

Weak measurement of arrival time

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The arrival time probability distribution is defined by analogy with classical mechanics. The difficulty of requiring knowledge of the values of noncommuting operators is circumvented using the concept of weak measurements. The proposed procedure is suitable for free particles as well as for those subjected to an external potential. It is shown that such an approach imposes an inherent limitation on the accuracy of the arrival time determination.

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I. INTRODUCTION

Time plays a special role in quantum mechanics. Unlike other observables, time remains a classical variable. It is well known that the self-adjoint operator of time does not exist for bounded Hamiltonians. The problems with time arise from the fact that in quantum mechanics many quantities cannot have definite values simultaneously. However, the detection of particles in time-of-flight and coincidence experiments is common, and quantum mechanics should give a method for the calculation of the arrival time. The arrival time distribution may be useful in solving the tunneling time problem, as well. Therefore, the quantum description of arrival time has attracted much attention [1–15].

Aharonov and Bohm introduced the arrival time operator [1]

$$\hat{T}_{AB} = \frac{m}{2} \left((X - \hat{x}) \frac{1}{\hat{p}} + \frac{1}{\hat{p}} (X - \hat{x}) \right). \quad (1)$$

By imposing several conditions (normalization, positivity, minimum variance, and symmetry with respect to the arrival point X) a quantum arrival time distribution for a free particle was obtained by Kijowski [2]. Kijowski's distribution may be associated with the positive operator valued measure generated by the eigenstates of \hat{T}_{AB} . However, Kijowski's set of conditions cannot be applied in a general case [2]. Nevertheless, arrival time operators can be constructed even if the particle is not free [15,16].

In this paper we take another approach. Since the mean arrival time even in classical mechanics can be infinite or the particle may not arrive at all, it is convenient to deal not with the mean arrival time and corresponding operator \hat{T} , but with the probability distribution of the arrival time. The probability distribution of the arrival time can be obtained from a suitable classical definition. The noncommutativity of the operators in quantum mechanics is circumvented by using the concept of weak measurements. Such an approach has several advantages. It gives, in principle, a procedure for measuring the quantity obtained. Since in classical mechanics all quantities can have definite values simultaneously, weak

measurements give the correct classical limit. The concept of weak measurements has already been applied to the time problem in quantum mechanics [17].

We proceed as follows. In Sec. II we discuss various definitions of the arrival time in classical mechanics. The weak measurement of the quantum arrival time distribution is presented in Sec. III. The properties of the quantity obtained are analyzed in Sec. IV. Section V summarizes our findings.

II. ARRIVAL TIME IN CLASSICAL MECHANICS

In classical mechanics the particle moves along the trajectory $H(x,p) = \text{const}$ as t increases. This allows us to work out the time of arrival at the point $x(t) = X$, by identifying the point (x_0, p_0) of the phase space where the particle is at $t=0$, and then following the trajectory that passes by this point, up to arrival at the point X . If multiple crossings are possible, one may define a distribution of arrival times with contributions from all crossings, when no distinction is made between first, second, and n th arrivals. In this article we will consider such a distribution.

We can ask whether there is a definition of the arrival time, that is valid in both classical and quantum mechanics. In our opinion, the words “the particle arrives from the left at the point X at the time t ” mean that (i) at time t the particle was in the region $x < X$ and (ii) at time $t + \Delta t$ ($\Delta t \rightarrow 0$) the particle is found in the region $x > X$. Now we apply this definition, given by (i) and (ii), to the time of arrival in the classical case.

Since quantum mechanics deals with probabilities, it is convenient to use a probabilistic description of the classical mechanics as well. Therefore, we will consider an ensemble of noninteracting classical particles. The probability density in the phase space is $\rho(x,p;t)$.

