization rate in the presence of the Coulomb potential is always larger than that in its absence. This tendency has already been observed in several papers [15,16]. Intuitively, one can easily suppose that reducing the binding energy gives rise to an increase of the photoionization rate. However, from our formula, one can understand this tendency in a more insightful way and interpret the phenomenon analytically.

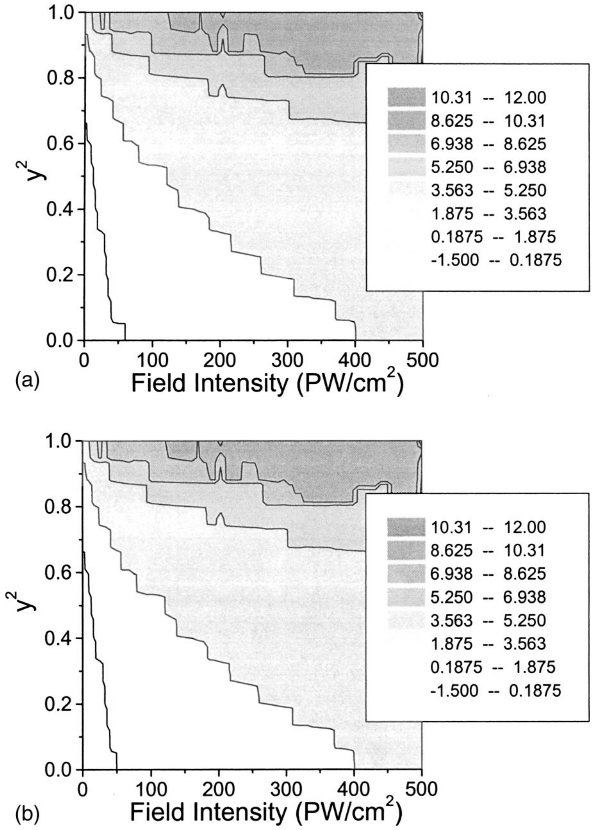

FIG. 1. Comparison of S calculated by (a) summation [Eq. (A11)] and (b) integration [Eq. (A14)]. The ordinate y^2 is $\cos^2\theta$ where θ is the angle between $\vec{p}$ and $\vec{F}$. In this figure, ln(S) is plotted. The wavelength of the incident laser is assumed to be 10 μm. For the summation in Eq. (A1), 500 000 terms from the lowest limit were summed up. Notice the quite good agreement between (a) and (b).

III. RESULTS AND DISCUSSION

In this section, first we show numerically the validity of the approximation (A14) for Eq. (A11). Next, we demonstrate the procedure used to obtain the photoionization rates by our formulas and compare some physical quantities in the first- and second-order Coulomb corrections.

Figure 1 compares S calculated by summation [Eq. (A11)] and that by integration [Eq. (A14)]. The ordinate y^2 corresponds to $\cos^2\theta$ where θ is the angle between $\vec{p}$ and $\vec{F}$. Note that $\ln(S)$ is plotted in the direction of the z axis. It is evident that both of them agree quite well over a wide range of laser intensity. This shows that a simple integration can be used for calculating the ionization rate; in this case, our formulation is much easier than Keldysh's.

If the field intensity is small, that is, δ in Eq. (A12) is small, it might be conjectured that the summation in Eq. (A11) and the integration in Eq. (A14) may be quite different. However, in such a case, γ becomes large and K in Eq. (A13) may be a large positive number in the exponent, since K is a monotonically increasing positive function of γ for all y values between -1 and 1 in the tunneling regime. Therefore, in the low field intensity range, S is quite small as can be seen in the figure.

On the other hand, when the field intensity is large, that is, δ in Eq. (A12) is large, it is conjectured that the summation in Eq. (A11) and the integration in Eq. (A14) may be of almost the same magnitude. We have confirmed that $\ln(S)$ is actually of the same magnitude down to two places of decimals at high field intensity. Although we do not show other calculations for various laser frequencies ω, the above argument holds in a wide range of ω.

