Chains of quasiclassical information for bipartite correlations and the role of twin observables

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Having the quantum correlations in a general mixed or pure bipartite state in mind, the part of information accessible by simultaneous measurement on both subsystems is shown *never to exceed* the part accessible by measurement on one subsystem, which, in turn is proved *not to exceed* the von Neumann mutual information. A particular pair of (opposite-subsystem) observables is shown to be responsible both for the amount of quasiclassical correlations and for that of the purely quantum entanglement in the *pure-state* case: the former via simultaneous subsystem measurements, and the latter through *the entropy of coherence or of incompatibility*, which is defined for the general case. The observables at issue are so-called *twin observables*. A general definition of the latter is given in terms of their detailed properties.

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As it is well known, quantum information theory is closely connected with the correlations inherent in an *arbitrary* bipartite state (mathematically: statistical operator) ρ_{12} of a composite system $1+2$. The correlations have surprisingly many facets and the relations among them are the subject of intense current investigation. This article is intended to make a contribution to the issue.

Let us define some quantitative elements of correlations. The subsystem states (reduced statistical operators) are $\rho_s \equiv \mathrm{Tr}_{s'}\rho_{12}$, $s, s'=1, 2$: $s\neq s'$ ("$\mathrm{Tr}_{s'}$" is a partial trace), and we have the three *von Neumann entropies*, $S(n)\equiv S(\rho_n)\equiv -\mathrm{Tr}_n(\rho_n \ln \rho_n)$, $n=1,2,12$. One of the basic correlation or entanglement entities is the *von-Neumann mutual information*,

$$I(1:2)\equiv S(1)+S(2)-S(12). \tag{1}$$

It is conjectured that it is the amount of *total* correlations [1].

For the purpose of notation, let us write down an arbitrary first subsystem and an arbitrary second-subsystem complete observable (Hermitian operator) with purely discrete spectra: $A_1=\Sigma_i a_i|i\rangle_1\langle i|_1$, $B_2=\Sigma_j b_j|j\rangle_2\langle j|_2$. The *measurement* of $A_1\otimes 1$ gives rise to the distant (as opposed to "direct") state decomposition $\rho_2=\Sigma_i p_i\rho_2^i$, where $p_i\equiv \mathrm{Tr}[\rho_{12}(|i\rangle_1\langle i|_1\otimes 1)]$ is the probability of the result a_i, and $\rho_2^i \equiv p_i^{-1}\,\mathrm{Tr}_1[\rho_{12}(|i\rangle_1\langle i|_1\otimes 1)]$ is the opposite-subsystem state corresponding to this result if $p_i>0$.

Entropy is concave [2] (Sec. II B there), i.e., $\Sigma_i p_i S(\rho_2^i)\leqslant S(2)$, and

$$I(m1\rightarrow 2)_A\equiv S(2)-\sum_i p_i S(\rho_2^i) \tag{2a}$$

is the *information gain* about subsystem 2 on account of the direct *measurement* of the observable A_1 on subsystem 1. Symmetrically, one defines the symmetric quantity $I(1\leftarrow m2)_B$.

One further defines [1,3]

$$I(m1\rightarrow 2)\equiv \sup\{I(m1\rightarrow 2)_A\}, \tag{2b}$$

the *supremum* taken over all complete A_1, as the largest amount of information (contained in the correlations) accessible by measurement of an observable on the first subsystem. Symmetrically, one defines the symmetric quantity $I(1\leftarrow m2)\equiv \sup\{I(1\leftarrow m2)_B\}$ over all second-subsystem complete measurements.

If one performs simultaneous measurement of $(A_1\otimes 1)$ and of $(1\otimes B_2)$ on ρ_{12} [denoted by $(A_1\wedge B_2)$], then one deals with a classical discrete joint probability distribution $p_{ij}\equiv \mathrm{Tr}[\rho_{12}(|i\rangle_1\langle i|_1\otimes |j\rangle_2\langle j|_2)]$. It implies, in its turn, the *mutual information* $I(m1:m2)_{A\wedge B}$ via the Gibbs-Boltzmann-Shannon entropies $H(A,B)\equiv -\Sigma_{ij}p_{ij}\ln p_{ij}$, $H(A)\equiv -\Sigma_i p_i \ln p_i$, $H(B)\equiv -\Sigma_j p_j \ln p_j$, where $p_i\equiv \Sigma_j p_{ij}$ and $p_j\equiv \Sigma_i p_{ij}$ are the marginal probability distributions. Then

$$I(m1:m2)_{A\wedge B}\equiv H(A)+H(B)-H(A,B). \tag{3a}$$

Finally,

$$I(m1:m2)\equiv \sup\{I(m1:m2)_{A\wedge B}\} \tag{3b}$$

over all choices of complete observables A_1 and B_2. This is the largest amount of information on a subsystem observable (contained in the quantum correlations) accessible by measurement of an observable on the opposite subsystem.

