Classical aspects of quantum walls in one dimension

Tamás Fülöp*
Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK), Tsukuba 305-0801, Japan

Taksu Cheon†
Laboratory of Physics, Kochi University of Technology, Tosa Yamada, Kochi 782-8502, Japan

Izumi Tsutsui‡
Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK), Tsukuba 305-0801, Japan
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We investigate the system of a particle moving on a half line $x \geqslant 0$ under the general walls at $x=0$ that are permitted quantum mechanically. These quantum walls, characterized by a parameter L, are shown to be realized as a limit of regularized potentials. We then study the classical aspects of the quantum walls by seeking a classical counterpart that admits the same time delay in scattering with the quantum wall, and also by examining the WKB exactness of the transition kernel based on the regularized potentials. It is shown that no classical counterpart exists for walls with $L<0$, and that the WKB exactness can hold only for $L=0$ and $L=\infty$.

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I. INTRODUCTION

Quantum systems with contact interactions (i.e., point interactions or reflecting boundaries) enjoy an increasing interest recently. On the theoretical side, they have been found to exhibit a number of intriguing features, many of which have been seen before only in connection with quantum field theories. Examples include renormalization [1–5], Landau poles [6], anomalous symmetry breaking [5], duality [7–9], supersymmetry [9], and spectral anholonomy [9–11]. On the experimental side, the rapid developments of nanotechnology forecast that nanoscale quantum devices can be designed and manufactured into desired specifications. The description of some of these systems will involve the theory of contact interactions. As a simple example, a piece of a single nanowire would act as a one-dimensional line with two reflecting end points between which a conduction particle moves almost freely, allowing for a quantum-mechanical description with boundaries. Other applications arise, for instance, in systems with impurities that act as point scatterers. All these areas of interest lend impetus to investigate quantum systems with contact interactions further to uncover their full potential, both theoretically and experimentally.

The topic of this paper is the quantum half-line system, which is perhaps the simplest among those with contact interactions. This system also appears frequently as the radial part of higher-dimensional systems [12]. (For recent experimental studies, see Ref. [13] and references therein.) We consider a quantum particle that moves freely on a half-line $x \geqslant 0$ with the end point $x=0$ acting as a reflecting boundary, or an impenetrable wall. This system is known (see Sec. II) to admit a one-parameter family of distinct walls characterized by the boundary conditions,

$$\psi(0)+L\,\psi'(0)=0, \tag{1.1}$$

where L is a parameter that takes all real values including $L=\infty$. Clearly, the standard wall in which we impose $\psi(0)=0$ is obtained for $L=0$ but it is just one of the various walls allowed, and therefore the first question one may ask is whether those nonstandard walls with $L\neq 0$ can arise in actual physical settings.

To answer this, we study how those nonstandard walls can be realized as a limit of finite (regularizing) potentials. The potentials we consider are steplike and may readily be manufactured using, e.g., thin layers of different types of semiconductors. We shall show that it is indeed possible to realize such nonstandard walls out of the steplike potentials if we fine tune the limiting procedure. We then turn to the question whether such nonstandard walls are available only quantum mechanically or not. This will be examined by looking at the time delay of the particle in scattering, which is the time difference between the moments of incidence and reflection at the wall. It will be shown that quantum nonstandard walls with $L<0$, which are characterized by positive time delay, have no classical counterpart possessing the same time delay, which implies that these walls are purely quantum. We also consider the validity of the semiclassical WKB approximation for the transition kernel under nonstandard walls, where now one takes into account the possible two classical paths, the direct path and the bounce path, in the path integral [14]. This is of interest because it has been known that, for the standard wall as well as that of $L=\infty$, the WKB approximation becomes exact if a sign factor is properly attached to the contribution of the bounce path. We shall see that for these two values of L the required sign factor can be accounted for by the bounce effect, showing that the WKB approximation is in fact exact, whereas for other L the WKB exactness

*Email address: fulopt@poe.elte.hu
†Email address: cheon@mech.kochi-tech.ac.jp
‡Email address: izumi.tsutsui@kek.jp

cannot hold. Before presenting these results, we provide the basics of the quantum system on the half line below.

