Bohm's interpretation and maximally entangled states

Thomas Durt[1] and Yves Pierseaux[2]
[1]*TENA, TONA Free University of Brussels, Pleinlaan 2, B-1050 Brussels, Belgium*
[2]*Attached to ELEM, TENA, Free University of Brussels, Pleinlaan 2, B-1050 Brussels, Belgium*
(Received 20 August 2002; published 22 November 2002)

Several no-go theorems showed the incompatibility between the locality assumption and quantum correlations obtained from maximally entangled spin states. We analyze these no-go theorems in the framework of Bohm's interpretation. The mechanism by which nonlocal correlations appear during the results of measurements performed on distant parts of entangled systems is explicitly put into evidence in terms of Bohmian trajectories. It is shown that a GHZ-like contradiction of the type $+1=-1$ occurs for well-chosen initial positions of the Bohmian trajectories and that it is this essential nonclassical feature that makes it possible to violate the locality condition.

DOI: 10.1103/PhysRevA.66.052109 PACS number(s): 03.65.Ud, 03.65.Ta

I. INTRODUCTION

Several no-go theorems show that it is impossible to conciliate locality and quantum mechanics. The essence of these theorems is that it is impossible to simulate the results of observations carried out on distant parts of an entangled quantum system in terms of a local common cause mechanism. This impossibility can be expressed through the violation of an inequality [1–3] or, even "stronger," of an equality [4,5]. Besides it is well known that the Bohmian interpretation, according to which particles have at any time a well-defined position [6], makes it possible to simulate all the quantum predictions for massive particles in the nonrelativistic regime. Therefore, it is interesting to understand better how nonlocality appears in the Bohmian picture. It is well known that for entangled systems the velocity associated with the Bohmian trajectories can be nonlocally influenced [7,8]. The goal of our paper is to analyze in detail the nature of the Bohmian trajectories in the specific situations encountered in the formulation of three well-known no-go theorems, Bell's theorem [1], Mermin's theorem [5], and the (GHZ) Greenberger-Horne-Zeilinger theorem [4], in which, respectively, two, three, and four spin-(1/2) particles are assumed to be prepared in maximally entangled states. The knowledge of the initial positions of these particles is sufficient in Bohm's formulation in order to predict the whole trajectories and also the results of local spin measurements performed with Stern-Gerlach devices. Note, however, that the specification of the initial positions together with the full measurement setup (not only its local parts) is required in order to explain all the quantum correlations. This property reflects the nonlocality of Bohmian dynamics, a fundamentally nonclassical feature of Bohm's interpretation by which Bell was led, together with the argument of Einstein, Podolsky, and Rosen (EPR) [9], to his famous inequalities [1]. We shall show explicitly in Sec. II of this paper how, for well-chosen initial positions, a GHZ-like contradiction occurs for a situation during which Bell's inequalities get violated. This result confirms Hardy's formulation of Bell's inequalities [10] according to which the violation of Bell's inequalities is proportional to the probability that a GHZ-like contradiction of the type $+1=-1$ occurs [4]. We shall also show explicitly in Sec. III of this paper how in the situation considered in the no-go theorems of Mermin [5] and Greenberger, Horne, Shimony, and Zeilinger [4] the GHZ-like contradiction occurs for all the initial positions.

II. BELL'S THEOREM IN A BOHMIAN DESCRIPTION

A. Simultaneous Stern-Gerlach measurements performed on a pair of entangled particles

In the situation described in Bell's theorem [1], two particles are prepared in an entangled spin state and spin measurements are performed simultaneously on each particle. The full wave function associated with such a state can be put in the form

$$\begin{aligned}|\Psi(\mathbf{r}_L,\mathbf{r}_R,\ t)\rangle = {} & a\,\psi_{++}(\mathbf{r}_L,\mathbf{r}_R,t)|+\rangle_L\otimes|+\rangle_R\\ & +b\,\psi_{+-}(\mathbf{r}_L,\mathbf{r}_R,t)|+\rangle_L\otimes|-\rangle_R\\ & +c\,\psi_{-+}(\mathbf{r}_L,\mathbf{r}_R,t)|-\rangle_L\otimes|+\rangle_R\\ & +d\,\psi_{--}(\mathbf{r}_L,\mathbf{r}_R,t)|-\rangle_L\otimes|-\rangle_R, \qquad (1)\end{aligned}$$

where a,b,c,d are complex constants representative of the spin entanglement, the indices L and R represent arbitrary spatial reference frames in two distant regions, the left and right regions, $\mathbf{r}_{L/R}$ represent the position vector expressed in these frames, and $|+/-\rangle_{L/R}$ represent the up/down spin states along the (not necessary parallel) Z axes of these frames. We shall assume for convenience that Stern-Gerlach devices are disposed at the same distance from the source, along the axes of propagation $X_{L/R}$ of the particles, that the point of penetration inside them coincides with the origins of the spatial frames, and with the origin of time, and that the initial state consists of a pair of Gaussian shaped particles of mass m propagating at a speed v_0:

$$\begin{aligned}|\Psi(\mathbf{r}_L,\mathbf{r}_R,t)\rangle = {} & \frac{1}{(2\sqrt{\pi}\,\delta r_0)^{6/2}}\exp\left(\frac{-\mathbf{r}_L^2-\mathbf{r}_R^2}{4\,\delta r_0^2}\right)\\ & \times\exp[ik_o(x_L+x_R)]. \qquad (2)\end{aligned}$$

When the $Z_{L/R}$ axis is parallel to the L/R magnetic field, it can be shown that the equation of evolution is separable into the coordinates $(x_L, y_L, z_L, x_R, y_R, z_R)$. For reasons of convenience, we shall also assume that the Stern-Gerlach setups are similar in the left and right regions. Then, it is easy to show that the solution of the equation of evolution given in Appendix B can be expressed in terms of the solution of the single-particle case:

$$|\Psi(\mathbf{r}_L,\mathbf{r}_R,t)\rangle = a\psi_{+L}(\mathbf{r}_L,t)\psi_{+R}(\mathbf{r}_R,t)|+\rangle_L\otimes|+\rangle_R + b\psi_{+L}(\mathbf{r}_L,t)\psi_{-R}(\mathbf{r}_R,t)|+\rangle_L\otimes|-\rangle_R + c\psi_{-L}(\mathbf{r}_L,t)\psi_{+R}(\mathbf{r}_R,t)|-\rangle_L\otimes|+\rangle_R + d\psi_{-L}(\mathbf{r}_L,t)\psi_{-R}(\mathbf{r}_R,t)|-\rangle_L\otimes|-\rangle_R, \tag{3}$$

with

$$\psi_{+/-L/R}(\mathbf{r}_{L/R},t) = \psi^x_{+/-L/R}(x_{L/R})\psi^y_{+/-L/R} \times (y_{L/R},t)\psi^z_{+/-L/R}(z_{L/R}), \tag{4}$$

where $\psi^x_{+/-}$, $\psi^y_{+/-}$, and $\psi^z_{+/-}$ are the solutions of the single-particle case that were originally derived by Bohm [11]. This derivation is reproduced in detail in Appendix A.

B. Bohm's interpretation

Bohm's interpretation [6] is, in summary, the following: whenever we can associate with the equation of evolution of a quantum system a conservation equation of the form $\partial_t\rho = \mathrm{div}(\mathcal{J})$, where ρ is a positive definite density of probability, and $\mathrm{div}(\mathcal{J})$ is the divergence of a current vector, we can interpret this density as a distribution of localized material points moving with the velocity $\mathcal{J}/\rho$. It is thus possible to formulate a hidden variable theory for the system: it would consist of a spatial distribution of material points, which initially coincides with the quantum distribution (given by ρ); these points move with a velocity equal to $\mathcal{J}/\rho$. In virtue of the conservation equation, the spatial distribution deduced from this evolution coincides then for all times with the quantum distribution. According to de Broglie, all the measurements being, as a last resort, position measurements, this hidden variable theory is, for what concerns practical purposes, equivalent to orthodox quantum mechanics.