The claimed chains of information inequalities, valid for every bipartite state ρ_{12}, go as follows:

$$0\leqslant I(m1:m2)\leqslant I(m1\rightarrow 2)\leqslant \min\{I(1:2),S(2)\}, \tag{4a}$$

$$0\leqslant I(m1:m2)\leqslant I(1\leftarrow m2)\leqslant \min\{I(1:2),S(1)\}. \tag{4b}$$

Both in inequalities (4a) and (4b) one has equality in the first inequality *if and only if* the state ρ_{12} is *uncorrelated*, i.e., $\rho_{12}=\rho_1\otimes\rho_2$.

The role of $S(2)$ in the last inequality in the expression (4a) is obvious from Eq. (2a), and symmetrically for (4b).

In the classical discrete case both chains (4a) and (4b) contain only equalities, and one has $I(1:2)\leqslant S(1),S(2)$. As

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to the corresponding inequality in the quantum case, one cannot do better than $I(1{:}2)\leqslant 2S(1),2S(2)$ [4].

The inequality $I(m1{:}m2)\leqslant I(1{:}2)$ implied by expression (4a), together with the stated necessary and sufficient condition for equality in the first inequality in expression (4a), was proved in 1973 by Lindblad [5] (Theorem 2 there). The third inequality in expression (4b) with $I(1{:}2)$ was claimed and a proof was presented in Ref. [3]. (It is perhaps useful to have an independent verification like the one in this article.)

The second inequality in expression (4a) is being proved in this article. For the sake of completeness, let me prove the entire chain.

The inequalities in expression (4a) are, essentially, *a consequence* of a *result of Lindblad* of classical value [6] (see Corollary there), and expression (4b) follows symmetrically. To explain this claim, let me introduce the so-called *relative entropy* of a quantum state (statistical operator) σ with relation to another state (statistical operator) ρ,

$$S(\sigma|\rho)\equiv \mathrm{Tr}\sigma\ln\sigma-\mathrm{Tr}\sigma\ln\rho.$$

One has $0\leqslant S(\sigma|\rho)$ with equality if and only if $\sigma=\rho$.

Lindblad's result involves the *ideal measurement* of an arbitrary complete or incomplete observable (Hermitian operator) A with a purely discrete spectrum. Let its unique spectral form, i.e., the one without repetition in the characteristic values, be $A=\Sigma_i a_i P^i$. Denoting by $T_A\sigma$ the state into which σ changes due to the nonselective ideal measurement of A in it, one has

$$T_A\sigma=\sum_i P^i\sigma P^i \tag{5}$$

[7], and Lindblad's result states that

$$S(T_A\sigma|T_A\rho)\leqslant S(\sigma|\rho). \tag{6}$$

One should note that also the right-hand side (RHS) of Eq. (5) is a statistical operator. Hence, for any other observable $B=\Sigma_j b_j Q^j$, Eq. (6) implies, what may be called *the Lindblad chain* $S(T_BT_A\sigma|T_BT_A\rho)\leqslant S(T_A\sigma|T_A\rho)\leqslant S(\sigma|\rho)$. One may even extend the measurements to operations [8].

The von Neumann mutual information in any bipartite state ρ_{12} can be expressed *in terms of relative entropy*,

$$I(1{:}2)=S(\rho_{12}|\rho_1\otimes\rho_2). \tag{7}$$

[This known claim is easily checked utilizing $\ln(\rho_1\otimes\rho_2)=(\ln\rho_1)\otimes 1+1\otimes(\ln\rho_2)$, which, in turn, is easily seen in spectral forms.]

I am going to demonstrate that the claimed chain of inequalities (4a) is a consequence of the Lindblad chain,

$$0\leqslant S(T_AT_B\rho_{12}|T_AT_B(\rho_1\otimes\rho_2))\leqslant S(T_A\rho_{12}|T_A(\rho_1\otimes\rho_2))$$
$$\leqslant S(\rho_{12}|\rho_1\otimes\rho_2) \tag{8}$$

with subsystem observables A_1 and B_2 that are *complete* in some subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ containing the ranges of the operators ρ_1 and ρ_2, respectively.

In order to recognize the meaning of *the first relative entropy* in inequality (8), we make use of the auxiliary claim that for complete or incomplete subsystem observables $A_1=\Sigma_i a_i P_1^i$ and $B_2=\Sigma_j b_j Q_2^j$ (unique spectral forms), and for any bipartite state ρ_{12}, one has

$$T_A\rho_1=\mathrm{Tr}_2(T_AT_B\rho_{12}),\quad T_B\rho_2=\mathrm{Tr}_1(T_AT_B\rho_{12}), \tag{9}$$

where ρ_s, $s=1,2$, are the subsystem states of ρ_{12}. Relations (9) are proved in Appendix A.