According to Keldysh, neglecting the Coulomb interaction in the final state, which is well known, changes the power of F in the preexponential expression without changing the exponential itself. However, our formulas including $\vec{p}$ dependence of the preexponential factors show this tendency even if the Coulomb effect vanishes in the tunneling limit. For instance, in the tunneling limit ($\gamma=0$) for cases (i) and (iii) in Appendix B, we have

$$w_0=2^{3/4}\sqrt{3\pi}\frac{I_0}{\hbar}\left(\frac{m^{1/2}I_0^{3/2}}{\hbar eF}\right)^{1/2}\times\frac{1}{1+\sqrt{2mI_0^3}/\hbar eF}\exp\left(-\frac{4\sqrt{2mI_0^3}}{3\hbar eF}\right). \quad (3.1)$$

On the other hand, for cases (ii) and (iv) in Appendix B, we obtain

$$w_0=2^{1/4}\sqrt{3\pi}\frac{I_0}{\hbar}\left(\frac{\hbar eF}{m^{1/2}I_0^{3/2}}\right)^{1/2}\times\exp\left(-\frac{4\sqrt{2mI_0^3}}{3\hbar eF}\right). \quad (3.2)$$

For comparison, we show the Keldysh formula in the tunneling limit,

$$w_0=\frac{\sqrt{3\pi}}{2}\frac{I_0}{\hbar}\left(\frac{m^{1/2}I_0^{3/2}}{eF\hbar}\right)^{1/2}\exp\left(-\frac{4\sqrt{2mI_0^3}}{3\hbar eF}\right). \quad (3.3)$$

Equation (3.1) shows that the photoionization rate increases, saturates, and decreases to zero eventually, while Eq. (3.2) has a tendency to increase unilaterally to infinity. This is the same tendency as that of the Keldysh and Amnosov-Delone-Krainov (ADK) models [23]. Note that $\tilde{A}$ vanishes in the tunneling limit in our approximation. The remarkable point to note is that in Eq. (3.1) the factor $1+\sqrt{2mI_0^3}/\hbar eF$ is relatively large in the tunneling limit below the barrier suppression ionization (BSI) region. This kind of factor was not found in the theories of Keldysh and ADK. It should be noted that this factor may amount to 2 or more. This suggests that this term is not negligible.

Next, we show how to estimate the photoionization rate in the tunneling regime. The procedure is as follows. For the approximation of Eq. (2.8) to be valid, we have to check that

$$\cos(\omega/w_0)\approx 1, \quad (3.4)$$

where the photoionization rate w_0 is calculated by Eqs. (2.32), (A6), (B1), or (B3). Equation (3.4) indicates that the photoionization process should terminate much more rapidly than the laser cycle.

To assure that we are in the tunneling region,

$$F<\frac{Z^3e^5m^2}{16\hbar^4} \quad (3.5)$$

and

$$\gamma<0.5. \quad (3.6)$$

Equation (3.5) ensures that the photoionization is not within the BSI region [23–26]. On the other hand, Eq. (3.6) prohibits the photoionization process from lying in the multiphoton regime. The value 0.5 was adopted following the assumption of Ref. [24]. If Eqs. (3.4)–(3.6) are satisfied at the same time, the most exact photoionization rate will be obtained.

As was pointed out in the last section, our formulas show that the photoionization rate is larger in the presence of the Coulomb potential than in its absence. This can be roughly explained as follows. In its absence, the barrier width is $I_0/(eF)$ and the barrier height is I_0. On the other hand, in its presence, the barrier width is $\sqrt{I_0^2-4Ze^3F}/(eF)$ and the barrier height is $I_0-\sqrt{(Ze/F)}$. It is evident that the barrier width and height are smaller when the Coulomb potential is present than when it is absent so that the tunneling rate in the presence of the Coulomb potential is larger than that in its absence. Our formula (B3) with Eqs. (B10) and (B11) reflects this fact. It should be noted that in the high-intensity limit the effect of the Coulomb potential is negligibly small.

The second-order Coulomb correction is less significant than the first correction in the tunneling regime. Let us compare some physical quantities in the first- and second-order corrections.

First, because $I_0=Z^2e^4m/(2\hbar^2)$, $\tilde{A}=-2\hbar^3\omega\gamma/(Z^2me^4)$ and $\tilde{L}\equiv\hbar^2F^2\beta^2/(4m\omega^2)=\hbar^6\omega^2\gamma^4/(2Z^4m^2e^8)$. The former is the contribution of the first-order Coulomb correction to the shift of the resonance structure of the photoionization rate versus laser intensity and the latter that of the second-order correction. Due to the assumption that

$$\gamma\ll 1 \quad \text{and} \quad \hbar\omega/I_0=2\hbar^3\omega/(Z^2me^4)\ll 1, \quad (3.7)$$

$\tilde{L}$ is much smaller than $\tilde{A}$. Therefore, it can be seen that the contribution from the second-order Coulomb correction is much smaller than that from the first-order correction.