Let us denote the region $x < X$ as Γ_1 and the region $x > X$ as Γ_2 . The probability that the particle arrives from region Γ_1 at region Γ_2 at a time between t and $t + \Delta t$ is proportional to the probability that the particle is in region Γ_1 at time t and in region Γ_2 at time $t + \Delta t$. This probability is

$$\Pi_+(t)\Delta t = \frac{1}{N_+} \int_{\Omega} dp dx \rho(x,p;t), \quad (2)$$

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where N_+ is the constant of normalization and the region of phase space Ω has the following properties: (i) the coordinates of the points in Ω are in the space region Γ_1 and (ii) if the phase trajectory goes through a point of the region Ω at time t then the particle at time $t + \Delta t$ is in the space region Γ_2 . Since Δt is infinitesimal, the change of coordinate during the time interval Δt is equal to $(p/m)\Delta t$. Therefore, the particle arrives from region Γ_1 at region Γ_2 only if the momentum of the particle at the point X is positive. The phase space region Ω consists of the points with positive momentum p and with coordinates between $X - p/m\Delta t$ and X . Then from Eq. (2) we have the probability of arrival time

$$\Pi_+(t)\Delta t = \frac{1}{N_+} \int_0^\infty dp \int_{X-(p/m)\Delta t}^X dx \rho(x,p;t). \quad (3)$$

Since Δt is infinitesimal and the momentum of every particle is finite, we can replace x in Eq. (3) by X and obtain the equality

$$\Pi_+(t,X) = \frac{1}{N_+} \int_0^\infty dp \frac{p}{m} \rho(X,p;t). \quad (4)$$

The obtained arrival time distribution $\Pi_+(t,X)$ is well known and has appeared quite often in the literature (see, e.g., the review [15] and references therein).

The probability current in classical mechanics is

$$J(x;t) = \int_{-\infty}^{+\infty} \frac{p}{m} \rho(x,p;t) dp. \quad (5)$$

From Eqs. (4) and (5) it is clear that the time of arrival is related to the probability current. This relation, however, is not straightforward. We can introduce the ‘‘positive probability current’’

$$J_+(x;t) = \int_0^\infty \frac{p}{m} \rho(x,p;t) dp \quad (6)$$

and rewrite Eq. (4) as

$$\Pi_+(t,X) = \frac{1}{N_+} J_+(X;t). \quad (7)$$

The proposed [14,18,19] various quantum versions of J_+ even in the case of the free particle can be negative (the so-called backflow effect). Therefore, the classical expression (7) for the time of arrival becomes problematic in quantum mechanics.

Similarly, for arrival from the right we obtain the probability density

$$\Pi_-(t,X) = \frac{1}{N_-} J_-(X;t), \quad (8)$$

where the negative probability current is

$$J_-(x;t) = \int_{-\infty}^0 \frac{|p|}{m} \rho(x,p;t) dp. \quad (9)$$

We see that our definition, given at the beginning of this section, leads to the proper result in classical mechanics. We will try to use the definition of the arrival time, given in this section by conditions (i) and (ii), for analysis of the arrival time probability distribution in quantum mechanics.

III. WEAK MEASUREMENT

The proposed definition of the arrival time probability distribution can be used in quantum mechanics only if the determination of the region in which the particle is does not disturb the motion of the particle. This can be achieved using the weak measurements of Aharonov, and others [20–25].

We have the detector in the initial state $|\Phi\rangle$. The detector interacts with the particle only in the region Γ_1 . In order for weak measurements to provide meaningful information, the measurements must be performed on an ensemble of identical systems. Each system with its own detector is prepared in the same initial state. The readings of the detectors are collected and averaged.

We take the operator describing the interaction between the particle and the detector of the form [17,26,27]

$$\hat{H}_I = \lambda \hat{q} \hat{P}_1, \quad (10)$$

where \hat{P}_1 is the projection operator projecting into the region Γ_1 and λ characterizes the strength of the interaction with the detector. The interaction operator (10) only slightly differs from the one used by Aharonov, Albert, and Vaidman [21]. A similar interaction operator was considered by von Neumann [28] and has been widely used in strong measurement models (e.g., [29–34] and many others).