Further, one can argue with Lindblad [5] (Theorem 2 there) as follows. Making use of Eqs. (1), (7), and (9), one obtains $S(T_AT_B\rho_{12}|T_AT_B(\rho_1\otimes\rho_2))=S(T_A\rho_1)+S(T_B\rho_2)-S(T_AT_B\rho_{12})$. Taking T_A and T_B in explicit form [cf. (5) *mutatis mutandis*], we see that we have a mixture of *orthogonal* pure states. The so-called *mixing property* of entropy allows us to write it as the sum of the so-called *mixing entropy* (that of the statistical weights) and the average entropy [2] (see Secs. II F and II B, there). Since pure states have zero entropy, one obtains

$$LHS=H(A)+H(B)-H(A,B)=I(m1{:}m2)_{A\wedge B}.$$

Next, we turn to *the second relative entropy* in inequality (8). Utilizing again relations (9) (this time with $B\equiv 1$), (7), and (1), one obtains

$$S(T_A\rho_{12}|T_A(\rho_1\otimes\rho_2))=S(T_A\rho_1)+S(\rho_2)-S(T_A\rho_{12}). \tag{10}$$

Since $A_1=\Sigma_i a_i|i\rangle_1\langle i|_1$, for $p_i\equiv\mathrm{Tr}(|i\rangle_1\langle i|_1\rho_{12})>0$, one has

$$|i\rangle_1\langle i|_1\rho_{12}|i\rangle_1\langle i|_1=p_i|i\rangle_1\langle i|_1\otimes\rho_2^i, \tag{11a}$$

$$\rho_2^i\equiv p_i^{-1}\,\mathrm{Tr}_1(|i\rangle_1\langle i|_1\rho_{12}|i\rangle_1\langle i|_1). \tag{11b}$$

(The tensor factor "$\otimes 1$" is repeatedly omitted because no confusion can arise.) The validity of Eq. (11a) is straightforward to check in any pair of orthonormal and complete subsystem bases.

On account of Eq. (11a) and the fact that both $T_A\rho_1$ and $T_A\rho_{12}$ are orthogonal mixtures of states [cf. Eq. (5)] with the same statistical weights, we can apply the mixing property of entropy both to $S(T_A\rho_1)$ and to $S(T_A\rho_{12})$. Then, the left-hand side (LHS) of Eq. (10) becomes equal to

$$H(A)+S(\rho_2)-\left(H(A)+\sum_i p_iS(\rho_2^i)\right)=I(m1\to 2)_A$$

[cf. Eq. (2a)].

The chain (8) can now be rewritten as

$$0\leqslant I(m1{:}m2)_{A\wedge B}\leqslant I(m1\to 2)_A\leqslant I(1{:}2). \tag{12}$$

(The symmetric chain is derived symmetrically.)

The inequality

$$I(m1{:}m2)_{A\wedge B}\leqslant I(m1\to 2)_A$$

has the obvious *physical interpretation* that, in general, only part of the quantum information gain about subsystem 2 due to the measurement of A_1 can be realized as an information about a concrete complete observable B_2.

The same inequality implies that the *quantum information gain* $I(m1\rightarrow 2)_A$ is an upper bound to any concrete information $I(m1{:}m2)_{A\wedge B}$ about some B_2.

Taking the *suprema* in inequality (12), and having Eq. (2a) in mind, one obtains the expression (4a).

In Ref. [1] $I(m1\rightarrow 2)$ is interpreted as the quasiclassical part of the amount of quantum correlations in any bipartite state ρ_{12}. The authors define the so-called relative entropy of entanglement $E_{RE}(\rho_{12})\equiv \inf\{S(\rho_{12}|\sigma_{12})\}$, where the infimum is taken over all separable states σ_{12}, as a measure of (purely quantum) entanglement (cf. also Ref. [9]). Since $I(1{:}2)=S(\rho_{12}|\rho_1\otimes\rho_2)$, obviously, $E_{RE}(\rho_{12})\leqslant I(1{:}2)$.

Essentially the same view of $I(m2\rightarrow 1)$ as in Ref. [1] is, independently, taken in Ref. [3]. The latter authors call the difference

$$\delta(m2\rightarrow 1)\equiv I(1{:}2)-I(m2\rightarrow 1) \tag{13}$$

"*quantum discord*," and they interpret it as the truly quantum part of the total amount of correlations $I(1{:}2)$. (It is inaccessible to subsystem measurement.)