II. BASICS OF THE QUANTUM SYSTEM ON THE HALF LINE

The system of a (nonrelativistic) free particle on a half line $x \in [0,\infty)$ is governed by the Hamiltonian $H = -\hbar^2/(2m) d^2/dx^2$, supplemented by some boundary condition imposed at the wall $x=0$. The boundary condition is determined by the requirement that H be self-adjoint on the positive half line $x \geqslant 0$ and, mathematically, this is done by finding proper domains of the operator H on which it is self-adjoint. The result is that there exists a U(1) family of domains of states specified by Eq. (1.1) (see, e.g., Ref. [12], Appendix D therein), which can be readily understood by a direct inspection as well. Indeed, one sees by partial integration that for H to be self-adjoint one must have $\psi^* \psi' = \psi'^* \psi$ at $x=0$ for any state ψ on which H acts. If $\psi'(0) \neq 0$, this implies $\psi(0)/\psi'(0) = [\psi(0)/\psi'(0)]^* = -L$ with L being some real constant, which is just the condition (1.1).[1] The case $\psi'(0)=0$ which also fulfills the requirement can be included by allowing $L=\infty$ in Eq. (1.1). The whole family is U(1) because of the range of the parameter: $L \in (-\infty,\infty) \cup \{\infty\} \cong \mathrm{U}(1)$.

Under the boundary condition (1.1) the positive-energy states are

$$\varphi_k(x) = \frac{1}{\sqrt{2\pi}} (e^{-ikx} + e^{i\delta_k} e^{ikx}) \tag{2.1}$$

with $\delta_k = 2 \operatorname{arccot} kL$. In addition, for $L>0$, we also have one negative-energy state,

$$\varphi_{\text{bound}}(x) = \left(\frac{2}{L}\right)^{1/2} e^{-x/L} \quad (L>0), \tag{2.2}$$

which is a bound state localized at the wall with its characteristic size L. The existence of the bound state (2.2) can also be ensured from the minimum-energy condition. Namely, for any normalized state ψ the expectation value of the energy reads

$$\langle \psi, H\psi \rangle = \frac{\hbar^2}{2m} \frac{1}{L^2} \int_0^\infty \mathrm{d}x |\psi(x) + L\psi'(x)|^2 - \frac{\hbar^2}{2m} \frac{1}{L^2}, \tag{2.3}$$

where L is the parameter in Eq. (1.1). The lower bound $-\hbar^2/2m 1/L^2$ is attained if there exists a state satisfying $\psi(x) + L\psi'(x) = 0$ for all $x \geqslant 0$, which is just the bound state (2.2).

As seen in the bound state, the parameter L furnishes a physical scale in many of the properties of the system. An example for this is provided by the time delay that occurs when an incoming particle is reflected from the wall. The time delay in quantum scattering processes has been studied extensively (see, e.g., [18,19] and references therein). Its definition and calculation are done for our system as follows.[2] Let us consider a wave packet formed out of the positive-energy states (2.1),

$$\begin{aligned} \psi(x,t) &= \int_0^\infty \mathrm{d}k f(k) e^{ikx_0} e^{-(i\hbar k^2/2m)t} \varphi_k(x) \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}k\; f(k) e^{ikx_0} e^{-(i\hbar k^2/2m)t} e^{-ikx} \\ &\quad + \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}k f(k) e^{ikx_0} e^{i\delta_k} e^{-(i\hbar k^2/2m)t} e^{ikx}, \end{aligned} \tag{2.4}$$

where $f(k)$ is a real function peaked at $k_0>0$. The first term describes the incident packet whose maximum starts from x_0 at $t=0$ and moves to the left with velocity magnitude $v_0 = \hbar k_0/m$, as can be seen from a stationary phase argument,

$$\begin{aligned} &\mathrm{d}/\mathrm{d}k[-\hbar k^2/(2m)t + kx_0 - kx]|_{k=k_0} \\ &= 0 \Rightarrow x_{\max}^{(1)}(t) = x_0 - (\hbar k_0/m)t. \end{aligned} \tag{2.5}$$

Similarly, the reflected packet given by the second term moves as

$$x_{\max}^{(2)}(t) = -x_0 + (\hbar k_0/m)t + 2L/[1+(k_0L)^2]. \tag{2.6}$$

As t increases, the first packet moves towards the wall at $x=0$, and its maximum reaches it at $t_1 = x_0/v_0$. Meanwhile, the second packet comes from the left (if we allow $x<0$ as well) moving to the right and arrives at the wall at $t_2 = (x_0 - 2L/[1+(k_0L)^2])/v_0$. The difference between the two instants gives the time delay,

$$\tau = t_2 - t_1 = -\frac{2mL}{\hbar k_0[1+(k_0L)^2]}. \tag{2.7}$$

For $L=0$ and $L=\infty$, this time delay is zero, as one would expect on the ground that for such cases there is no parameter in the system possessing the dimension of time. Note that for negative L the time delay is positive, whereas for positive L it is negative.