The measurement time is τ . We assume that the interaction strength λ and the time τ are small. The very small parameter λ ensures that the particle’s evolution is undisturbed. The operator \hat{q} acts in the Hilbert space of the detector. We require a continuous spectrum of the operator \hat{q} . For simplicity, we can consider this operator as the coordinate of the detector. The momentum conjugate to q is p_q .

Since the interaction strength λ and the duration of the measurement τ are small, the probability $\langle \hat{P}_1 \rangle$ of finding the particle in the region Γ_1 does not change significantly during the measurement. The action of the Hamiltonian (10) results in a small change of the mean detector’s momentum $\langle \hat{p}_q \rangle - \langle \hat{p}_q \rangle_0 = -\lambda \tau \langle \hat{P}_1 \rangle$, where $\langle \hat{p}_q \rangle_0 = \langle \Phi(0) | \hat{p}_q | \Phi(0) \rangle$ is the mean momentum of the detector at the beginning of the measurement and $\langle \hat{p}_q \rangle = \langle \Phi(\tau) | \hat{p}_q | \Phi(\tau) \rangle$ is the mean momentum of the detector after the measurement. Therefore, in analogy with Ref. [21], we define the ‘‘weak value’’ of the probability of finding the particle in the region Γ_1 ,

$$W(1) \equiv \langle \hat{P}_1 \rangle = \frac{\langle \hat{p}_q \rangle_0 - \langle \hat{p}_q \rangle}{\lambda \tau}. \quad (11)$$

In order to obtain the arrival time probability using the definition from Sec. II, we measure the momenta p_q of each detector after the interaction with the particle. After time Δt we perform the final, postselection measurement on the par-

ticles of our ensemble and measure if the particle is found in region Γ_2 . Then we collect the outcomes p_q only for the particles found in region Γ_2 .

The projection operator projecting into region Γ_2 is \hat{P}_2 . In the Heisenberg representation this operator is

$$\tilde{P}_2(t) = \hat{U}(t)^\dagger \hat{P}_2 \hat{U}(t), \quad (12)$$

where \hat{U} is the evolution operator of the free particle. After the measurement the state of the particle and the detector is $\hat{U}_M(\tau)|\Phi\rangle|\Psi\rangle$, where $|\Psi\rangle$ is the initial state of the particle and \hat{U}_M is the evolution operator of the particle interacting with the detector. The joint probability that the detector has momentum p_q and the particle after time Δt is found in region Γ_2 is

$$W(p_q, 2) = \langle \Psi | \langle \Phi | \hat{U}_M(\tau)^\dagger | p_q \rangle \langle p_q | \tilde{P}_2(\Delta t) \hat{U}_M(\tau) | \Phi \rangle | \Psi \rangle, \quad (13)$$

where $|p_q\rangle$ is the eigenfunction of the momentum operator \hat{p}_q , and \hat{P}_2 is the projection operator projecting into the region Γ_2 . In quantum mechanics the probability that two quantities simultaneously have definite values does not always exist. If the joint probability does not exist then the concept of conditional probability is meaningless. However, in our case the operators \hat{P}_2 and $|p_q\rangle\langle p_q|$ act in different spaces and commute; therefore, the joint probability $W(p_q, 2)$ exists.

Let us define the conditional probability, i.e., the probability that the momentum of the detector is p_q , provided that the particle after time Δt is found in region Γ_2 . This probability is given according to Bayes' theorem as

$$W(p_q|2) = \frac{W(p_q, 2)}{W(2)}, \quad (14)$$

where $W(2)$ is the probability that the particle after time Δt is found in region Γ_2 . The average momentum of the detector on condition that the particle after time Δt is found in region Γ_2 equals