Next we apply the derived chain of quasiclassical informations to pure states. They represent a simple enough case to gain detailed insight.

Quasiclassical informations in bipartite pure states. We turn now to a general *pure state* $\rho_{12}\equiv|\Phi\rangle_{12}\langle\Phi|_{12}$. Let us write $|\Phi\rangle_{12}$ as a Schmidt decomposition [10,11] into biorthogonal state vectors,

$$|\Phi\rangle_{12}=\sum_i r_i^{1/2}|i\rangle_1|i\rangle_2 . \tag{14}$$

Taking

$$A_1\equiv\sum_i a_i|i\rangle_1\langle i|_1,\quad 0\neq a_i\neq a_{i'}\neq 0\quad \text{for}\quad i\neq i', \tag{15a}$$

$$B_2\equiv\sum_i b_i|i\rangle_2\langle i|_2,\quad 0\neq b_i\neq b_{i'}\neq 0\quad \text{for}\quad i\neq i', \tag{15b}$$

one obtains for the induced classical discrete probability distribution [cf. Eq. (3a)], $p_{ij}=\delta_{ij}r_i$. Then

$$\begin{aligned} I(m1{:}m2)_{A\wedge B}&=H(A)=H(B)=H(A,B)=S(1)=S(2)\\ &=I(m1\rightarrow 2)=I(1\leftarrow m2) \end{aligned} \tag{16}$$

[cf. inequalities (4a) and (4b) without $I(1{:}2)$]. It is seen from Eq. (3b) that $I(m1{:}m2)_{A\wedge B}$ is a lower bound to all quantities in the chains (4a) and (4b), and it reaches its highest possible value $S(1)=S(2)$ in $|\Phi\rangle_{12}$ [cf. Eq. (16)]. Hence, it equals not only $I(m1{:}m2)$, but also $I(m1\rightarrow 2)$ and $I(1\leftarrow m2)$.

Besides, also

$$\delta(m1\rightarrow 2)=\delta(1\leftarrow m2)=S(1)=S(2) \tag{17}$$

[because $I(1{:}2)=2S(1)=2S(2)$]. The same quantity, called entropy of entanglement and denoted by $E(|\Phi\rangle_{12})$ was obtained in Ref. [12].

Returning to the above quasiclassical informations in $|\Phi\rangle_{12}$, one can say that the pair (A_1,B_2) of opposite-subsystem observables (15a) and (15b) actually *realize, in simultaneous measurement, the entire part of the total correlations that is available for subsystem measurement*. This pair of observables has noteworthy properties. Next, we resort to a sketchy presentation of them in the general case.

Twin observables with respect to a general bipartite state. Let us now turn to a concise but sufficiently detailed definition of *twin observables*, which is wider than the one given in previous work [11,13]. All necessary proofs are provided in Appendix B.

Let ρ_{12} be an arbitrary given bipartite state, and let A_1 and B_2 be opposite-subsystem observables (Hermitian operators) having the following three properties with respect to ρ_{12}.

(i) The operators *commute* with the corresponding reduced statistical operators, $[A_1,\rho_1]=0$, $[B_2,\rho_2]=0$.

On account of the commutations, the [topological closures $\bar{\mathcal{R}}(\rho_i)$ of the] ranges $\mathcal{R}(\rho_i)$, $i=1,2$, are *invariant* subspaces for A_1 and B_2, respectively, and the operators have *purely discrete spectra* in them. These are precisely the *detectable* parts of the respective spectra of A_1 and B_2, i.e., they consist of those characteristic values that have positive probability in ρ_{12}.

(ii) The detectable parts of the spectra of A_1 and B_2 consist of *an equal number* of characteristic values, i.e., they are of the same power.

(iii) One can establish a one-to-one map between the two detectable parts of the spectra *such that* the corresponding characteristic values, denoted by the same index *i*, *satisfy for all values of i* one of the following four conditions.

(a) *The information-theoretic condition,*

$$p_{ii'}\equiv \mathrm{Tr}\,\rho_{12}P_1^iP_2^{i'}=\delta_{i,i'}p_i ,$$

where P_1^i is the characteristic projector of A_1 corresponding to the detectable characteristic value a_i and symmetrically for $P_2^{i'}$ and $b_{i'}$ of B_2; and $p_i\equiv \mathrm{Tr}\,\rho_1P_1^i$ is the probability of P_1^i in ρ_{12}.

(b) *The measurement-theoretic condition,*

$$P_1^i\rho_{12}P_1^i=P_2^i\rho_{12}P_2^i .$$

(c) *The condition in terms of quantum logic,*

$$\mathrm{Tr}\,[\rho_2(P_1^i)P_2^i]=1,$$

where $\rho_2(P_1^i)\equiv p_i^{-1}\mathrm{Tr}_1\rho_{12}P_1^i$ is *the conditional state* of subsystem 2 when the event P_1^i occurs.