Secondly, the Keldysh parameter is affected only by the second-order Coulomb correction. From Eq. (2.30),

$$\bar{\gamma}=\frac{\gamma}{1+\hbar^3\omega\gamma^3/(Z^2me^4)}. \tag{3.8}$$

The part $\hbar^3\omega\gamma^3/(Z^2me^4)$ is the contribution from the second-order correction. If we compare this with $\tilde{A}$, it is evident that the second-order correction is smaller by many orders of magnitude than the first-order correction.

Thirdly, the "modified" ponderomotive energies $\bar{e}^2F^2/(4m\omega^2)$ and $E^2F^2/(4m\omega^2)$ are introduced by including the second-order Coulomb correction. The contribution of the second-order correction $\hbar\beta=Zm^2\omega^4/F^3$ is also much smaller than the elementary electron charge e.

From the above arguments, it is concluded that the second-order Coulomb correction is negligibly small compared to the first-order correction in the tunneling limit.

IV. CONCLUDING REMARKS

Including the principal effect of the long-range Coulomb potential in the Volkov function (the approximate Coulomb-Volkov function), we derived analytical formulas for the photoionization rate for hydrogenlike atoms following Keldysh's procedure [1], not depending on the crude quasiclassical analysis for the preexponential factors that was adopted by Keldysh. An important point is that our formulas are quite simple and insightful and do not require huge computer memory for the calculation, and physical meaning can easily be deduced. For example, comparing Eq. (B3) with the assumption of Eqs. (B10) and (B11), and with the assumption $\tilde{A}=0$ (no Coulomb effect) one can draw the important conclusion that the photoionization rate in the presence of the Coulomb potential is always larger than that in the absence of the Coulomb potential.

As for the derivation method, Keldysh used the saddle-point method for the contour integration of Eq. (2.27). Our derivation shows that we do not have to use the saddle-point method; instead, we can perform the contour integration by the residue theorem. We have also reduced the infinite summation to integration as shown in Eq. (A8), which renders the final photoionization rate formula quite simple. Our simple formulas Eqs. (A6) and (B3) are free from the infinite summation $S(\gamma,\tilde{I}_0/\hbar\omega)$ in Eq. (16) of Ref. [1] and Dawson's integral of Eq. (A2).

In summary, the role of the first-order Coulomb correction is to lower the ionization potential and enhance the photoionization rates and the second-order correction modifies the Keldysh parameter, decreases the binding energy, and increases the photoionization rates.

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APPENDIX A: DEFINITION OF $N(\gamma,\bar{\gamma},\omega,I_0,\tilde{A},\tilde{B})$ AND INTEGRATED FORM OF EQ. (2.32)

The preexponential factor $N(\gamma,\bar{\gamma},\omega,I_0,\tilde{A},\tilde{B})$ in Eq. (2.32) is given by

$$\begin{aligned}N(\gamma,\bar{\gamma},\omega,I_0,\tilde{A},\tilde{B})&=4\sqrt{2I_0\omega/\hbar}\left(\frac{\bar{\gamma}}{\gamma}\right)^2\left(\frac{\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}}\right)^{3/2}\frac{C_1}{C_2}\sum_{n=0}^{\infty}\\&\times\exp\left[-2\left(\left\langle\frac{\tilde{I}_0''}{\hbar\omega}+1\right\rangle-\frac{\tilde{I}_0''}{\hbar\omega}+n\right)C_3\right]\\&\times\Theta\left[\left\{\frac{2\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}}C_2^2\left(\left\langle\frac{\tilde{I}_0''}{\hbar\omega}+1\right\rangle-\frac{\tilde{I}_0''}{\hbar\omega}+n\right)\right\}^{1/2}\right],\end{aligned} \tag{A1}$$

where the symbol $\langle x\rangle$ denotes the integer part of the number x and $\Theta(x)$ is Dawson's integral,