From the eigenfunctions (2.1) and (2.2) the Feynman kernel describing the transition of the particle from $x=a$ at $t=0$ to $x=b$ at $t=T$ can be calculated (see Refs. [15–17]). The result is

$$K(b,T;a,0) = \sqrt{m/2\pi i\hbar T}[e^{im/2\hbar T(b-a)^2} \mp e^{im/2\hbar T(b+a)^2}], \tag{2.8}$$

[1] The fact that the constant L is universal for any state ψ can be seen by considering Eq. (1.1) for all linear combinations of two states ψ_1 and ψ_2 with L_1 and L_2, from which one deduces $L_1 = L_2$ immediately.

[2] Compare this with the classical-mechanical definition of time delay, presented in Sec. IV.

for $L=0$ ("$-$" sign) and $L=\infty$ ("$+$" sign). For $L<0$ the kernel is given by

$$\sqrt{m/2\pi i\hbar T}\Bigg[e^{im/2\hbar T(b-a)^2}+e^{im/2\hbar T(b+a)^2} - \frac{2}{|L|}\int_0^\infty dz e^{-z/|L|}e^{im/2\hbar T(b+a+z)^2}\Bigg], \tag{2.9}$$

and for $L>0$ by

$$\sqrt{m/2\pi i\hbar T}\Bigg[e^{im/2\hbar T(b-a)^2}+e^{im/2\hbar T(b+a)^2} - \frac{2}{L}\int_0^\infty dz e^{-z/L}e^{im/2\hbar T(b+a-z)^2}\Bigg]+\frac{2}{L}e^{i\hbar T/2mL^2}e^{-(b+a)/L}. \tag{2.10}$$

The salient feature of the result is that, for $L=0$ and $L=\infty$, the kernel (2.8) almost coincides with that obtained by the WKB semiclassical approximation, because the two terms in Eq. (2.8) correspond to the free kernels for the direct path from $(a,0)$ to (b,T) and for the bounce path that hits the wall once during the transition, respectively. The only problem for the complete WKB exactness is the appearance of the $\mp$ sign factor attached to the contribution from the bounce path. We shall show later that this sign factor can be attributed to the classical action $\Delta S_{\text{bounce}}=\hbar\pi$ gained by the bounce effect at the wall so that $e^{(i/\hbar)\Delta S_{\text{bounce}}}=\mp 1$.

III. REALIZATION OF THE WALL

We now discuss how to realize the wall characterized by Eq. (1.1) in actual physical settings. For this, we shall adopt a regularization method that is analogous to those used earlier for point singularities [4,12]. We extend the space to the entire line $-\infty<x<\infty$ and seek a potential $V(x)$ with finite support such that, in the limit of vanishing support, the boundary condition (1.1) at $x=0$ can be realized. Obviously, since no probability flow is admitted through the wall at $x=0$, such a regularized potential has to become infinitely high for $x<0$ in the limit. A simple choice for the potential fulfilling the demand is

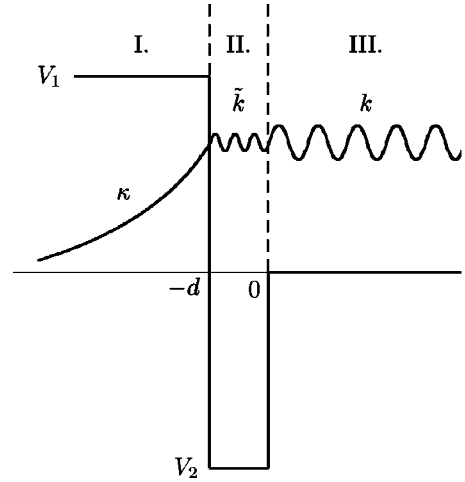

FIG. 1. The regularized potential (3.1) and the eigenfunction (3.2).

$$V(x)=\begin{cases} V_1, & x<-d \quad (\text{domain I})\\ V_2, & -d<x<0 \quad (\text{domain II})\\ 0, & x>0 \quad (\text{domain III})\end{cases} \tag{3.1}$$

with constants $V_1>0$ and $V_2<0$. Here, the scale of the support is given by the regularization parameter d, and V_1 and V_2 are assumed to be functions of d such that $V_1, |V_2|\to\infty$ as $d\to 0$.