(d) *The algebraic condition,*

$$P_1^i\rho_{12}=P_2^i\rho_{12} .$$

The four conditions in property (iii) are *equivalent*.

If A_1 and B_2 do have the mentioned three properties, then we call them twin observables for ρ_{12}. If all characteristic values of A_1 and B_2 in $\overline{\mathcal{R}}(\rho_1)$ and $\overline{\mathcal{R}}(\rho_2)$, respectively, are *nondegenerate*, i.e., if $\forall\ i{:}\mathrm{Tr}P_s^iQ_s=1$, where Q_s is the range projector of ρ_s, $s=1,2$, we say that A_1 and B_2 are *complete* twin observables with respect to ρ_{12}.

Following are the comments on the four conditions in property (iii).

(a) The probability distribution $p_{ii'}=\delta_{i,i'}p_i$ is the best possible classical information channel: a so-called lossless and noiseless one. It is obvious that the correspondence between the detectable parts of the spectra is *unique*.

(b) The detectable characteristic values a_i of A_1 and b_i of B_2 are *equally probable* in ρ_{12}. Besides, the ideal measurement of A_1 and that of B_2 [actually of $(A_1\otimes 1_2)$ and of $(1_1\otimes B_2)$] convert ρ_{12} into the same state (cf. the general formula of Lüders for ideal measurement [7]). This makes possible the so-called *distant measurement* [11]: One can measure B_2 in ρ_{12} without any dynamical influence on the second subsystem by just measuring A_1 on the first subsystem (or vice versa) in the state ρ_{12} of the bipartite system.

(c) For an arbitrary event (projector) E_2 for subsystem 2 one can write

$$\mathrm{Tr}[\rho_{12}P_1^iE_2]=p_i\mathrm{Tr}[\rho_2(P_1^i)E_2],$$

i.e., one can factorize coincidence probability into probability of the condition P_1^i and conditional probability of the event E_2 (in analogy with classical physics). The conditional state $\rho_2(P_1^i)$, when giving probability one, *extends* the absolute implication in quantum logic (which is $E\leqslant F\Leftrightarrow EF=E$, E and F projectors) by *state-dependent implication* [14]. This makes P_1^i and P_2^i to imply each other ρ_{12} dependently.

(d) Since the detectable characteristic values of twin observables A_1 and B_2 are arbitrary, one can choose them equal: $\forall\ i{:}a_i=b_i$. Then the algebraic condition *strengthens* into

$$A_1\rho_{12}=B_2\rho_{12}.$$

This case was studied in detail in previous work [11,13]. It was shown that the stronger algebraic condition implies all three above properties, i.e., it by itself makes A_1 and B_2 twin observables (as defined in this article) with the additional property (iv): $\forall\ i{:}a_i=b_i$. It was also shown that in the pure state case the multiplicities of a_i and b_i necessarily coincide, but they need not be equal in the mixed-state case.

Without property (iv) twin observables have a wider scope of potential application.

Let us return to the above discussion of quasiclassical informations inherent in a given pure state vector $|\Phi\rangle_{12}$. In view of the information-theoretic condition in property (iii) of twin observables, it clearly follows from the above discussion of Eqs. (15a) and (15b) that one is dealing with twin observables.

One can say that it is the pair (A_1,B_2) of twin observables given by Eqs. (15a) and (15b) that realizes, in simultaneous measurement, the entire quasiclassical information.

The ideal nonselective measurements of A_1, that of B_2, and that of $A_1\wedge B_2$ each convert $|\Phi\rangle_{12}$ into one and the same mixed state

$$\rho'_{12}\equiv\sum_i r_i|i\rangle_1\langle i|_1\otimes|i\rangle_2\langle i|_2 \tag{18}$$

[cf. Eq. (14)].

As it is easily seen, the same pair of observables (15a) and (15b) are *complete twin observables* not only with respect to $|\Phi\rangle_{12}$, but also regarding ρ'_{12}. Also Eq. (16) holds true for the latter. Again, the same pair of twin observables "carry" the entire subsystem-measurement-accessible part of information. But instead of Eq. (17), we have zero quantum discord. There is no subsystem-measurement-inaccessible part of information. [No wonder, we are dealing with a biorthogonal separable mixed state in Eq. (18).]

In view of the fact that twin observables have a variety of particular properties, one may wonder if the pair given by Eqs. (15a) and (15b) is, perhaps, of some relevance also for the quantum discord in $|\Phi\rangle_{12}$ [cf. Eq. (14)]. To reach an answer in the affirmative, we must first introduce entropy of coherence.