In Appendix B, we show how to deduce such a conservation equation for the pair of particles considered here. From this equation of conservation, it is straightforward to deduce the conserved density [Eq. (B8)]

$$\rho(\mathbf{r}_L,\mathbf{r}_R,t) = \langle\Psi(\mathbf{r}_L,\mathbf{r}_R,t)|\Psi(\mathbf{r}_L,\mathbf{r}_R,t)\rangle = |a\psi_{+L}(\mathbf{r}_L,t)\psi_{+R}(\mathbf{r}_R,t)|^2 + |b\psi_{+L}(\mathbf{r}_L,t)\psi_{-R}(\mathbf{r}_R,t)|^2 + |c\psi_{-L}(\mathbf{r}_L,t)\psi_{+R}(\mathbf{r}_R,t)|^2 + |d\psi_{-L}(\mathbf{r}_L,t)\psi_{-R}(\mathbf{r}_R,t)|^2. \tag{5}$$

This density depends, in general, on the two positions $(\mathbf{r}_L,\mathbf{r}_R)$. The six-dimensional velocity [Eq. (B9)] associated with the conserved density through the relation $\mathcal{J}=\boldsymbol{\rho}\cdot\boldsymbol{v}$ can be split into two local three-dimensional velocities $\boldsymbol{v}_L, \boldsymbol{v}_R$ defined as follows:

$$\boldsymbol{v}_L(\mathbf{r}_L,\mathbf{r}_R,t) = \frac{\hbar}{m\rho}\{\mathrm{Im}[\psi_{+L}(\mathbf{r}_L,t)^*\boldsymbol{\nabla}_L\psi_{+L}(\mathbf{r}_L,t)] \times[|a\psi_{+R}(\mathbf{r}_R,t)|^2+|b\psi_{-R}(\mathbf{r}_R,t)|^2] + \mathrm{Im}[\psi_{-L}(\mathbf{r}_L,t)^*\boldsymbol{\nabla}_L\psi_{-L}(\mathbf{r}_L,t)] \times[|c\psi_{+R}(\mathbf{r}_R,t)|^2+|d\psi_{-R}(\mathbf{r}_R,t)|^2]\}, \tag{6}$$

$$\boldsymbol{v}_R(\mathbf{r}_L,\mathbf{r}_R,t) = \frac{\hbar}{m\rho}\{\mathrm{Im}[\psi_{+R}(\mathbf{r}_R,t)^*\boldsymbol{\nabla}_R\psi_{+R}(\mathbf{r}_R,t)] \times[|a\psi_{+L}(\mathbf{r}_L,t)|^2+|c\psi_{-L}(\mathbf{r}_L,t)|^2] + \mathrm{Im}[\psi_{-R}(\mathbf{r}_R,t)^*\boldsymbol{\nabla}_R\psi_{-R}(\mathbf{r}_R,t)] \times[|b\psi_{+L}(\mathbf{r}_L,t)|^2+|d\psi_{-L}(\mathbf{r}_L,t)|^2]\}, \tag{7}$$

where $\boldsymbol{\nabla}_{L(R)}$ is the gradient on the spatial left (right) coordinates and $\mathrm{Im}(z)$ is the imaginary part of z. In general, these velocities contain a nonvanishing contribution depending on the position of the particle situated at the other side of the source, and are thus influenced by what happens in a region from which they are separated by a spacelike distance. In the formulation of Bell's theorem, the particles are assumed to be prepared in the so-called singlet state for which $a=d=(1/\sqrt{2})\sin(\theta/2)$ and $b=-c=(1/\sqrt{2})\cos(\theta/2)$, where θ is the angle between the magnetic fields in the left and right Stern-Gerlach devices. When these axes are parallel, we recover the standard expression

$$|\Psi(\mathbf{r}_L,\mathbf{r}_R,t)\rangle = \frac{1}{\sqrt{2}}\psi_{+L}(\mathbf{r}_L,t)\psi_{-R}(\mathbf{r}_R,t)|+\rangle_L\otimes|-\rangle_R - \frac{1}{\sqrt{2}}\psi_{-L}(\mathbf{r}_L,t)\psi_{+R}(\mathbf{r}_R,t)|-\rangle_L\otimes|+\rangle_R. \tag{8}$$

It is worth noting that when the axes of the devices are parallel, the results of spin measurements are always opposite. This perfect correlation is due to the fact that the singlet state is maximally entangled, and that its expression is invariant under a simultaneous rotation of the axes of quantization, when both axes are parallel. The specification of a, b, c and d allows us to express explicitly the Bohmian dynamics during the passage through the Stern-Gerlach devices. We obtain, after some lengthy but simple computations, the following velocities for the left as well as for the right x and y components:

$$\frac{dx}{dt} = v_0 + \frac{k^2}{1+k^2t^2}t(x-v_0t), \tag{9}$$

$$\frac{dy}{dt}=\frac{k^2}{1+k^2t^2}ty \tag{10}$$

(where $k=\hbar/2m\delta^2 r_0$ expresses the diffusion inherent to the Schrödinger equation). The solutions of these dynamical equations are

$$x=v_0t+x_0\sqrt{1+k^2t^2}, \tag{11}$$

$$y=y_0\sqrt{1+k^2t^2}. \tag{12}$$

In Appendix A we show that, during a typical Stern-Gerlach experiment, the diffusion is negligible during the time of flight through the Stern-Gerlach devices. The trajectories along these axes are thus essentially free translations, and present no special interest for the understanding of the measurement process.

In the singlet state, the dynamics of the z components is given by

$$\frac{dz_L}{dt}=\frac{k^2}{1+k^2t^2}tz_L+\frac{+s^2\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(z_L+z_R)}{2}\right)+c^2\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(z_L-z_R)}{2}\right)-c^2\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-z_L+z_R)}{2}\right)-s^2\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-z_L-z_R)}{2}\right)}{+s^2\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(z_L+z_R)}{2}\right)+c^2\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(z_L-z_R)}{2}\right)+c^2\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-z_L+z_L)}{2}\right)+s^2\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-z_R-z_L)}{2}\right)}$$
$$\times\left\{\frac{-k^2}{1+k^2t^2}\alpha t^3+2\alpha t\right\}, \tag{13}$$

$$\frac{dz_R}{dt}=\frac{k^2}{1+k^2t^2}tz_R$$
$$+\frac{+s^2\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(z_L+z_R)}{2}\right)-c^2\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(z_L-z_R)}{2}\right)+c^2\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-z_L+z_L)}{2}\right)-s^2\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-z_R-z_L)}{2}\right)}{+s^2\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(z_L+z_R)}{2}\right)+c^2\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(z_L-z_R)}{2}\right)+c^2\exp\left(\frac{\beta t^2}{1+\ \ k^2t^2}\frac{(-z_L+z_R)}{2}\right)+s^2\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-z_L-z_R)}{2}\right)}$$
$$\times\left\{\frac{-k^2}{1+k^2t^2}\alpha t^3+2\alpha t\right\}, \tag{14}$$

where $\alpha=a_1\mu/2m$, $\beta=a_1\mu/m\delta r_0^2$, $c=\cos(\theta/2)$ and $s=\sin(\theta/2)$.