To find the appropriate dependence of $V_1(d)$ and $V_2(d)$, let us consider an energy eigenstate φ in the potential (3.1) with energy $E<V_1$ (see Fig. 1),

$$\varphi(x)=\begin{cases} \varphi_{\rm I}(x)=Ne^{\kappa x}, & x<-d, \quad \kappa=\sqrt{(2m/\hbar^2)(V_1-E)},\\ \varphi_{\rm II}(x)=Ae^{i\tilde{k}x}+Be^{-i\tilde{k}x}, & -d<x<0, \quad \tilde{k}=\sqrt{(2m/\hbar^2)(|V_2|+E)},\\ \varphi_{\rm III}(x)=Ce^{ikx}+De^{-ikx}, & x>0, \quad k=\sqrt{2mE/\hbar^2}\end{cases} \tag{3.2}$$

(for $E<0$, $\varphi_{\rm III}(x)=Me^{-\sqrt{2m|E|}/\hbar x}$). Under such finite potentials (i.e., without infinity or singularity), the wave function and its derivative are required to be continuous. The condition which is dynamically important is provided by the continuity of the ratio φ'/φ which is free from the ambiguity of overall normalization. From this continuity condition, we obtain

$$\kappa=\frac{i\tilde{k}(Ae^{-i\tilde{k}d}-Be^{i\tilde{k}d})}{Ae^{-i\tilde{k}d}+Be^{i\tilde{k}d}}, \quad \frac{\varphi'_{\rm III}}{\varphi_{\rm III}}(0)=\frac{i\tilde{k}(A-B)}{A+B} \tag{3.3}$$

at $x=-d$ and $x=0$. Note that both $\tilde{k}$ and κ are d dependent $\tilde{k}=\tilde{k}(d)$, $\kappa=\kappa(d)$ through $V_1(d)$ and $V_2(d)$ and so are the two ratios in Eq. (3.3). If we introduce

$$R(d)=\frac{\varphi'_{\rm III}}{\varphi_{\rm III}}(0), \quad \alpha=\arctan\frac{\kappa}{\tilde{k}}, \quad \beta=\tilde{k}d, \tag{3.4}$$

then from Eq. (3.3) we find

$$R(d)=\tilde{k}\frac{(Ae^{-i\beta}-Be^{i\beta})\cos\beta-i(Ae^{-i\beta}+Be^{i\beta})\sin\beta}{(Ae^{-i\beta}+Be^{i\beta})\cos\beta-i(Ae^{-i\beta}-Be^{i\beta})\sin\beta}$$
$$=\tilde{k}\tan(\alpha-\beta). \tag{3.5}$$

The boundary condition (1.1) is realized if

$$R(d)\to-\frac{1}{L}\quad\text{as}\ \ d\to0, \tag{3.6}$$

independently of the energy E. In what follows, we present a set of regularized potentials fulfilling this requirement.

To this end, we first define

$$\alpha_0=\lim_{d\to0}\alpha,\quad \beta_0=\lim_{d\to0}\beta, \tag{3.7}$$

and note that, since $V_1(d)\to\infty$ as $d\to0$, we always have $\kappa\to\infty$, whereas since $0<\alpha<\pi/2$ by definition, we have $0\leqslant\alpha_0\leqslant\pi/2$. Note also that, if $V_2(d)$ used in our regularization is such that $\beta\to\infty$, then $\tan(\alpha-\beta)$ will oscillate between $-\infty$ and ∞ so $R(d)$ will not have a limit. We therefore confine ourselves to cases in which β has a finite (zero or nonzero) limit β_0. Now, let us suppose $\beta_0\neq\alpha_0$ (mod π), that is, $\tan(\alpha-\beta)\to\tan(\alpha_0-\beta_0)\neq0$. Then, if $|V_2|\to\infty$ we have $\tilde{k}\to\infty$ and, consequently, $R(d)\to\pm\infty$. If $|V_2|$ remains finite, on the other hand, then we find $\alpha_0=\pi/2$ and $\beta_0=0$ and hence $R(d)\to\infty$. We thus see that these regularizations yield necessarily the standard wall $L=0$.

The foregoing argument shows that nonstandard walls with $L\neq0$ can be realized only by such realizations in which V_1 and V_2 are fine tuned as

$$\beta_0=\alpha_0(\mathrm{mod}\,\pi). \tag{3.8}$$

We shall suppose Eq. (3.8) from now on, and consider the limit of $R(d)$ for the cases $\alpha_0=0$, $0<\alpha_0<\pi/2$ and $\alpha_0=\pi/2$, separately.