Entropy of coherence or of incompatibility. To begin with, we should notice that the difference between Eqs. (14) and (18) lies in *coherence*, which is present in the former and absent in the latter. One may wonder if coherence can be given a precise and general definition.

I suggest to consider the following quantity as *the amount of coherence or of incompatibility* between a given observable $A=\Sigma_ia_iP^i$ (in the unique spectral form) and a given quantum state ρ, and call it *the entropy of coherence or of incompatibility*,

$$E_C(A,\rho)\equiv S(T_A\rho)-S(\rho) \tag{19}$$

[cf. Eq. (5)], i.e., the increase of entropy in ideal nonselective measurement of A in ρ.

That the RHS of Eq. (19) is always nonnegative and zero if and only if A and ρ commute (compatibility) was proved in Ref. [15] (pp. 380–387) for complete A. That for any state ρ and for any incomplete observable A there always exists a complete one B such that the former is a function of the latter and such that $T_A\rho=T_B\rho$ was proved in Ref. [16] (Theorem 2 there). Hence, the RHS of Eq. (19) is always nonnegative also for incomplete observables, and it is zero if and only if $[A,\rho]=0$. [Namely, the commutation is sufficient for $T_A\rho=\rho$, and hence for zero LHS of Eq. (19). On the other hand, the mentioned zero implies, as stated, commutation with B, and hence also with A.]

Utilizing the mixing property of entropy, we can rewrite Eq. (19) as

$$E_C(A,\rho)=H(A)-\left(S(\rho)-\sum_i w_iS(\rho_i)\right), \tag{20}$$

where $\forall\ i{:}w_i\equiv\mathrm{Tr}\,P^i\rho$, $\rho_i\equiv P_i\rho P_i/w_i$ (for $w_i>0$) and $H(A)\equiv H(w_i)$ is the mixing entropy, which is, simultaneously, also the *entropy of the observable* A in ρ.

It was proved in Ref. [17] (Theorem 2 there) that, whenever $S(\rho)<\infty$, the second term on the RHS of Eq. (20) is, in its turn, always nonnegative, and zero if and only if $\forall\ i:S(\rho_i)=S(\rho)$. (This condition is satisfied, e.g., when ρ and all ρ_i are pure states, like in the case of measurement in a pure state.) On the other hand, the above discussion shows that the mentioned second term never exceeds the first; and they are equal if and only if $[A,\rho]=0$.

If A is *complete* and ρ mixed or pure, then the states ρ_i are pure and

$$E_C(A,\rho)=H(A)-S(\rho). \tag{21}$$

If ρ is *pure* and A is incomplete or complete, the states ρ_i are again pure, and

$$E_C(A,\rho)=H(A). \tag{22}$$

If *both A is complete*, i.e., $\forall\ i:P^i=|i\rangle\langle i|$, *and ρ is pure*, i.e., $\rho=|\phi\rangle\langle\phi|$, then

$$E_C(A,\rho)=H(|f_i|^2), \tag{23a}$$

where

$$|\phi\rangle=\sum_i f_i|i\rangle \tag{23b}$$

is the relevant expansion.

Now we may face the question if the twin observables given by Eqs. (15a) and (15b) have anything to do with quantum discord in $|\Phi\rangle_{12}$.

Purely quantum information and coherence in bipartite pure states. The entropy of coherence of $(A_1\otimes 1)$ given by Eq. (15a) or of its twin observable $(1\otimes B_2)$ [cf. Eq. (15b)] in $|\Phi\rangle_{12}$ [cf. Eq. (14)] is $H(A)=H(B)=S(1)=S(2)$, which equals the relative entropy of entanglement $E_{RE}(|\Phi\rangle_{12})$ or the quantum discord $\delta(m2\rightarrow 1)$ in this state. In ρ'_{12} given by Eq. (18) the analogous coherence entropies are zero (because $[(A_1\otimes 1),\rho'_{12}]=[(1\otimes B_2),\rho'_{12}]=0$).

Thus, in *every pure bipartite state* $|\Phi\rangle_{12}$ it is not only true that a pair of twin observables A_1 and B_2 "carries" the quasiclassical part of correlations, i.e., the one accessible by subsystem measurement, but it is also true that *the same twin observables "carry" also the subsystem-measurement-inaccessible part of correlations*, i.e., the quantum entanglement, via the amount of coherence of any of the twin observables in the bipartite state.