Numerical computations (confirmed by a careful evaluation of the terms present in the last equations) show that, when the two magnets act simultaneously, there exist four attractors for the trajectories, corresponding to the results of spin measurements ($\mathcal{U}$ refers to "up" and $\mathcal{D}$ to "down") $(\mathcal{U}_L,\mathcal{U}_R)$, $(\mathcal{U}_L,\mathcal{D}_R)$, $(\mathcal{D}_L,\mathcal{U}p_R)$, and $(\mathcal{D}_L,\mathcal{D}_R)$. During the passage through the device, the trajectories can be shown to fall very quickly in the basin of one of these attractors. For instance, in the attractor basin of the $(\mathcal{U}_L,\mathcal{U}_R)$ outcome, the exponential factor $e[\beta t^2/(1\ +\ k^2t^2)(z_L+z_R)/2]$ will dominate all the other exponential factors, and the z velocities will obey the following equations:

$$\frac{dz_L}{dt}=\frac{k^2}{1+k^2t^2}tz_L+1\left\{\frac{-k^2}{1+k^2t^2}\alpha t^3+2\alpha t\right\},$$

$$\frac{dz_R}{dt}=\frac{k^2}{1+k^2t^2}tz_R+1\left\{\frac{-k^2}{1+k^2t^2}\alpha t^3+2\alpha t\right\}.$$

In virtue of the fact that in a well-conceived device, we can neglect the diffusion terms relatively to the classical drift terms, we can take k to be equal to 0 in the previous equation, which gives the classical expression for the velocities (2α is the magnitude of the classical acceleration undergone inside the magnetic field of the device):

$$\frac{dz_L}{dt}=+2\alpha t,$$

$$\frac{dz_R}{dt}=+2\alpha t.$$

Similarly, in say the attractor basin of the $(\mathcal{D}_L,\mathcal{U}_R)$ outcome, we shall get

$$\frac{dz_L}{dt}=-2\alpha t,$$

$$\frac{dz_R}{dt}=+2\alpha t.$$

This means that after a short time, the trajectories become and remain quasiclassical. The choice of the attractor basin is deterministically specified by the initial positions. It is worth noting that when the initial positions lie in between the regions that are characterized by this quasiclassical behavior, the deflection will be quite less important. These positions can be associated with the tails of the outgoing wave packets and their weight is negligible, because the diffusion of the Gaussian wave packets is negligible relative to the average deflection undergone during the passage through the Stern-Gerlach device.

C. GHZ-like contradiction in the two-particles case

As a last resort, the final outcome is thus *nearly always* unambiguously determined by the initial positions z_L and z_R, together with the specifications of the directions of the magnets in the left and right regions. Let us assume that we choose to change the direction of the magnet in the right device. Then, the final outcome is determined by the initial positions z_L and z'_R. z'_R depends on y_R and z_R and on the angle between the old quantization axis Z_R and the new one Z'_R. It could happen that for the same initial position coordinates z_L, y_R, z_R, the attractor basin of the left trajectory changes due to the change of z_R to z'_R. This means that the choice of the basis of detection in the right region can nonlocally influence the outcome of the measurement in the left region. Therefore although the results of all possible measurements can be deterministically foreseen on the basis of the knowledge of the initial positions, which constitutes a deterministic mechanism, this mechanism is nonlocal, contrarily to the common cause mechanisms considered by Bell [1], which are assumed to be deterministic and local. This helps to understand how Bohmian mechanics makes it possible to violate Bell's inequalities. The probability of an outcome can be obtained, after integration of the Bohmian velocities, by performing a weighted average on the predictions associated with this outcome for particular initial positions. The weight is equal to the initial probability of presence, a Gaussian distribution in our case. In accordance with Bohm's interpretation, the probability obtained so is exactly the same as the standard quantum probability. In Ref. [12], the attractor basins of the outcomes related to different noncompatible spin measurements in the left and right regions were explicitly determined for a well-chosen initial position and the following result was shown. If we choose four directions (Z_L, Z'_L, Z_R, Z'_R) such that the directions Z_L, Z'_L, Z'_R are coplanar and are all separated by angles of 120°, while Z'_L and Z_R are parallel, that the system of two particles is prepared in the singlet state, and that the initial coordinates of position of the pair are defined as follows: $(y_L, z_L)(t=0)=10^{-3}$ cm $\times(\sqrt{3}/2,1/2)$ and $(y_R, z_R)(t=0)=1.1\times10^{-3}$ cm $\times(\sqrt{3}/2,1/2)$, then we shall observe, after integration of the Bohmian velocities, the following outcomes: $(\mathscr{U}_{Z_L},\mathscr{U}_{Z_R}),(\mathscr{U}_{Z'_L},\mathscr{U}_{Z'_R})$, and $(\mathscr{D}_{Z'_L},\mathscr{U}_{Z_R})$. If the locality assumption was valid, the results of the measurements ought to be predetermined before the measurements take place. This can be shown by an EPR-like reasoning [9], which is, roughly summarized, the following: in the singlet state, the value of the spin component along an arbitrary axis of reference in the left region can be deduced from the observation of the corresponding value in the right region because the singlet state exhibits perfect (100%) correlations; if we assume that distant measurements may not influence the local spin values, then these values existed necessarily before their measurement took place. This means that the local value (in the left region but also in the right region by a similar reasoning) of the spin along an arbitrary direction ought to be predetermined before the measurement.

Obviously, if the locality assumption was valid, we could deduce from the predictions relative to the two first joint measurements $((\mathscr{U}_{Z_L},\mathscr{U}_{Z_R}),(\mathscr{U}_{Z'_L},\mathscr{U}_{Z'_R}))$ that the third result ought to be $(\mathscr{U}_{Z'_L},\mathscr{U}_{Z_R})$ too, which is not the case here [we find $(\mathscr{D}_{Z'_L},\mathscr{U}_{Z_R})$]. In Ref. [12] we showed how Bohmian nonlocality is related to the nonfactorizability condition postulated by Bell as a consequence of the locality condition. For the purposes of the present paper, we went further. We also integrated the velocities when the magnets are parallel to the direction (Z_L, Z'_R), and we found the outcome $(\mathscr{U}_{Z_L},\mathscr{U}_{Z'_R})$. By assigning a value $+1$ to the outcome spin up and -1 to the outcome spin down, we arrive at a GHZ-like contradiction of the type $+1=-1$ that we shall study in detail in the following section. This can be done by considering the product of the outcomes of $Z_L\cap Z_R$ and $Z'_L\cap Z'_R$ on one side and the product of the outcomes of $Z'_L\cap Z_R$ and $Z_L\cap Z'_R$ on the other side. If the outcomes associated with the four experiments were predetermined, these products ought clearly to be equal, but for the choice of initial position considered here, they are, respectively, equal to $+1$ and -1, which can be put in the paradoxical form $+1=-1$.

It can be shown [12] that when the four directions (Z_L, Z'_L, Z_R, Z'_R) are coplanar and are all separated by angles of 120°, while Z'_L and Z_R are parallel, and that the system of two particles is prepared in the singlet state, Bell's and Clauser-Horne's inequalities are violated. It can also be shown that a GHZ-like contradiction occurs only for a fraction of the initial positions that differs significantly from unity. For instance, when the initial coordinates of position of the pair are defined as follows: $(y_L, z_L)(t=0)=10^{-3}$ cm$\times(\sqrt{3}/2,1/2)$ and $(y_R, z_R)(t=0)=-1\times10^{-3}$ cm$\times(\sqrt{3}/2,1/2)$, then we shall observe, after integration of the Bohmian velocities, the following outcomes: $(\mathscr{U}_{Z_L},\mathscr{D}_{Z_R}),(\mathscr{U}_{Z'_L},\mathscr{D}_{Z'_R})$, $(\mathscr{U}_{Z_L},\mathscr{D}_{Z'_R})$, and $(\mathscr{U}_{Z'_L},\mathscr{D}_{Z_R})$. In this case, no GHZ contradiction occurs, and the results of local measurements do not depend on which measurements are performed in a distant region. These results confirm Hardy's analysis, who showed [10] that the degree of violation of Bell's and Clauser-Horne's inequalities is proportional to the probability of encountering GHZ-like paradoxical situations.