(i) Case $\alpha_0=0$. We then have, as $d\to0$, $\alpha\approx\tan\alpha=\kappa/\tilde{k}\to0$ and $\beta-\beta_0\to0$ and hence $\tan(\alpha-\beta)=\tan(\alpha-\beta+\beta_0)\approx\kappa/\tilde{k}-\beta+\beta_0$. Thus the ratio is approximated as

$$R(d)\approx\kappa-\tilde{k}(\beta-\beta_0). \tag{3.9}$$

Now, if $\beta_0=0$ then the right-hand side (rhs) reads $\kappa-\tilde{k}^2d$. Hence, to get a finite $R(d)$, $\tilde{k}^2d$ has to compensate the divergence of κ. This can be done if κ and $\tilde{k}$ behave as

$$\kappa\sim cd^{\nu}-\frac{1}{L},\quad \tilde{k}\sim c^{1/2}d^{(\nu-1)/2}\quad(-1<\nu<0), \tag{3.10}$$

which is realized if, for instance, we put

$$V_1(d)=\frac{\hbar^2}{2m}\left(c^2d^{2\nu}-\frac{2c}{L}d^{\nu}\right),\quad V_2(d)=-\frac{\hbar^2}{2m}cd^{\nu-1}, \tag{3.11}$$

with a constant $c>0$. It is then readily confirmed that this regularized potential (3.11) does lead to $R(d)$ fulfilling Eq. (3.6) for all $E>0$. If $\beta_0>0$, on the other hand, then $\beta_0d^{-1}(\beta-\beta_0)$ on the rhs of Eq. (3.9) has to cancel the divergence of κ. This means $\tilde{k}\sim\beta_0d^{-1}+(1/\beta_0)\kappa$. The needed finite term $-1/L$ can be provided again by κ if $\kappa\sim c_1d^{\nu}-1/L$. This is achieved, for example, by

$$V_1(d)=\frac{\hbar^2}{2m}\left(c^2d^{2\nu}-\frac{2c}{L}d^{\nu}\right),$$
$$V_2(d)=-\frac{\hbar^2}{2m}(\beta_0^2d^{-2}+2cd^{\nu-1}). \tag{3.12}$$

It is again easy to confirm that Eq. (3.12) yields $R(d)$ fulfilling Eq. (3.6) for $\nu>-\frac{1}{2}$.

(ii) Case $0<\alpha_0<\pi/2$. In this case, we have $\tilde{k}\sim\beta_0d^{-1}$ and $\kappa\sim(\beta_0\tan\beta_0)d^{-1}$. Using the Taylor expansion

$$\alpha=\arctan(\kappa/\tilde{k})\approx\alpha_0+\cos^2\alpha_0(\kappa/\tilde{k}-\tan\alpha_0), \tag{3.13}$$

we find

$$R(d)\approx\tilde{k}\tan[\alpha_0-\beta_0+\cos^2\alpha_0(\kappa/\tilde{k}-\tan\alpha_0)]$$
$$\approx\cos^2\alpha_0(\kappa-\tilde{k}\tan\alpha_0). \tag{3.14}$$

Hence the choice

$$\kappa\sim(\beta_0\tan\beta_0)d^{-1}-(1/\cos^2\beta_0)\frac{1}{L} \tag{3.15}$$

may lead to Eq. (3.6). A possible regularized potential realizing Eq. (3.15) is

$$V_1(d)=\frac{\hbar^2}{2m}\left[(\beta_0^2\tan^2\beta_0)d^{-2}-\frac{2}{L}(\beta_0\tan\beta_0/\cos^2\beta_0)d^{-1}\right],$$
$$V_2(d)=-\frac{\hbar^2}{2m}\beta_0^2d^{-2}, \tag{3.16}$$

which can be shown to give $R(d)$ satisfying Eq. (3.6).