$$\begin{aligned} \langle \hat{p}_q \rangle_2 &= \int p_q W(p_q|2) dp_q \\ &= \frac{1}{W(2)} \langle \Psi | \langle \Phi | \hat{U}_M(\tau)^\dagger \hat{p}_q \tilde{P}_2(\Delta t) \hat{U}_M(\tau) | \Phi \rangle | \Psi \rangle, \end{aligned} \quad (15)$$

where $\tilde{P}_2(\Delta t) = \hat{U}(\Delta t)^\dagger \hat{P}_2 \hat{U}(\Delta t)$ is the operator \hat{P}_2 in the Heisenberg picture. According to Eq. (11)

$$W(1|2) = \frac{\langle \hat{p}_q \rangle_0 - \langle \hat{p}_q \rangle_2}{\lambda \tau} = \frac{\langle \hat{p}_q \rangle_0 W(2) - \langle \hat{p}_q \rangle_2 W(2)}{\lambda \tau W(2)} \quad (16)$$

is the weak value of the probability of finding the particle in region Γ_1 on condition that the particle after time Δt is in region Γ_2 . The probability that the particle is in region Γ_1 and after time Δt is in region Γ_2 equals

$$W(1, 2) = W(2)W(1|2). \quad (17)$$

When the measurement time τ is sufficiently small, the influence of the Hamiltonian of the particle can be neglected and the evolution operator of the particle and the detector can be expressed as

$$\hat{U}_M(\tau) = \exp\left(-\frac{i}{\hbar} \lambda \tau \hat{q} \hat{P}_1\right).$$

We expand the operator $\hat{U}_M(\tau)$ in a series of the parameter λ , assuming that λ is small. In the first-order approximation, using Eqs. (16) and (17), we obtain

$$\begin{aligned} W(1, 2) &\approx \frac{1}{2} \langle \tilde{P}_2(\Delta t) \hat{P}_1 + \hat{P}_1 \tilde{P}_2(\Delta t) \rangle + \frac{i}{\hbar} \langle \hat{p}_q \rangle \langle \hat{q} \rangle \\ &\quad - \text{Re} \langle \hat{q} \hat{p}_q \rangle \langle [\hat{P}_1, \tilde{P}_2(\Delta t)] \rangle. \end{aligned} \quad (18)$$

The probability $W(1, 2)$ is constructed using conditions (i) and (ii) from Sec. II: the weak measurement is performed to determine if the particle is in region Γ_1 and after time Δt the strong measurement determines if the particle is in region Γ_2 . Therefore, according to Sec. II, the quantity $W(1, 2)$ after normalization can be considered as the weak value of the arrival time probability distribution.

Equation (18) consists of two terms and we can introduce two quantities

$$\Pi^{(1)} = \frac{1}{2\Delta t} \langle \hat{P}_1 \tilde{P}_2(\Delta t) + \tilde{P}_2(\Delta t) \hat{P}_1 \rangle \quad (19)$$

and

$$\Pi^{(2)} = \frac{1}{2i\Delta t} \langle [\hat{P}_1, \tilde{P}_2(\Delta t)] \rangle. \quad (20)$$

Then

$$W(1, 2) = \Pi^{(1)} \Delta t - \frac{2\Delta t}{\hbar} (\langle \hat{p}_q \rangle \langle \hat{q} \rangle - \text{Re} \langle \hat{q} \hat{p}_q \rangle) \Pi^{(2)}. \quad (21)$$

If the commutator $[\hat{P}_1, \tilde{P}_2(\Delta t)]$ in Eqs. (19)–(21) is not zero, then, even in the limit of the very weak measurement, the measured value depends on the particular detector. This fact means that in such a case we cannot obtain a definite value for the arrival time probability. Moreover, the coefficient $(\langle \hat{p}_q \rangle \langle \hat{q} \rangle - \text{Re} \langle \hat{q} \hat{p}_q \rangle)$ may be zero for a specific initial state of the detector, e.g., for a Gaussian distribution of the coordinate q and momentum p_q .