Besides, Peres [13] showed that for two particles in the singlet state, such a paradoxical situation also characterized the quantum average values of well-chosen observables. The interpretation of Peres's paradox in the framework of Bohm's theory was already studied by Dewdney in Ref. [8], so we shall not repeat his work in the present paper. Furthermore, it is impossible to express Peres's paradox in terms of local Stern-Gerlach measurements, only because it involves nonlocal measurements. Therefore, presently, there is no explicit formulation of this paradox in terms of Bohmian trajectories. Nevertheless, such a formulation is possible, in principle [8]. It is worth noting that the implications of Bohm's theory regarding nonlocality were already recognized by Bohm and Bell, and that the elucidation of certain no-go theorems was qualitatively developed in the past (see, for instance, Refs. [7,8] and references therein). Computer simulations of the dynamics inside the Stern-Gerlach devices (with two particles and parallel magnets only) can be found in Ref. [8], as well as a qualitative discussion of their connection with Peres's paradox. The analytical expression of Bohm's velocities in the general two-particle case (with parallel and nonparallel magnets) was given in Ref. [12], as well as the connection between Bohmian nonlocality and nonfactorizability. Nevertheless, the occurrence of a GHZ-like contradiction in the two-particle case as well as the explicit study of the topology of the attractor basins for the three- and four-particle cases and their connection to GHZ-like contradictions were not, to the knowledge of the authors, published elsewhere. This brings us to the following section.

III. MERMIN AND GHZ NO-GO THEOREMS IN A BOHMIAN DESCRIPTION

The two paradoxes that we shall study in this section are of the following form: for well-chosen local observables and quantum states, the quantum correlations are such that if the results of the local measurements are fixed in advance and do not depend on the measurements that are performed in distant regions we come to a contradiction of the type $+1=-1$. We showed in the preceding section that in the two-particle case such a contradiction could occur for well-chosen initial positions. However, after averaging over the initial positions of the particles, for which the contradiction occurs sometimes but not always, the paradox is expressed by the violation of an inequality in accordance with Hardy's analysis [10]. The novelty of Mermin's and GHZ paradoxes [4,5] is that the contradiction does not take the form of the violation of an inequality but really of an equality. As we shall now show, the meaning of this result, interpreted in terms of Bohm's trajectories is that all the initial positions lead to the same paradoxical situation, and not only some of them, as in the two-particle case.

A. Mermin's no-go theorem: GHZ-like contradiction in the three-particle case

In Mermin's theorem [5], three particles are prepared in a maximally entangled spin state of the form

$$|\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,t)\rangle=\frac{1}{\sqrt{2}}\psi_{+1}(\mathbf{r}_1,t)\psi_{+2}(\mathbf{r}_2,t)\psi_{+3}(\mathbf{r}_3,t)|+\rangle_1 \otimes|+\rangle_2\otimes|+\rangle_3-\frac{1}{\sqrt{2}}\psi_{-1}(\mathbf{r}_1,t)\psi_{-2} \times(\mathbf{r}_2,t)\psi_{-3}(\mathbf{r}_3,t)|-\rangle_1\otimes|-\rangle_2\otimes|-\rangle_3. \tag{15}$$

The particles are sent along three coplanar directions separated by an angle of 120° $|+\rangle_i$ ($|-\rangle_i$) represents a spin-up (spin-down) state along the direction of propagation of the ith particle. Four "product" observables, each of which is associated with a simultaneous spin measurement performed on the three particles, are considered: $\Sigma_x^1\Sigma_y^2\Sigma_y^3$, $\Sigma_y^1\Sigma_x^2\Sigma_y^3$, $\Sigma_y^1\Sigma_y^2\Sigma_x^3$, and $\Sigma_x^1\Sigma_x^2\Sigma_x^3$, where Σ_x^i represents a Stern-Gerlach mesurement performed on the ith particle with the magnet parallel to the X direction, which is orthogonal to the plane of propagation of the particles, and Σ_y^i represents a Stern-Gerlach mesurement performed on the ith particle with the magnet parallel to the Y_i direction, which is orthogonal to the direction of propagation of the ith particle and to X. The Σ matrices possess two eigenvalues, $+1$ and -1. The eigenvalue $+1$ corresponds to the outcome spin up and -1 to the outcome spin down. $|\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,t)\rangle$ is the simultaneous eigenstate of the four product observables for the eigenvalues $+1,+1,+1,-1$. This imposes severe constraints on the results of l ocal spin measurements. For instance, without performing the measurement of Σ_x^1, we can deduce what would be the outcome observed during this measurement from the results of the measurements of Σ_x^2 and Σ_x^3. By an EPR-like reasoning similar to that described in the preceding section, the locality assumption implies that the outcome of the measurement of Σ_x^1 must be determined before the measurement took place and does not depend on which measurement is performed in distant regions on the other particles. A similar reasoning holds for the measurement of Σ_y^1, Σ_x^2, Σ_y^2, Σ_x^3, and Σ_y^3. Let us denote as S_x^i and S_y^i the prediction assigned to the measurement of Σ_x^i and Σ_y^i; the prediction $+1$ corresponds to an "up" deflection inside the Stern-Gerlach device, and -1 to a "down" deflection. The constraints imposed by the properties of the maximally entangled state in which the particles are prepared can be expressed as follows:

$$\begin{aligned} S_x^1S_y^2S_y^3&=+1,\\ S_y^1S_x^2S_y^3&=+1,\\ S_y^1S_y^2S_x^3&=+1,\\ S_x^1S_x^2S_x^3&=-1. \end{aligned} \tag{16}$$

The product of the three first equations gives

$$(S_y^1)^2(S_y^2)^2(S_y^3)^2S_x^1S_x^2S_x^3=+1. \tag{17}$$

S_y^i being equal to $+$ or -1, we get

$$S_x^1S_x^2S_x^3=+1, \tag{18}$$

which together with the fourth equation leads to a contradiction of the type $+1=-1$. We encountered already such a contradiction in the preceding section, and showed that it could be explained in terms of the nonlocal properties of Bohmian trajectories. This is also true in the present case. By a straightforward generalization of the two-particle case, it can be shown that the Bohmian velocities during the measurement of one spin component along X, one component along Y, and another one along Y' are the following:

$$\frac{dx}{dt}=\frac{k^2}{1+k^2t^2}tx+\frac{+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(x+y+y')}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(x-y-y')}{2}\right)-\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x-y+y')}{2}\right)-\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x+y-y')}{2}\right)}{+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(x+y+y')}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(x-y-y')}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x-y+y')}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x+y-y')}{2}\right)}\times\left\{\frac{-k^2}{1+k^2t^2}\alpha t^3+2\alpha t\right\}, \tag{19}$$

$$\frac{dy}{dt}=\frac{k^2}{1+k^2t^2}ty+\frac{+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(x+y+y')}{2}\right)-\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(x-y-y')}{2}\right)-\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x-y+y')}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x+y-y')}{2}\right)}{+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(x+y+y')}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(x-y-y')}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x-y+y')}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x+y-y')}{2}\right)}\times\left\{\frac{-k^2}{1+k^2t^2}\alpha t^3+2\alpha t\right\}, \tag{20}$$

$$\frac{dy'}{dt}=\frac{k^2}{1+k^2t^2}ty'+\frac{+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(x+y+y')}{2}\right)-\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(x-y-y')}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x-y+y')}{2}\right)-\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x+y-y')}{2}\right)}{+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(x+y+y')}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(x-y-y')}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x-y+y')}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x+y-y')}{2}\right)}\times\left\{\frac{-k^2}{1+k^2t^2}\alpha t^3+2\alpha t\right\}. \tag{21}$$