$$\Theta(x)=\exp(-x^2)\int_0^x\exp(y^2)dy. \tag{A2}$$

Other quantities in Eq. (A1) are defined by

$$\begin{aligned}C_1=\Bigg\{&1-\frac{\tilde{A}(2+\bar{\gamma}^2)}{2I_0(1+\bar{\gamma}^2)}+\frac{\tilde{A}^2}{2I_0\hbar\omega}\frac{\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}}\\&+\frac{2\bar{\gamma}\sqrt{1+\bar{\gamma}^2}}{I_0\hbar\omega}\tilde{A}\tilde{B}-\frac{2+3\bar{\gamma}^2}{I_0}\tilde{B}+\frac{2\bar{\gamma}(1+\bar{\gamma}^2)^{3/2}}{I_0\hbar\omega}\tilde{B}^2\Bigg\}^2,\end{aligned} \tag{A3}$$

$$C_2=\sqrt{1+\frac{\tilde{A}}{2I_0(1+\bar{\gamma}^2)}+\tilde{B}\frac{1+2\bar{\gamma}^2}{I_0}}, \tag{A4}$$

$$\begin{aligned}C_3=\sinh^{-1}\bar{\gamma}-\frac{\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}}+\frac{\tilde{A}\bar{\gamma}^3}{2I_0(1+\bar{\gamma}^2)^{3/2}}\\-\frac{\tilde{B}\bar{\gamma}^3}{I_0\sqrt{1+\bar{\gamma}^2}}.\end{aligned} \tag{A5}$$

Aside from the Keldysh-like form of Eq. (A1), it is insightful to integrate it over n. By so doing, Eq. (2.32) reduces to

$$w_0=\sqrt{2\pi I_0\omega/\hbar}\left(\frac{\overline{\gamma}}{\gamma}\right)^2\frac{\overline{\gamma}^2}{1+\overline{\gamma}^2}\frac{C_1}{C_4\sqrt{C_3}}\times\exp\left[-\frac{2}{\hbar\omega}\left\{\tilde{I}_0\left(\sinh^{-1}\overline{\gamma}-\frac{\overline{\gamma}\sqrt{1+\overline{\gamma}^2}}{1+2\overline{\gamma}^2}\right)+\tilde{A}\sinh^{-1}\overline{\gamma}+\tilde{B}(\sinh^{-1}\overline{\gamma}+\overline{\gamma}\sqrt{1+\overline{\gamma}^2})\right\}\right], \quad (A6)$$

where

$$C_4=\sinh^{-1}\overline{\gamma}+\frac{\tilde{A}\overline{\gamma}}{2I_0\sqrt{1+\overline{\gamma}^2}}+\frac{\tilde{B}\overline{\gamma}\sqrt{1+\overline{\gamma}^2}}{I_0}. \quad (A7)$$

Note that $\tilde{I}_0''$ disappears in Eq. (A6), while Eq. (A1) contains this term. The resonance structures detected by using Eq. (A1) are expected to show a shift from the counterpart in Ref. [1] and that including the first-order Coulomb correction (see Appendix B).

An important approximation involved in the present theory is to change the summation with respect to n (proportional to the excess photon number absorbed above threshold) to integration,

$$\sum_{n>\langle\tilde{I}_0''/\hbar\omega\rangle}^{\infty}\sqrt{n\hbar\omega-\tilde{I}_0''}\exp\left\{-\frac{2}{\hbar\omega}(n\hbar\omega-\tilde{I}_0'')(G-Hy^2)\right\}\approx\int_{\tilde{I}_0''/\hbar\omega}^{\infty}dn\sqrt{n\hbar\omega-\tilde{I}_0''}\times\exp\left\{-\frac{2}{\hbar\omega}(n\hbar\omega-\tilde{I}_0'')(G-Hy^2)\right\}, \quad (A8)$$

where

$$G=\sinh^{-1}\overline{\gamma}+\frac{\tilde{A}\overline{\gamma}}{2I_0\sqrt{1+\overline{\gamma}^2}}+\frac{\tilde{B}\overline{\gamma}\sqrt{1+\overline{\gamma}^2}}{I_0} \quad (A9)$$

and

$$H=\frac{\overline{\gamma}}{\sqrt{1+\overline{\gamma}^2}}+\frac{\tilde{A}\overline{\gamma}}{2I_0(1+\overline{\gamma}^2)^{3/2}}+\frac{\tilde{B}\overline{\gamma}(1+2\overline{\gamma}^2)}{I_0\sqrt{1+\overline{\gamma}^2}}. \quad (A10)$$