The quantities $W(1, 2)$, $\Pi^{(1)}$, and $\Pi^{(2)}$ are real. However, it is convenient to consider the complex quantity

$$\Pi_C = \Pi^{(1)} + i\Pi^{(2)} = \frac{1}{\Delta t} \langle \hat{P}_1 \tilde{P}_2(\Delta t) \rangle. \quad (22)$$

We call it the ‘‘complex arrival probability.’’ We can introduce the corresponding operator

$$\hat{\Pi}_+ = \frac{1}{\Delta t} \hat{P}_1 \tilde{P}_2(\Delta t). \quad (23)$$

By analogy, the operator

$$\hat{\Pi}_- = \frac{1}{\Delta t} \tilde{P}_2 \hat{P}_1(\Delta t) \quad (24)$$

corresponds to arrival from the right.

The introduced operator $\hat{\Pi}_+$ has some properties of the classical positive probability current. From the conditions $\hat{P}_1 + \hat{P}_2 = 1$ and $\tilde{P}_1(\Delta t) + \tilde{P}_2(\Delta t) = 1$ we have

$$\hat{\Pi}_+ - \hat{\Pi}_- = \frac{1}{\Delta t} (\tilde{P}_2(\Delta t) - \hat{P}_2).$$

In the limit $\Delta t \rightarrow 0$ we obtain the probability current $\hat{J} = \lim_{\Delta t \rightarrow 0} (\hat{\Pi}_+ - \hat{\Pi}_-)$, as in classical mechanics. However, the quantity $\langle \hat{\Pi}_+ \rangle$ is complex and the real part can be negative, in contrast to the classical quantity J_+ . The reason for this is the noncommutativity of the operators \hat{P}_1 and $\tilde{P}_2(\Delta t)$. When the imaginary part is small, the quantity $\langle \hat{\Pi}_+ \rangle$ after normalization can be considered as the approximate probability distribution of the arrival time.

IV. ARRIVAL TIME PROBABILITY

The operator $\hat{\Pi}_+$ is obtained without specification of the Hamiltonian of the particle and is suitable for free particles and for particles subjected to an external potential as well. In this section we consider the arrival time of the free particle.

The calculation of the weak arrival time distribution $W(1,2)$ involves the average $\langle \hat{\Pi}_+ \rangle$. Therefore, it is useful to have the matrix elements of the operator $\hat{\Pi}_+$. It should be noted that the matrix elements of the operator $\hat{\Pi}_+$ as well as the operator itself are only auxiliary and do not have independent meaning.

In the basis of momentum eigenstates $|p\rangle$, normalized according to the condition $\langle p_1 | p_2 \rangle = 2\pi\hbar \delta(p_1 - p_2)$, the matrix elements of the operator $\hat{\Pi}_+$ are

$$\begin{aligned} \langle p_1 | \hat{\Pi}_+ | p_2 \rangle &= \frac{1}{\Delta t} \langle p_1 | \hat{P}_1 \hat{U}(\Delta t)^\dagger \hat{P}_2 \hat{U}(\Delta t) | p_2 \rangle \\ &= \frac{1}{\Delta t} \int_{-\infty}^X dx_1 \int_X^\infty dx_2 e^{-i(\hbar)p_1 x_1} \langle x_1 | \hat{U}(\Delta t)^\dagger | x_2 \rangle \\ &\quad \times e^{i(\hbar)p_2 x_2 - (i/\hbar)(p_2^2/2m)\Delta t}. \end{aligned} \quad (25)$$