(iii) Case $\alpha_0=\pi/2$. We still have $\tilde{k}\sim\beta_0d^{-1}$ but now $\kappa/\tilde{k}\to\infty$ so $\alpha\approx\pi/2-\tilde{k}/\kappa$, and therefore

$$R(d)\approx\tilde{k}\tan\left[\frac{\pi}{2}-\frac{\tilde{k}}{\kappa}-(\beta-\beta_0)-\beta_0\right]\approx\tilde{k}\left[-\frac{\tilde{k}}{\kappa}-(\beta-\beta_0)\right]. \tag{3.17}$$

The realization (3.6) will be attained if, for example, we have $\kappa/\tilde{k}^2\to\infty$ and provide $-1/L$ through $\tilde{k}$ by assuming $\tilde{k}\sim\beta_0d^{-1}+1/L\,1/\beta_0$. This is the case with the regularization,

$$V_1(d)=\frac{\hbar^2}{2m}c_1^2d^{2\nu}\quad(\nu<-2),$$
$$V_2(d)=-\frac{\hbar^2}{2m}\left(\beta_0^2d^{-2}+\frac{2}{L}d^{-1}\right). \tag{3.18}$$

To summarize, the regularization by means of the steplike potential (3.1) leads generically to the standard wall $L=0$. It can also lead to nonstandard walls $L\neq 0$ but only as exceptional cases under the fine tuning Eq. (3.8). It is worth emphasizing that the crucial factor in determining the limit of $R(d)$, i.e., the boundary condition at $x=0$, is not the leading asymptotic behavior of V_1 and V_2 in $d\to 0$ but always a subleading term. A similar phenomenon has been observed for the regularization of the Dirac delta point interactions in three space dimensions [12].

The regularizations we used are based on a steplike potential. Needless to say, other types of potentials can also be used for realizing the walls. One can, for instance, look for a potential that leads to the realization for any L without involving the mass parameter m. Such a regularization may be more desirable than that we constructed—where the potentials turned out to be m dependent—for the reason that potentials should be independent of the particle. Nonetheless, our simple regularization may well exhibit a universal feature of the realization of the (standard and nonstandard) walls, as we can see, for example, the bound state is accommodated in the negative middle part of the steplike potential we used.

IV. CLASSICAL COUNTERPARTS

Having seen that the quantum walls characterized by L can be realized by means of regularized potentials, we now turn to the question whether those walls have classical counterparts or not. We investigate this in the phenomena of time delay discussed in Sec. II, by asking if there is a classical system with some appropriate potential $V(x)$ that can account for the same amounts of time delay as those observed under the walls. Note that systems with the regularized potentials discussed above are not applicable for this purpose, because, in those systems, the time a classical particle spends in a potential (3.1) tends necessarily to zero as $d\to 0$ (since, as $V_2\to -\infty$, the distance run by the particle becomes zero while its velocity becomes infinity).

To find a potential for the classical particle that reproduces the quantum time delay, we shall first consider the walls with $L>0$. In this case the time delay (2.7) is negative, and if the classical picture is available, the incident particle with velocity magnitude $v=\hbar k/m$ must return earlier by

$$|\tau|=\frac{2L}{v}\,\frac{1}{1+\left(\dfrac{mL}{\hbar}v\right)^2} \tag{4.1}$$

than we would expect when it collided with the wall at $x=0$. Observe that, for small v the (minus) delay $|\tau|$ approaches $2L/v$. This suggests that a slow particle sees the wall at (around) $x=L$, not $x=0$. Consequently, the reflecting potential $V(x)$ is expected to begin to grow at $x=L$. For definiteness, let us search for the potential in the qualitative form as shown in Fig. 2. (This fixes an arbitrariness in the choice of the potential. As we will see, demanding a positive, monotonically decreasing potential determines the potential uniquely.) Now, let us introduce

$$\tilde{\tau}=\frac{2L}{v}+\tau=\sqrt{2mL^2E}\Bigg/\left(\frac{\hbar^2}{2mL^2}+E\right), \tag{4.2}$$

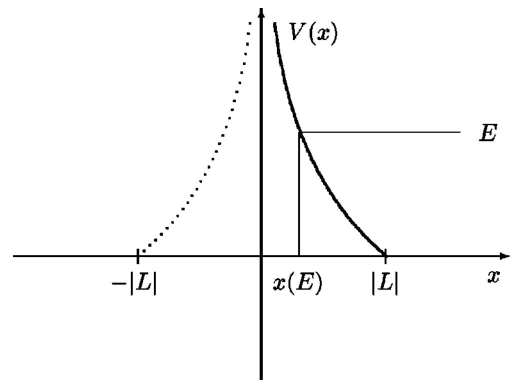

FIG. 2. The realizing potential (4.6) is shown by the solid line for $L>0$. For $L<0$ the obtained potential becomes the dotted line and is unphysical.

(where $E=\frac{1}{2}mv^2$ is the incoming energy), which is the time spent by the particle in the region on the left of the point $x=L$. Our problem is then an inverse problem: Determine a potential $V(x)$ from a given $\tilde{\tau}(E)$ as a function of E. This can be answered if we follow the well-known argument of Landau and Lifshitz [20] used for the problem of determining a well-shaped potential from the period time with which a particle moves.