This approximation needs to be examined. For this purpose, we consider

$$S=\sum_n\sqrt{n-\delta}\exp[-K(n-\delta)], \quad (A11)$$

where

$$\delta=\frac{\tilde{I}_0''}{\hbar\omega} \quad (A12)$$

and

$$K=2(G-Hy^2). \quad (A13)$$

S in Eq. (A11) should be compared with

$$S=\int_\delta^\infty dn\sqrt{n-\delta}\exp[-K(n-\delta)]=\tfrac{1}{2}\sqrt{\pi/K^3}. \quad (A14)$$

An important feature of the approximation (A14) is that it is independent of δ. In Sec. III, we show numerically the validity of the approximation Eq. (A14) for Eq. (A11).

The formula Eq. (A6) is quite simple and the infinite summation of Eq. (A1) is unnecessary. By adopting Dawson's integral, as the tunneling limit approaches, the computation time for obtaining converged results becomes quite large because the summation over n requires many iterations. In that case, the above formulas integrated over n, such as Eq. (A6), are quite convenient.

APPENDIX B: PHOTOIONIZATION RATES IN THE FIRST-ORDER COULOMB CORRECTION

In this appendix, we show explicitly the terms appearing in Eq. (2.32) for different treatments of the preexponential factors. Here, we show four cases that we can consider.

Generally, the photoionization rate can be expressed as

$$w_0=4\sqrt{2I_0\omega/\hbar}\left(\frac{\gamma}{\sqrt{1+\gamma^2}}\right)^{3/2}\frac{[1-\tilde{A}(2+\gamma^2)/2I_0(1+\gamma^2)+(\tilde{A}^2/2I_0\hbar\omega)(\gamma/\sqrt{1+\gamma^2})]^2}{\sqrt{1+\tilde{A}/2I_0(1+\gamma^2)+B}}\exp\left[-\frac{2}{\hbar\omega}\left\{\tilde{I}_0'\left(\sinh^{-1}\gamma-\frac{\gamma\sqrt{1+\gamma^2}}{1+2\gamma^2}\right)+\tilde{A}\sinh^{-1}\gamma\right\}\right]\sum_{n=0}^{\infty}\exp\left[-2\left(\left\langle\frac{\tilde{I}_0'}{\hbar\omega}+1\right\rangle-\frac{\tilde{I}_0'}{\hbar\omega}+n\right)\left\{\sinh^{-1}\gamma-\frac{\gamma}{\sqrt{1+\gamma^2}}+\frac{\tilde{A}\gamma^3}{2I_0(1+\gamma^2)^{3/2}}+C-\frac{\gamma}{\sqrt{1+\gamma^2}}B\right\}\right]\Theta\left[\left\{\frac{2\gamma}{\sqrt{1+\gamma^2}}B\left(\left\langle\frac{\tilde{I}_0}{\hbar\omega}+1\right\rangle-\frac{\tilde{I}_0}{\hbar\omega}+n\right)\right\}^{1/2}\right], \quad (B1)$$

where $\Theta(x)$ is the Dawson integral defined by Eq. (A2) and

$$\tilde{I}_0'=I_0+\frac{e^2F^2}{4m\omega^2}=I_0\left(1+\frac{1}{2\gamma^2}\right). \quad (B2)$$

Furthermore, if we perform the summation over n in the above equation, then the following simpler expression is obtained:

$$w_0=2\sqrt{\frac{\pi I_0\omega}{\hbar}}\frac{\gamma^2}{1+\gamma^2}\left(1-\frac{\tilde{A}(2+\gamma^2)}{2I_0(1+\gamma^2)}+\frac{\tilde{A}^2}{2I_0\hbar\omega}\frac{\gamma}{\sqrt{1+\gamma^2}}\right)^2$$
$$\times\frac{\exp\left[-\frac{2}{\hbar\omega}\{\tilde{I}_0[\sinh^{-1}\gamma-\gamma\sqrt{1+\gamma^2}/(1+2\gamma^2)]+\tilde{A}\sinh^{-1}\gamma\}\right]}{[\sinh^{-1}\gamma+(\tilde{A}/2I_0)(\gamma/\sqrt{1+\gamma^2})+2C]\sqrt{2\{\sinh^{-1}\gamma-\gamma/\sqrt{1+\gamma^2}+(\tilde{A}/2I_0)[\gamma^3/(1+\gamma^2)^{3/2}]+C-(\gamma/\sqrt{1+\gamma^2})B\}}}. \tag{B3}$$