They possess four attractor basins that correspond to the outcomes $(+++)$, $(+--)$, $(--+)$, and $(-+-)$. During the measurement of the fourth observable $\Sigma_x^1\Sigma_x^2\Sigma_x^3$, the Bohmian trajectories obey

$$\frac{dx_1}{dt}=\frac{k^2}{1+k^2t^2}tx_1+\frac{-\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x_1-x_2-x_3)}{2}\right)-\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x_1+x_2+x_3)}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(+x_1-x_2+x_3)}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(+x_1+x_2-x_3)}{2}\right)}{+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x_1-x_2-x_3)}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x_1+x_2+x_3)}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(+x_1-x_2+x_3)}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(+x_1+x_2-x_3)}{2}\right)}\times\left\{\frac{-k^2}{1+k^2t^2}\alpha t^3+2\alpha t\right\}, \tag{22}$$

$$\frac{dx_2}{dt}=\frac{k^2}{1+k^2t^2}tx_2+\frac{-\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x_1-x_2-x_3)}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x_1+x_2+x_3)}{2}\right)-\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(+x_1-x_2+x_3)}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(+x_1+x_2-x_3)}{2}\right)}{+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x_1-x_2-x_3)}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x_1+x_2+x_3)}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(+x_1-x_2+x_3)}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(+x_1+x_2-x_3)}{2}\right)}\times\left\{\frac{-k^2}{1+k^2t^2}\alpha t^3+2\alpha t\right\}, \tag{23}$$

$$\frac{dx_3}{dt}=\frac{k^2}{1+k^2t^2}tx_3+\frac{-\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x_1-x_2-x_3)}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x_1+x_2+x_3)}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(+x_1-x_2+x_3)}{2}\right)-\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(+x_1+x_2-x_3)}{2}\right)}{+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x_1-x_2-x_3)}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(-x_1+x_2+x_3)}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(+x_1-x_2+x_3)}{2}\right)+\exp\left(\frac{\beta t^2}{1+k^2t^2}\frac{(+x_1+x_2-x_3)}{2}\right)}\times\left\{\frac{-k^2}{1+k^2t^2}\alpha t^3+2\alpha t\right\}. \tag{24}$$

They possess four attractor basins that correspond to the outcomes $(---)$, $(-++)$, $(+-+)$, and $(++-)$. Very quickly, the Bohmian trajectory will fall into one of these basins and follow a nearly classical dynamics. For each choice of initial position (except for the initial positions that belong to the regions that separate two basins, which are regions of negligible weight that correspond to the tails of the Gaussian packets), the product of the predicted outcomes associated with the observables Σ_x^1, Σ_x^2, and Σ_x^3 is equal to $+1$ when we consider their realization during the first three experiments and to -1 when we consider the last one during which they are simultaneously measured.

B. Greenberger, Horne, and Zeilinger's no-go theorem: GHZ-like contradiction in the four-particle case

For what concerns the GHZ paradox [4], we shall first reformulate it in a simplified form that is closer to Mermin's formulation. Four particles are now prepared in a maximally entangled spin state of the form

$$|\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,t)\rangle = \frac{1}{\sqrt{2}}\psi_{+1}(\mathbf{r}_1,t)\psi_{+2}(\mathbf{r}_2,t)\psi_{-3}(\mathbf{r}_3,t)\psi_{-4}(\mathbf{r}_4,t)|+\rangle_1 \otimes|+\rangle_2\otimes|-\rangle_3\otimes|-\rangle_4-\frac{1}{\sqrt{2}}\psi_{-1}(\mathbf{r}_1,t)\psi_{-2}(\mathbf{r}_2,t) \times\psi_{+3}(\mathbf{r}_3,t)\psi_{+4}(\mathbf{r}_4,t)|-\rangle_1\otimes|-\rangle_2\otimes|+\rangle_3\otimes|+\rangle_4. \tag{25}$$

The particles are sent along four different coplanar directions and $|+\rangle_i$ ($|-\rangle_i$) represents a spin-up (spin-down) state along the direction of propagation of the ith particle. Four product observables, each of which is associated with simultaneous spin measurements performed on the four particles, are considered: $\Sigma_x^1\Sigma_x^2\Sigma_x^3\Sigma_x^4$, $\Sigma_y^1\Sigma_x^2\Sigma_y^3\Sigma_x^4$, $\Sigma_y^1\Sigma_x^2\Sigma_x^3\Sigma_y^4$, and $\Sigma_x^1\Sigma_x^2\Sigma_y^3\Sigma_y^4$, where Σ_x^i represents a Stern-Gerlach mesurement performed on the ith particle with the magnet parallel to the X direction, which is orthogonal to the plane of propagation of the particles, and Σ_y^i represents a Stern-Gerlach measurement performed on the ith particle with the magnet parallel to the Y direction, which is orthogonal to the direction of propagation of the ith particle and to X. $|\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,t)\rangle$ is the simultaneous eigenstate of the four product observables for the eigenvalues $-1,-1,-1,+1$. As before, this imposes severe constraints on the results of local spin measurements. By performing an EPR-like reasoning similar to the previous ones, the assumption of locality implies that these results must be determined before the measurements take place and may not depend on which measurement is performed in distant regions on the other particles. Let us denote by S_x^i and S_y^i the prediction assigned to the measurement of Σ_x^i and Σ_y^i; the prediction $+1$ corresponds to an up deflection inside the Stern-Gerlach device, and -1 to a down deflection. These constraints are expressed as follows:

$$S_x^1S_x^2S_x^3S_x^4=-1,$$
$$S_y^1S_x^2S_y^3S_x^4=-1,$$
$$S_y^1S_x^2S_x^3S_y^4=-1,$$
$$S_x^1S_x^2S_y^3S_y^4=+1. \tag{26}$$

The product of the three last equations gives

$$S_x^1S_x^2S_x^3S_x^4=+1, \tag{27}$$

which together with the first equation leads to a contradiction of the type $+1=-1$. The analysis of Bohm's trajectories is similar to the two- and three-particle cases, so that we shall not repeat it entirely here. For instance, during the measurement of $\Sigma_x^1\Sigma_x^2\Sigma_x^3\Sigma_x^4$ we get that the velocity of the first particle obeys the following equation:

$$\frac{dx_1}{dt}=\frac{k^2}{1+k^2t^2}tx_1+\left\{-\sinh\left[\frac{\beta t^2}{1+k^2t^2}(-x_1-x_2-x_3+x_4)\right] -\sinh\left[\frac{\beta t^2}{1+k^2t^2}(-x_1+x_2+x_3+x_4)\right]+\sinh\left[\frac{\beta t^2}{1+k^2t^2} \times(+x_1-x_2+x_3+x_4)\right]+\sinh\left[\frac{\beta t^2}{1+k^2t^2}(+x_1+x_2 -x_3+x_4)\right]\right\}\Big/\left\{+\cosh\left[\frac{\beta t^2}{1+k^2t^2}(-x_1-x_2-x_3 +x_4)\right]+\cosh\left[\frac{\beta t^2}{1+k^2t^2}(-x_1+x_2+x_3+x_4)\right] +\cosh\left[\frac{\beta t^2}{1+k^2t^2}(+x_1-x_2+x_3+x_4)\right] +\cosh\left[\frac{\beta t^2}{1+k^2t^2}(+x_1+x_2-x_3+x_4)\right]\right\} \times\left\{\frac{-k^2}{1+k^2t^2}\alpha t^3+2\alpha t\right\}. \tag{28}$$