We start by writing the relationship between the potential and $\tilde{\tau}$ as

$$\tilde{\tau}(E)=\sqrt{2m}\int_{x(E)}^{L}\frac{\mathrm{d}x}{\sqrt{E-V(x)}}=\sqrt{2m}\int_0^E\left(-\frac{\mathrm{d}x(V)}{\mathrm{d}V}\right)\frac{\mathrm{d}V}{\sqrt{E-V}}. \tag{4.3}$$

Dividing by $\sqrt{W-E}$ with W being an auxiliary parameter, and integrating with respect to E from 0 to W leads to

$$\int_0^W\frac{\tilde{\tau}(E)\mathrm{d}E}{\sqrt{W-E}}=\sqrt{2m}\int_0^W\mathrm{d}V\left(-\frac{\mathrm{d}x}{\mathrm{d}V}\right)\int_V^W\frac{\mathrm{d}E}{\sqrt{(W-E)(E-V)}}. \tag{4.4}$$

The inner integral (the one with respect to E) gives π, while on the lhs we can evaluate the integral explicitly [cf. Eq. (4.2)]. From the result,

$$\pi\sqrt{2m}L[1-1/\sqrt{1+(2mL^2/\hbar^2)W}]=\pi\sqrt{2m}[L-x(W)], \tag{4.5}$$

we obtain $x(W)=L[1+(2mL^2/\hbar^2)W]^{-1/2}$, inverting it yields[3]

$$V(x)=\frac{\hbar^2}{2mL^2}\left(\frac{L^2}{x^2}-1\right). \tag{4.6}$$

We can see that this wall-realizing potential sits on the positive half line. This is unavoidable: Indeed, if a potential is identically zero on the whole positive half line and is

[3]We remark that while this potential reproduces the time delay classically, it does not reproduce the boundary condition (1.1) and hence cannot serve as a potential to realize the walls quantum mechanically.

nonzero only on the negative half line then the time delay is necessarily non-negative. The most we can reach is that the penetration of the wall-realizing potential to the positive half line is finite. Equation (4.6) presents such a solution. We will see that, for $L<0$, we have to pay more.

For $L<0$, the time delay is positive, i.e., the quantum wave packet returns later than expected:

$$\tau=\frac{2|L|}{v}\frac{1}{1+\left(\frac{m|L|}{\hbar}v\right)^2}=\sqrt{2m}|L|\frac{1}{\sqrt{E}\left(1+\frac{2mL^2}{\hbar^2}E\right)}. \tag{4.7}$$

This is the time delay we try to reproduce with the corresponding classical particle as its classical time delay

$$\tau_{\mathrm{cl},x_0}(E)=\sqrt{2m}\int_{x(E)}^{x_0}\frac{\mathrm{d}x}{\sqrt{E-V(x)}}-\frac{2x_0}{\sqrt{2E/m}}, \tag{4.8}$$

where x_0 is the initial position of the particle. For small v, Eq. (4.7) becomes $2|L|/v$, which suggests that a slow particle enters the $x<0$ region and sees the wall near $x=-|L|$. For this, the realizing potential $V(x)$ is expected to start to increase at $x=-|L|$, and to keep increasing for smaller x. However, if one repeats the same argument used for the $L>0$ case, one ends up with Eq. (4.6) again, with now the left branch of this function (see Fig. 2). The obvious problem with this branch, i.e., it increases for x to the right of $-|L|$ and is unphysical, may be understood intuitively as follows. For high energies E, the particle is expected to move approximately freely, and since the particle travels at least until $x=-|L|$, the $E\to\infty$ asymptotics of the time delay would be at least $2|L|/v$. However, the time delay we have to reproduce has only a v^{-3} asymptotic behavior. This means that the coefficient of the v^{-1} term must vanish for $E\to\infty$, implying that in the limit the particle reaches only until $x=0$.