(i) If we assume that all of the preexponential factors depend on $\vec{p}$,

$$B=\frac{\hbar\omega}{2I_0b_4^2}\frac{\gamma^3}{(1+\gamma^2)^{3/2}}\times\left\{1-\frac{3\tilde{A}}{2I_0(1+\gamma^2)}+\frac{\tilde{A}^2\gamma}{\hbar\omega I_0\sqrt{1+\gamma^2}}\right\}^2+\frac{\hbar\omega}{2I_0b_4}\frac{1}{\gamma(1+\gamma^2)^{3/2}}\times\left\{-(4\gamma^2+1)+\frac{3\tilde{A}(2\gamma^6+2\gamma^4+7\gamma^2+2)}{2I_0(1+\gamma^2)}+\frac{\tilde{A}^2\gamma(\gamma^4-4\gamma^2-1)}{\hbar\omega I_0\sqrt{1+\gamma^2}}\right\} \tag{B4}$$

and

$$C=-\frac{\hbar\omega}{2(1+\gamma^2)I_0b_4}\left\{(1+2\gamma^2)-\frac{\tilde{A}(4\gamma^4+13\gamma^2+6)}{2I_0(1+\gamma^2)}+\frac{\tilde{A}^2\gamma(1+2\gamma^2)}{\hbar\omega I_0\sqrt{1+\gamma^2}}\right\}. \tag{B5}$$

(ii) If we assume that u_s depends on $\vec{p}$ but $\cos\theta_{pF}$ does not ($\cos\theta_{pF}=1$),

$$B=\frac{\hbar\omega}{2I_0b_4^2}\frac{\gamma^3}{(1+\gamma^2)^{3/2}}\times\left\{1-\frac{3\tilde{A}}{2I_0(1+\gamma^2)}+\frac{\tilde{A}^2\gamma}{\hbar\omega I_0\sqrt{1+\gamma^2}}\right\}^2+\frac{\hbar\omega}{2I_0b_4}\frac{\gamma}{(1+\gamma^2)^{3/2}}\left\{(\gamma^2-2)+\frac{\tilde{A}(2\gamma^4-8\gamma^2+5)}{2I_0(1+\gamma^2)}+\frac{2\tilde{A}^2}{\hbar\omega I_0}\frac{\gamma(\gamma^2-1)}{\sqrt{1+\gamma^2}}\right\} \tag{B6}$$

and

$$C=-\frac{\hbar\omega\gamma^2}{2(1+\gamma^2)I_0b_4}\left\{1-\frac{3\tilde{A}}{2I_0(1+\gamma^2)}+\frac{\tilde{A}^2\gamma}{\hbar\omega I_0\sqrt{1+\gamma^2}}\right\}. \tag{B7}$$

(iii) If we assume that $\cos\theta_{pF}$ depends on $\vec{p}$ but u_s does not ($u_s=i\gamma$),

$$B=-\frac{\hbar\omega\sqrt{1+\gamma^2}}{2\gamma I_0b_4}\left\{1+\frac{\tilde{A}\gamma^2}{I_0(1+\gamma^2)}+\frac{\tilde{A}^2\gamma}{\hbar\omega I_0(1+\gamma^2)}-\frac{3\tilde{A}}{I_0\sqrt{1+\gamma^2}}\right\} \tag{B8}$$

and

$$C=-\frac{\hbar\omega}{2I_0b_4}\left\{1-\frac{\tilde{A}(2\gamma^2+3)}{I_0(1+\gamma^2)}+\frac{\tilde{A}^2\gamma}{\hbar\omega I_0\sqrt{1+\gamma^2}}\right\}. \tag{B9}$$

(iv) If we assume that $\cos\theta_{pF}$ and u_s are $\vec{p}$ independent ($\cos\theta_{pF}=1$ and $u_s=i\gamma$),

$$B=0 \tag{B10}$$

and

$$C=0. \tag{B11}$$

In Eqs. (B4)–(B9), b_4 is defined by

$$b_4=-1+\frac{\tilde{A}(2+\gamma^2)}{2I_0(1+\gamma^2)}-\frac{\tilde{A}^2\gamma}{2I_0\hbar\omega\sqrt{1+\gamma^2}}. \tag{B12}$$

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