Similar equations are associated with the other velocities and to the other measurements. Only the factor that contains exponential terms changes from equation to equation. It is easy to guess the form of this factor because it obeys the following simple rules. To each outcome corresponds, at the numerator and at the denominator as well, a product of exponential factors of the form

$$\exp\left[\left(\frac{\beta t^2}{1+k^2t^2}\,\frac{(\pm x_1\pm x_2\pm x_3\pm x_4)}{2}\right)\right],$$

where x_i represents the projection on the axis parallel to the local ith magnet. The sign of x_i in the exponent represents the spin value asymptotically reached during the local measurement. The weight of this product is equal, at the numerator and at the denominator as well, to the probability of finding the corresponding outcome. When the sign of x_i in the exponent is negative, the whole product will also be negatively weighted at the numerator when we consider the equation of evolution of x_i. Otherwise, and also at the denominator, the products will have a positive weight. This can be checked directly in the previous equation and is also true for the two- and three-particle cases.

As in the three-particle case treated in the preceding section, the integration of the velocities leads to a paradoxical situation for any value of the initial position (excepted for some positions that belong to a region of negligible weight).

IV. CONCLUSIONS

In conclusion, we showed in this paper how the elucidation of several paradoxical situations related to the question of nonlocality can be realized in the framework of Bohm's interpretation and is finally reduced to the study of the topology of attractor basins of the Bohmian dynamics. One could object that this explanation is obvious, being given that no satisfying relativistically covariant formulation of Bohm's interpretation exists for situations in which more than one (entangled) particles are involved. This criticism is valid: in the present approach, we systematically considered the Schrödinger equation, in which a unique time appears, similar to the Newtonian, absolute, time. Now, a theorem by Hardy [14] shows the impossibility to build a Lorentz invariant realistic theory that would mimic the predictions of quantum mechanics. In order to preserve realism it is therefore necessary to reintroduce a kind of absolute or etheric time, attached to an absolute or special frame of reference, which answers the previous criticism. The interesting question is then: Could the Bohmian interpretation help us to conceive an experiment that would reveal the existence of this quantum ether. The original Bohmian interpretation is too *ad hoc* to allow for such a possibility, but slightly modified versions of the interpretation (in which the randomization of the hidden variable, here the position of the particle, according to the ψ^2 distribution is not guaranteed for all times) would allow us to conceive such experiments [15]. A realistic interpretation in the manner of would then open the door to the conception of a quantum version of the Michelson-Morley experiment.

APPENDIX A: BOHM'S SOLUTION FOR THE PASSAGE OF ONE SPIN-(1/2) PARTICLE THROUGH A STERN-GERLACH DEVICE

Let us describe the wave function of a spin-(1/2) particle as superposition of a spin-up and spin-down components along the direction Z of the magnetic field of the Stern-Gerlach device: $\Psi(\mathbf{r},t)=\psi_+(\mathbf{r},t)|+\rangle+\psi_-(\mathbf{r},t)|-\rangle$. The Pauli-Schrödinger equation describes the evolution of this wave function in the presence of an external magnetic field $\mathbf{B}$:

$$i\hbar\partial_t\Psi(\mathbf{r},t)=-\frac{\hbar^2}{2m}\Delta\Psi(\mathbf{r},t)-\mu\mathbf{B}\cdot\mathbf{\Sigma}\Psi(\mathbf{r},t), \quad \text{(A1)}$$

where $\mathbf{\Sigma}$ represents the Pauli matrices and μ is the gyromagnetic coupling constant of the (neutral) particle. In a Stern-Gerlach device, the field is always parallel to the Z direction, and its gradient is constant ($B_x=B_y=0$ and $B_z=a_0+a_1z$). Then, the evolutions of the two spin components decouple and we get

$$i\hbar\partial_t\psi_\pm(\mathbf{r},t)=-\frac{\hbar^2}{2m}\Delta\psi_\pm(\mathbf{r},t)\mp\mu(a_0+a_1z)\psi_\pm(\mathbf{r},t). \quad \text{(A2)}$$

When the incoming wave packet is initially Gaussian shaped, this equation is separable in Cartesian coordinates, and we find

$$i\hbar\partial_t\psi^x_\pm(x,t)=-\frac{\hbar^2}{2m}\partial_x^2\psi^x_\pm(x,t),$$

$$i\hbar\partial_t\psi^y_\pm(y,t)=-\frac{\hbar^2}{2m}\partial_y^2\psi^y_\pm(y,t),$$

$$i\hbar\partial_t\psi^z_\pm(z,t)=-\frac{\hbar^2}{2m}\partial_z^2\psi^z_\pm(z,t)\mp\mu(a_0+a_1z)\psi^z_\pm(z,t). \quad \text{(A3)}$$

The two first equations correspond to a free propagation, and in the case of an initial Gaussian wave packet, their solution is well known and given in many standard textbooks of quantum mechanics, so we shall not discuss how we obtain their contribution to the solution. The third equation is less common. We shall show, following Bohm himself [11], how to solve it. Let us try to find a generalized plane-wave solution of the form $\psi^z_\pm(z,t)=f_\pm(z,t)\exp[i(kz-\hbar k^2t/2m)]$. Then, f fulfills

$$i\hbar\partial_t f^z_\pm(z,t)=-\frac{\hbar^2}{2m}(\partial_z^2\pm 2ik\partial_z)f^z_\pm(z,t)$$
$$\mp\mu(a_0+a_1z)f^z_\pm(z,t). \quad \text{(A4)}$$

If we could neglect the derivatives, we would have $f^z_\pm(z,t)=\exp[\pm i(\mu t/\hbar)(a_0+a_1z)]$, but then the derivatives are not zero but yield $\hbar^2/2m[\pm kt(2a_1\mu/\hbar)+(a_1\mu t/\hbar)^2]$. This term depends on time only, so that we can compensate it by multiplying the postulated value of f by $\exp(i[\mp(a_1\mu/2m)kt^2-(\mu^2a_1^2/6m\hbar)t^3])$. We find, thus, the exact solution:

$$\psi^z_\pm(z,t)=\exp\left[i\left(kz-\hbar k^2t/2m\pm\frac{\mu t}{\hbar}(a_0+a_1z)\mp\frac{a_1\mu}{2m}kt^2\right.\right.$$
$$\left.\left.-\frac{\mu^2a_1^2}{6m\hbar}t^3\right)\right]. \quad \text{(A5)}$$

When $t=0$, this wave is plane: $\psi^z_\pm(z,0)=\exp(ikz)$. Thanks to the linearity of the Schrödinger equation, the general solution is a superposition of these generalized plane waves with as the weight the Fourier components at time zero as the weight. The initial packet being Gaussian shaped, we have

$$\psi_{\pm}^{z}(z,t=0)=\sqrt{\frac{1}{(2\sqrt{\pi}\delta r_0)}}\exp\left(\frac{-z^2}{4\delta r_0^2}\right).$$

Its Fourier transform is easily obtained thanks to the well-known formula

$$\int_{v\in\mathbf{R}}dv\,\exp(-av^2+bv)=\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right).$$

It gives $\sqrt{\delta r_0/\sqrt{\pi}}\exp(-\delta r_0^2k^2)$.