APPENDIX A

Proof of relations (9) is based on $\sum_j (Q_2^j)^2=1$, and on $\mathrm{Tr}_2[(\rho_{12}Q_2^j)Q_2^j]=\mathrm{Tr}_2[Q_2^j(\rho_{12}Q_2^j)]$,

$$T_A\rho_1\equiv\sum_i P_1^i(\mathrm{Tr}_2\,\rho_{12})P_1^i=\sum_i P_1^i\left[\mathrm{Tr}_2\left(\sum_j Q_2^j\rho_{12}Q_2^j\right)\right]P_1^i$$

$$=\mathrm{Tr}_2\sum_i P_1^i\left(\sum_j Q_2^j\rho_{12}Q_2^j\right)P_1^i=\mathrm{Tr}_2\,T_AT_B\rho_{12}.$$

The second relation in Eq. (9) is proved symmetrically.

APPENDIX B

Proofs for the initial claims in the definition of twin observables. As well known, statistical operators, in particular, the reduced ones, have purely discrete spectra and their spectral forms (with distinct characteristic values) read: $\rho_s=\sum_k r_k^s Q_s^k$, $s=1,2$. As a consequence of the commutations in property (i), one has $\forall\ k:[A_1,Q_1^k]=0,[B_2,Q_2^k]=0$. Since the range projectors Q_s of ρ_s are $Q_s=\sum_k Q_s^k$, $s=1,2$ (all r_k^s are positive), one has also $[A_1,Q_1]=0$, $[B_2,Q_2]=0$. Hence, the (topological closures of the) ranges $\mathcal{R}(\rho_s)$ ($\bar{\mathcal{R}}(\rho_s)=\mathcal{R}(Q_s)$), $s=1,2$ are invariant subspaces for A_1 and B_2, respectively. Further, since also the characteristic subspaces $\mathcal{R}(Q_1^k)$ of ρ_1 are invariant for A_1, and they are necessarily finite dimensional (because $\sum_k d_k^1 r_k^1=\mathrm{Tr}\,\rho_1=1$, where d_k^1 is the multiplicity of r_k^1), only discrete characteristic values of A_1 appear in $\mathcal{R}(\rho_1)$, and symmetrically for B_2.

Let $\sum_l a_l P_1^l$ be the discrete part of the spectral form (with distinct characteristic values) of A_1. This operator and A_1Q_1 act equally in $\bar{\mathcal{R}}(\rho_1)$. Further, as already proved, all spectral projectors of A_1 belonging to its (possible) continuous spectrum are subprojectors of the null-space projector $Q_1^\perp$. Hence, $A_1Q_1=\sum_l a_l(P_1^lQ_1)$. Omitting all terms in which $P_1^lQ_1=0$, and changing the index from l to i in the remaining sum, one obtains the spectral form $A_1Q_1=\sum_i a_i(P_1^iQ_1)$. Obviously, A_1 has those and only those characteristic values a_i in $\bar{\mathcal{R}}(\rho_1)$ for which $P_1^iQ_1\neq 0$.

On the other hand, the detectable discrete characteristic values a_n of A_1 in ρ_{12} are those for which $0<p_n\equiv\mathrm{Tr}(\rho_1P_1^n)$. One can always write $\rho_1=\rho_1Q_1$. Therefore, $p_n=\mathrm{Tr}[\rho_1(P_1^nQ_1)]$. If $P_1^nQ_1=0$, then $p_n=0$. If $P_1^nQ_1\neq 0$, and we substitute the spectral form $\rho_1=\sum_k r_k^1Q_1^k$, then $p_n=\sum_k r_k^1\mathrm{Tr}(P_1^nQ_1^k)$. (We omit Q_1 because $Q_1Q_1^k=Q_1^k$.) Since $\sum_k P_1^nQ_1^k=P_1^nQ_1$, which is nonzero by assumption, not all $P_1^nQ_1^k$ can be zero. The nonzero terms $r_k^1\mathrm{Tr}(P_1^nQ_1^kP_1^n)$ are obviously positive. Thus, $p_n>0$, and a_n is detectable. This bears out the claim that precisely the detectable values of A_1 in ρ_{12} appear as its characteristic values in $\bar{\mathcal{R}}(\rho_1)$. (Thus, we can write i instead of n like in the preceding passage.)

Proof of equivalence of the four conditions will be given via the following closed chain of implications: (a) $\Rightarrow$ (d) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

LINK (a) $\Rightarrow$ (d).