After performing the integration we obtain

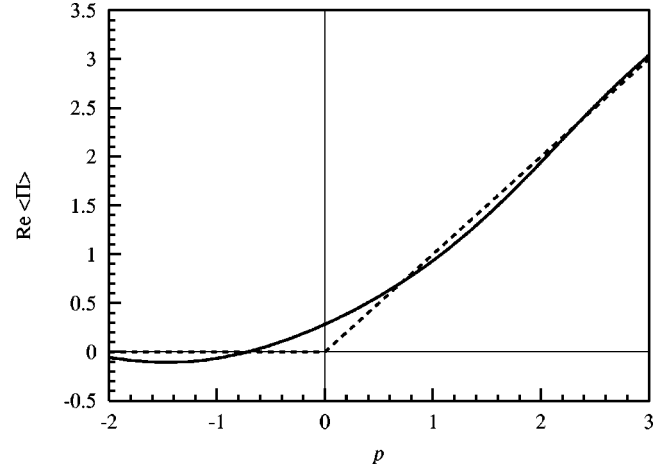


FIG. 1. The real part of the quantity $\langle p | \hat{\Pi}_+ | p \rangle$, according to Eq. (28). The corresponding classical positive probability current is shown with the dashed line. The parameters used are $\hbar = 1$, $m = 1$, and $\Delta t = 1$. In this system of units, the momentum p is dimensionless.

$$\begin{aligned} \langle p_1 | \hat{\Pi}_+ | p_2 \rangle &= \frac{i\hbar}{2\Delta t(p_2 - p_1)} \exp\left(\frac{i}{\hbar}(p_2 - p_1)X\right) \\ &\quad \times \left[e^{(i/\hbar)(\Delta t/2m)(p_1^2 - p_2^2)} \operatorname{erfc}\left(-p_1 \sqrt{\frac{i\Delta t}{2\hbar m}}\right) \right. \\ &\quad \left. - \operatorname{erfc}\left(-p_2 \sqrt{\frac{i\Delta t}{2\hbar m}}\right) \right], \end{aligned} \quad (26)$$

where $\sqrt{i} = \exp(i\pi/4)$. When

$$\frac{1}{\hbar} \frac{\Delta t}{2m} (p_1^2 - p_2^2) \ll 1,$$

$$p_1 \sqrt{\frac{\Delta t}{2\hbar m}} > 1, \quad p_2 \sqrt{\frac{\Delta t}{2\hbar m}} > 1,$$

the matrix elements of the operator $\hat{\Pi}_+$ are

$$\langle p_1 | \hat{\Pi}_+ | p_2 \rangle \approx \frac{p_1 + p_2}{2m} \exp\left(\frac{i}{\hbar}(p_2 - p_1)X\right). \quad (27)$$

This equation coincides with the expression for the matrix elements of the probability current operator.

From Eq. (26) we obtain the diagonal matrix elements of the operator $\hat{\Pi}_+$,

$$\begin{aligned} \langle p | \hat{\Pi}_+ | p \rangle &= \frac{p}{2m} \operatorname{erfc}\left(-p \sqrt{\frac{i\Delta t}{2\hbar m}}\right) \\ &\quad + \frac{\hbar}{\sqrt{i2\pi\hbar m\Delta t}} e^{-(i/\hbar)(p^2/2m)\Delta t}. \end{aligned} \quad (28)$$

The real part of the quantity $\langle p | \hat{\Pi}_+ | p \rangle$ is shown in Fig. 1 and the imaginary part in Fig. 2.

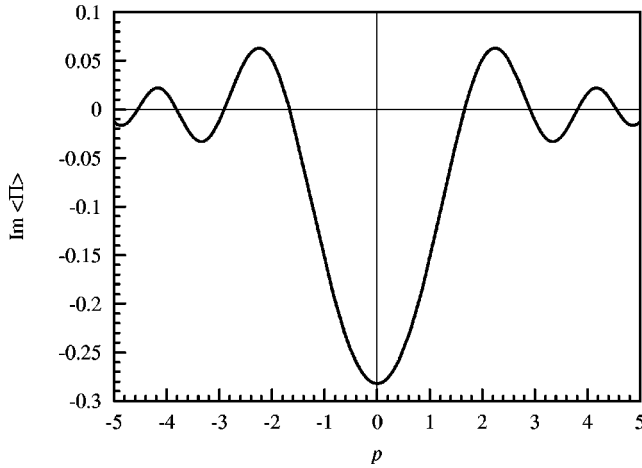


FIG. 2. The imaginary part of the quantity $\langle p | \hat{\Pi}_+ | p \rangle$. The parameters used are the same as in Fig. 1.