The wave function at time t is thus

$$\psi_{\pm}^{z}(z,t)=\frac{1}{\sqrt{2\pi}}\int dk\,\exp\left[i\left(kz-\hbar k^2t/2m\pm\frac{\mu t}{\hbar}(a_0+a_1z)\mp\frac{a_1\mu}{2m}kt^2-\frac{\mu^2a_1^2}{6m\hbar}t^3\right)\right]\sqrt{\frac{\delta r_0}{\sqrt{\pi}}}\exp(-\delta r_0^2k^2). \tag{A6}$$

We can let go out of the integral the terms independent of k, and reorder the other ones:

$$\psi_{\pm}^{z}(z,t)=\frac{\sqrt{2\delta r_0}}{2\sqrt{\pi^{3/2}}}\exp\left(\pm\frac{\mu t}{\hbar}(a_0+a_1z)-\frac{\mu^2a_1^2}{6m\hbar}t^3\right)\int dk\,\exp\left[-k^2(\delta r_0^2+i\hbar t/2m)+ik\left(z\mp\frac{a_1\mu}{2m}t^2\right)\right]. \tag{A7}$$

We get

$$\psi_{\pm}^{z}(z,t)=\sqrt{\frac{1}{2\sqrt{\pi}}\left(\delta r_0+i\frac{\hbar t}{2m\delta r_0}\right)}\exp\left[\pm\frac{\mu t}{\hbar}(a_0+a_1z)-\frac{\mu^2a_1^2}{6m\hbar}t^3\right]\exp-\frac{\left(z\mp\frac{a_1\mu}{2m}t^2\right)^2}{4(\delta r_0^2+i\hbar t/2m)}. \tag{A8}$$

Using the equalities

$$\frac{1}{\delta r_0^2+i\hbar t/2m}=\frac{(\delta r_0^2-i\hbar t/2m)}{(\delta r_0^4+\hbar^2t^2/4m^2)},$$

$$\sqrt{\frac{1}{2\sqrt{\pi}}\left(\delta r_0+i\frac{\hbar t}{2m\delta r_0}\right)}=\sqrt{\frac{1}{2\sqrt{\pi}\sqrt{\left(\delta r_0^2+\frac{\hbar^2t^2}{4m^2\delta r_0^2}\right)}}\exp\left(-i\arccos\frac{\delta r_0}{\sqrt{\left(\delta r_0^2+\frac{\hbar^2t^2}{4m^2\delta r_0^2}\right)}}\right)},$$

and

$$\delta r_t^2=\delta r_0^2\left(1+\frac{\hbar^2t^2}{4m^2\delta r_0^4}\right),$$

we get the contribution of $\psi_{\pm}^{z}(z,t)$ to the wave function:

$$\psi_{\pm}^{z}(z,t)=\frac{1}{(2\sqrt{\pi}\delta r_t)^{1/2}}\exp\left(-\frac{(z\mp a_1\mu t^2/2m)^2}{4\delta^2r_t}\right)\exp\left[i\left(\frac{\hbar t(z\mp a_1\mu t^2/2m)^2}{2m\delta^2r_04\delta r_t^2}-\frac{\arccos\left(\frac{\delta r_0}{\delta r_t}\right)}{2}\pm\frac{\mu t}{\hbar}(a_0+a_1z)-\frac{\mu^2a_1^2}{6m\hbar}t^3\right)\right]. \tag{A9}$$

Notice that $\psi_{\pm}^{y}(y,t)$ is obtained in exactly the same way, with the requirement that $a_0=a_1=0$.

In conclusion,

$$\psi_{\pm}(\mathbf{r},t)=\frac{1}{(2\sqrt{\pi}\delta r_t)^{3/2}}\exp\left(-\frac{(x-v_0t)^2}{4\delta r_t^2}\right)\exp\left[i\left(\frac{\hbar t(x-v_0t)^2}{2m\delta^2 r_0 4\delta r_t^2}+k_0x-\frac{\hbar k_0^2 t}{2m}-\frac{\arccos\left(\frac{\delta r_0}{\delta r_t}\right)}{2}\right)\right]$$
$$\times\exp\left(-\frac{y^2}{4\delta r_t^2}\right)\exp\left[i\left(\frac{\hbar t y^2}{2m\delta^2 r_0 4\delta r_t^2}-\frac{\arccos\left(\frac{\delta r_0}{\delta r_t}\right)}{2}\right)\right]\exp\left(-\frac{(z\mp a_1\mu t^2/2m)^2}{4\delta^2 r_t}\right)$$
$$\times\exp\left[i\left(\frac{\hbar t(z\mp a_1\mu t^2/2m)^2}{2m\delta^2 r_0 4\delta r_t^2}-\frac{\arccos\left(\frac{\delta r_0}{\delta r_t}\right)}{2}\right)\right]\exp\left(\pm\frac{\mu t}{\hbar}(a_0+a_1z)-\frac{\mu^2 a_1^2}{6m\hbar}t^3\right), \quad \text{(A10)}$$

with $\delta r_t^2=\delta r_0^2(1+\hbar^2t^2/4m^2\delta r_0^4)$. The interpretation of this solution, which appeared originally in Ref. [11], is straightforward: the wave packet of the particles with spin up (down) diffuses under the influence of the free part of the Schrödinger evolution (the Laplacian term) and is simultaneously uniformly accelerated upwards (downwards). It ''feels'' effectively a potential due to the gyromagnetic coupling, which varies linearly in z, and the changes of sign with the z spin.

After a time τ, the particles leave the Stern-Gerlach device, and the Gaussian packets move freely, diffusing around centers that conserve the velocity that they possessed when leaving the magnets. The dependence in z is thus, up to irrelevant global phases,

$$\exp\left(-\frac{(z\mp a_1\mu\tau t/m\pm a_1\mu\tau^2/2m)^2}{4\delta r_t^2}\right)$$
$$\times\exp\left[i\left(\frac{\hbar t(z\mp a_1\mu\tau t/m\pm a_1\mu\tau^2/2m)^2}{2m\delta^2 r_0 4\delta r_t^2}\right.\right.$$
$$\left.\left.\pm a_1\mu\tau/m(z\mp a_1\mu\tau^2/2m)\right)\right], \quad \text{(A11)}$$

while the dependence in x and y is the same as before.

In a typical single Stern-Gerlach experiment involving silver atoms. the mass of the incoming atom is equal to 1.8010^{-22} g, the gyromagnetic coupling constant μ is equal to 9.27×10^{-21} g cm^2 s^{-2} G^{-1}, the gradient of the magnetic field along the Z axis is equal to 10^4 G cm^{-1}, the length of the magnet is equal to 10 cm, the velocity of the incoming particle along the X axis is equal to 10^4 cm s^{-1}, the time of flight through the magnet τ is equal to 10^{-3} s, $\hbar$ is equal to 1.05×10^{-27} erg s, the spreading of the incoming particle δr_0 is equal to 10^{-3} cm. Then,

$$k=\frac{\hbar}{2m\delta^2 r_0}=2.91\ \mathrm{s}^{-1}, \quad \text{(A12)}$$

$$\alpha=a_1\mu/2m=2.58\times10^5\ \ \mathrm{cm\,s^{-2}}, \quad \text{(A13)}$$

$$\beta=a_1\mu/m\,\delta r_0^2=5.15\times10^{11}\ \ \mathrm{cm^{-1}\,s^{-2}}. \quad \text{(A14)}$$

The heights of the centers of the outgoing bundles moving upwords and downwards after the time τ are equal to $\pm\alpha\tau^2=\pm0.258$ cm. Their Z velocities are $\pm2\alpha\tau=\pm5.15$ m s^{-1}. The spreading is then, taking the diffusion into account, $\delta r_0\sqrt{1+k^2\tau^2}=10^{-3}\sqrt{1+(2.91\times10^{-3})^2}$ cm ($\approx10^{-3}$ cm), which is very small in comparison to the distance between the bundles (0.515 cm), so that these bundles are clearly separated. If we place a screen at a distance of 1 m from the magnet, it is reached after 10^{-2} s, and the spots are separated by a distance of the order of 11 cm, while the extent of a single spot is still of the order of 10^{-3} cm.