Let

$$\rho_{12}=\sum_k w_k|\Phi\rangle_{12}^k\langle\Phi|_{12}^k \tag{B1}$$

be a (convex linear) decomposition of ρ_{12} into ray projectors. (For instance, the $|\Phi\rangle_{12}^k$ can be the characteristic state vectors of ρ_{12}.) If a projector E is probability-one in ρ_{12}, then so is it in each $|\Phi\rangle_{12}^k$ [as seen from $1=\mathrm{Tr}(\rho_{12}E)=\sum_k w_k\mathrm{Tr}(|\Phi\rangle_{12}^k\langle\Phi|_{12}^kE)$ and $\sum_k w_k=1$]. Further,

$$1=\langle\Phi|_{12}^{k}E|\Phi\rangle_{12}^{k}\Rightarrow 0=\langle\Phi|_{12}^{k}E^{\perp}|\Phi\rangle_{12}^{k}\Rightarrow||E^{\perp}|\Phi\rangle_{12}^{k}||^{2}$$
$$=0\Rightarrow E^{\perp}|\Phi\rangle_{12}^{k}=0\Rightarrow E|\Phi\rangle_{12}^{k}=|\Phi\rangle_{12}^{k}.$$

The sum $\Sigma_i P_1^i$ ($\Sigma_i P_2^i$) of all detectable values of A_1 (B_2) is a probability-one projector in ρ_{12}. Therefore,

$$\forall\ k:|\Phi\rangle_{12}^{k}=\left(\sum_i P_1^i\right)|\Phi\rangle_{12}^{k}=\left(\sum_i P_2^i\right)|\Phi\rangle_{12}^{k},$$

and

$$|\Phi\rangle_{12}^{k}=\left(\sum_i P_1^i\right)\left(\sum_i P_2^i\right)|\Phi\rangle_{12}^{k}=\sum_{ii'} P_1^i P_2^{i'}|\Phi\rangle_{12}^{k}. \tag{B2}$$

Assuming the validity of condition (a), and utilizing Eq. (B1), we have

$$i\neq i' \quad\Rightarrow\quad 0=p_{ii'}\equiv \mathrm{Tr}\,\rho_{12}P_1^iP_2^{i'}$$
$$=\sum_k w_k\langle\Phi|_{12}^{k}P_1^iP_2^{i'}|\Phi\rangle_{12}^{k}.$$

Since $\forall\ k:w_k>0$, the second factor in each term in this sum, generally nonnegative, must be zero. This implies, by making use of the definiteness of the norm as above, that for distinct i and i'

$$\forall\ k:P_1^iP_2^{i'}|\Phi\rangle_{12}^{k}=0. \tag{B3}$$

Relations (B2) and (B3) imply

$$\forall\ k,i:P_1^i|\Phi\rangle_{12}^{k}=P_1^iP_2^i|\Phi\rangle_{12}^{k}=P_2^i|\Phi\rangle_{12}^{k}. \tag{B4}$$

Relation (B4) in conjunction with Eq. (B1) finally gives condition (d).

LINK (d) $\Rightarrow$ (b)

Making use of condition (d) and its adjoint in the LHS of condition (b), this condition is immediately derived.

LINK (b) $\Rightarrow$ (c)

The LHS of condition (c) can be rewritten as

$$\forall\ i:p_i^{-1}\mathrm{Tr}P_1^i(P_2^i\rho_{12}P_2^i)P_1^i.$$

If one utilizes condition (b), this expression becomes $p_i^{-1}p_i$, i.e., condition (c) follows.

LINK (c) $\Rightarrow$ (a)

Let us return to the argument given in the proof of the link [(a) $\Rightarrow$ (d)], and to Eq. (B1). It was shown that a probability-one projector E in ρ_{12} is such an event also in each $|\Phi\rangle_{12}^{k}$, and $\forall\ k:E|\Phi\rangle_{12}^{k}=|\Phi\rangle_{12}^{k}$. Then, Eq. (B1) implies

$$E\rho_{12}=\rho_{12}. \tag{B5}$$

Assuming the validity of (c), P_2^i is a probability-one projector in $\rho_2(P_1^i)$, hence, on account of the adjoint of Eq. (B5), one has

$$\rho_2(P_1^i)=\rho_2(P_1^i)P_2^i. \tag{B6}$$

The LHS of condition (a), due to Eq. (B6), implies

$$p_{ii'}\equiv\mathrm{Tr}(\rho_{12}P_1^iP_2^{i'})=p_i\mathrm{Tr}[\rho_2(P_1^i)P_2^{i'}]$$
$$=p_i\mathrm{Tr}[(\rho_2(P_1^i)P_2^i)P_2^{i'}]=\delta_{i,i'}p_i.$$

Thus, (a) is derived.

Proof of the stronger algebraic relation. Since $\rho_{12}=(\Sigma_iP_1^i)\rho_{12}$, one has $A_1\rho_{12}=(\Sigma_ia_iP_1^i)\rho_{12}$. Assuming then property (iv), i.e., $\forall\ i:a_i=b_i$; and utilizing condition (d), one further obtains

$$A_1\rho_{12}=\left(\sum_i b_iP_2^i\right)\rho_{12}=B_2\rho_{12}.$$

The last equality is due to the fact that for the second subsystem one has the symmetric argument. Thus, the stronger algebraic relation is derived.

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