Using the asymptotic expressions for the function erfc we obtain from Eq. (28) that

$$\lim_{p \rightarrow +\infty} \langle p | \hat{\Pi}_+ | p \rangle \rightarrow \frac{p}{m}$$

and $\langle p | \hat{\Pi}_+ | p \rangle \rightarrow 0$, when $p \rightarrow -\infty$, i.e., the imaginary part tends to zero and the real part approaches the corresponding classical value as the modulus of the momentum $|p|$ increases. Such behavior is evident from Figs. 1 and 2, also.

The asymptotic expressions for the function erfc are valid when the argument of the erfc is large, i.e., $|p| \sqrt{\Delta t / 2\hbar m} > 1$ or

$$\Delta t > \frac{\hbar}{E_k}. \quad (29)$$

Here E_k is the kinetic energy of the particle. The dependence of the quantity $\text{Re}\langle p | \hat{\Pi}_+ | p \rangle$ from Δt is shown in Fig. 3. For small Δt the quantity $\langle p | \hat{\Pi}_+ | p \rangle$ is proportional to $1/\sqrt{\Delta t}$. Therefore, unlike in classical mechanics, in quantum mechanics Δt cannot be zero. Equation (29) imposes the lower bound on the resolution time Δt . It follows that our model does not permit determination of the arrival time with resolution greater than \hbar/E_k . A relation similar to Eq. (29) based

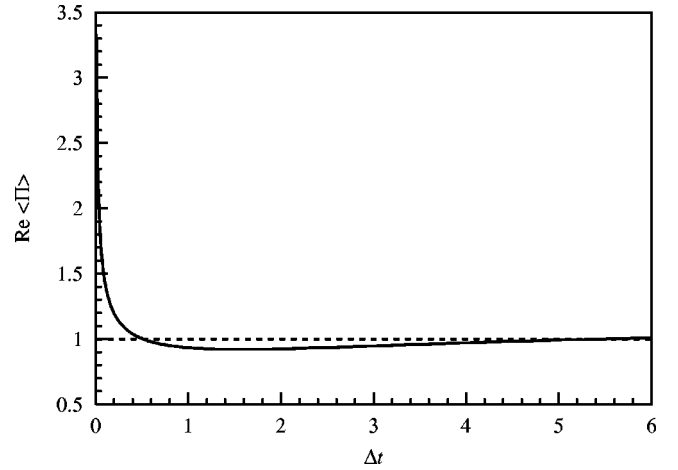


FIG. 3. The dependence of the quantity $\text{Re}\langle p | \hat{\Pi}_+ | p \rangle$ according to Eq. (28) on the resolution time Δt . The corresponding classical positive probability current is shown with the dashed line. The parameters used are $\hbar = 1$, $m = 1$, and $p = 1$. In these units, the time Δt is dimensionless.

on measurement models was obtained by Aharonov *et al.* [29]. The time-energy uncertainty relations associated with the time of arrival distribution, are also discussed in Refs. [8,35].

V. CONCLUSION

A definition of arrival time probability density for one-sided arrivals is proposed, relying on particles being in different regions at different instances. This definition is extended quantum mechanics, using the concept of weak measurements by Aharonov *et al.* The proposed procedure is suitable for free particles and for particles subjected to an external potential, as well. It gives not only a mathematical expression for the arrival time probability distribution but also a way of measuring the quantity obtained. However, this procedure gives no unique expression for the arrival time probability distribution.

In analogy with the complex tunneling time, the complex arrival time probability distribution is introduced [Eq. (22)]. It is shown that the proposed approach imposes an inherent limitation, Eq. (29), on the resolution time of the arrival time determination.

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