APPENDIX B: THE TWO-PARTICLE CASE

The Pauli-Schrödinger equation allows us to describe the evolution of the wave function inside the magnets of the devices:

$$i\hbar\partial_t\Psi(\mathbf{r}_L,\mathbf{r}_R,t)=-\frac{\hbar^2}{2m}(\Delta_L+\Delta_R)\Psi(\mathbf{r}_L,\mathbf{r}_R,t)+(\mu\mathbf{B}_L\cdot\mathbf{\Sigma}_L$$
$$+\mu\mathbf{B}_R\cdot\mathbf{\Sigma}_R)\Psi(\mathbf{r}_L,\mathbf{r}_R,t), \quad \text{(B1)}$$

with

$$\Delta_L(\Delta_R)[\psi_{\pm L}(\mathbf{r}_L,t)\psi_{\pm' R}(\mathbf{r}_R,t)|\pm\rangle_L\otimes|\pm'\rangle_R]$$
$$=[\Delta_L\psi_{\pm L}(\mathbf{r}_L,t)]\psi_{\pm' R}(\mathbf{r}_R,t)|\pm\rangle_L\otimes|\pm'\rangle_R[\psi_{\pm L}(\mathbf{r}_L,t)$$
$$\times(\Delta_R\psi_{\pm' R}(\mathbf{r}_R,t))|\pm\rangle_L\otimes|\pm'\rangle_R), \quad \text{(B2)}$$

where $\Delta_{L/R}$ is the Laplacian operator in the left/right coordinates, $\mathbf{B}_{L/R}$ is the magnetic field in the left/right regions, while the components of $\mathbf{\Sigma}_{L/R}$ are the Σ matrices of Pauli. When the magnetic fields are parallel to the Z axes, only the third Σ matrices appear, which are defined by

$$\mathbf{\Sigma}^3_{L(R)}\psi_{\pm L}(\mathbf{r}_L,t)\psi_{\pm' R}(\mathbf{r}_R,t)|\pm\rangle_L\otimes|\pm'\rangle_R$$
$$=\pm(\pm')\psi_{\pm L}(\mathbf{r}_L,t)\psi_{\pm' R}(\mathbf{r}_R,t)|\pm\rangle_L\otimes|\pm'\rangle_R. \quad \text{(B3)}$$

Then, it is easy to check that the Pauli-Schrödinger equation is separable into the six spatial coordinates and that the general solution, when the wave packet is initially Gaussian, can be expressed in terms of the solutions associated with the single-particle case that we obtained in the preceding appendix.

Let us now derive the conservation equation associated with this evolution law. We can write it in the form

$$i\hbar\partial_t\Psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\Delta\Psi(\mathbf{r},t) + \mu\mathbf{B}\cdot\mathbf{\Sigma}\Psi(\mathbf{r},t), \qquad \text{(B4)}$$

where the Laplacian is a six-dimensional Laplacian, the magnetic field a six-dimensional field. It is worth noting that $\mathbf{\Sigma}$ is a self-adjoint operator on the spin space (isomorph to $\mathbf{C}^4$). The adjoint (transposed conjugate) of the previous equation is

$$-i\hbar\partial_t\Psi^\dagger(\mathbf{r},t) = -\frac{\hbar^2}{2m}\Delta\Psi^\dagger(\mathbf{r},t) + \Psi^\dagger(\mathbf{r},t)\mu\mathbf{B}\cdot\mathbf{\Sigma}, \qquad \text{(B5)}$$

where we used the fact that $\mathbf{\Sigma}=\mathbf{\Sigma}^\dagger$. Let us now multiply the equation of evolution by $\Psi^\dagger$, its adjoint equation by Ψ, and take their difference, we get

$$i\hbar\partial_t\Psi^\dagger(\mathbf{r},t)\Psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\Psi^\dagger(\mathbf{r},t)[\Delta\Psi(\mathbf{r},t)] - [\Delta\Psi^\dagger(\mathbf{r},t)]\Psi(\mathbf{r},t). \qquad \text{(B6)}$$

We can rewrite the second member of the last equation as the six dimensional divergency of a six dimensional flux:

$$\Psi^\dagger(\mathbf{r},t)[\Delta\Psi(\mathbf{r},t)] - [\Delta\Psi^\dagger(\mathbf{r},t)]\Psi(\mathbf{r},t) = \mathrm{div}[\Psi^\dagger(\mathbf{r},t)\cdot\mathrm{grad}\Psi(\mathbf{r},t) - \mathrm{grad}\,\Psi^\dagger(\mathbf{r},t)\cdot\Psi(\mathbf{r},t)]. \qquad \text{(B7)}$$

Now, $\Psi^\dagger(\mathbf{r},t)\cdot\mathrm{grad}\Psi(\mathbf{r},t)$ is the complex conjugate of $\mathrm{grad}\,\Psi^\dagger(\mathbf{r},t)\cdot\Psi(\mathbf{r},t)$ so that we obtain the equation of conservation $\partial_t\rho=\mathrm{div}(\mathcal{J})$, where

$$\rho = \Psi^\dagger(\mathbf{r},t)\Psi(\mathbf{r},t) \qquad \text{(B8)}$$

or, in another notation,

$$\rho(\mathbf{r}_L,\mathbf{r}_R,t) = \langle\Psi(\mathbf{r}_L,\mathbf{r}_{R,t)}|\Psi(\mathbf{r}_L,\mathbf{r}_{R,t)}\rangle \qquad \text{(B9)}$$

and

$$\mathcal{J} = (\hbar/m)\mathrm{Im}[\Psi^\dagger(\mathbf{r},t)\mathrm{grad}\,\Psi(\mathbf{r},t)], \qquad \text{(B10)}$$

where $\mathrm{Im}(z)$ is the imaginary part of z.

The three- and four-particle cases can be treated in a similar way.

[1] J.S. Bell, Phys. Scr. **1**, 195 (1964).
[2] J.F. Clauser and M.A. Horne, Phys. Rev. D **10**, 526 (1974).
[3] I. Pitowsky, *Quantum Probability. Quantum Logic*, Lecture Notes in Physics (Springer-Verlag, Berlin, 1989).
[4] D.M. Greenberger, M.A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).
[5] N.D. Mermin, Phys. Today **43**, 9 (1990).
[6] D. Bohm, Phys. Rev. **85**, 166 (1952).
[7] C. Dewdney, P.R. Holland, A. Kyprianidis, and J.P. Vigier, Nature (London) **336**, 53 (1988).
[8] C. Dewdney, J. Phys. A **25**, 3615 (1992).
[9] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[10] L. Hardy, Phys. Lett. A **161**, 21 (1991).
[11] D. Bohm, *Quantum Theory* (Prentice-Hall, Englewood Cliffs, NJ, 1951).
[12] T. Durt, Found. Phys. **27**, 415 (1997).
[13] A. Peres, Phys. Lett. A **151**, 107 (1990).
[14] L. Hardy, Phys. Rev. Lett. **68**, 2981 (1992).
[15] T. Durt, Helv. Phys. Acta **72**, 356 (1999).