The situation cannot be helped with any additional potential in $-|L|<x<0$ or in $0<x$, nor by any other modification. Actually, it can be proven that no classically acceptable reflecting potential can fulfil the requirement that the time delay (4.7) be reproduced exactly for all $x_0>x_{\mathrm{thresh}}$, that is, for all initial positions of the incoming particle above a finite, possibly positive threshold position x_{thresh}. To see this, let us consider an arbitrary piecewise differentiable potential, even possibly diverging at the discontinuity points. Then the classical force $-V'(x)$ exists everywhere except for finitely many points, while at a discontinuity point an incoming classical trajectory can be continued with the outgoing trajectory that has the same energy E as the incoming one. The potential is further required to act as a completely reflecting wall, that is, for every positive energy E, there has to be a turning point $x(E)$ (as in Fig. 2). Note that then the function $x(E)$ is necessarily nonincreasing, and its inverse is $V(x)$ locally, i.e., it reproduces at least parts of the function $V(x)$.

First let us discuss the case when V is differentiable (and hence continuous) everywhere. The x_0 independence of the time delay $\tau_{\mathrm{cl},x_0}(E)$ [cf. Eq. (4.8)] implies

$$\frac{\mathrm{d}}{\mathrm{d}x_0}\tau_{\mathrm{cl},x_0}(E)=\sqrt{2m}\left[\frac{1}{\sqrt{E-V(x_0)}}-\frac{1}{\sqrt{E}}\right]=0 \tag{4.9}$$

and thus that $V=0$ above x_{thresh}. Let x_{pos} denote the lowest x above which the potential is nonpositive. Naturally, one has $x_{\mathrm{pos}}\leqslant x_{\mathrm{thresh}}$ and can write $x_{\mathrm{pos}}=\sup\{x|V(x)>0\}$, from which one finds $x_{\mathrm{pos}}=\lim_{E\searrow 0}x(E)$, that is, x_{pos} is the "turning point for zero energy."

If there exists an energy E_* with a turning point on the negative half line, $x(E_*)<0$, then for larger energies E the time delay is at least

$$\sqrt{2m}\int_{x(E_*)}^{x_0}\frac{\mathrm{d}x}{\sqrt{E-V}}-\sqrt{2m}\frac{x_0}{\sqrt{E}}, \tag{4.10}$$

which is obtained by omitting the time of traveling through the interval $[x(E),x(E_*)]$. Since V is continuous on the interval $[x(E_*),x_0]$, it is bounded and hence the high-energy asymptotics of Eq. (4.10) is

$$\sqrt{2m}\frac{x_0-x(E_*)}{\sqrt{E}}-\sqrt{2m}\frac{x_0}{\sqrt{E}}=\sqrt{2m}\frac{|x(E_*)|}{\sqrt{E}}\sim\frac{1}{\sqrt{E}}. \tag{4.11}$$

This is in contradiction with the asymptotics $E^{-3/2}$ of the demanded time delay (4.7). Consequently, all turning points have to be on the non-negative half line,

$$x(E)\geqslant\lim_{E'\to\infty}x(E')=:x_\infty\geqslant 0. \tag{4.12}$$

Next we prove that in $(x_\infty,x_{\mathrm{pos}}]$ the potential V decreases strictly. Namely, if we assume the contrary then there will be at least one point x_1 in this interval that is not a turning point [see Fig. 3(a)]. Within $[x_1,x_{\mathrm{pos}}]$, let x_2 denote the turning point with the highest energy E_2. Then, in the function $\tau_{\mathrm{cl},x_0}(E)$ there will be a discontinuity at $E=E_2$:

$$\frac{1}{\sqrt{2m}}\left[\lim_{E\searrow E_2}\tau_{\mathrm{cl},x_0}(E)-\lim_{E\nearrow E_2}\tau_{\mathrm{cl},x_0}(E)\right]=\lim_{E\searrow E_2}\int_{x(E)}^{x_0}\frac{\mathrm{d}x}{\sqrt{E-V}}-\lim_{E\nearrow E_2}\int_{x(E)}^{x_0}\frac{\mathrm{d}x}{\sqrt{E-V}}=\lim_{E\searrow E_2}\left[\int_{x(E)}^{x_2}\frac{\mathrm{d}x}{\sqrt{E-V}}+\int_{x_2}^{x_0}\frac{\mathrm{d}x}{\sqrt{E-V}}\right]$$
$$-\lim_{E\nearrow E_2}\int_{x(E)}^{x(0)}\frac{\mathrm{d}x}{\sqrt{E-V}}=\lim_{E\searrow E_2}\int_{x(E)}^{x_2}\frac{\mathrm{d}x}{\sqrt{E-V}}>\lim_{E\searrow E_2}\int_{x_1}^{x_2}\frac{\mathrm{d}x}{\sqrt{E-V}}=\int_{x_1}^{x_2}\frac{\mathrm{d}x}{\sqrt{E_2-V}}>0. \tag{4.13}$$

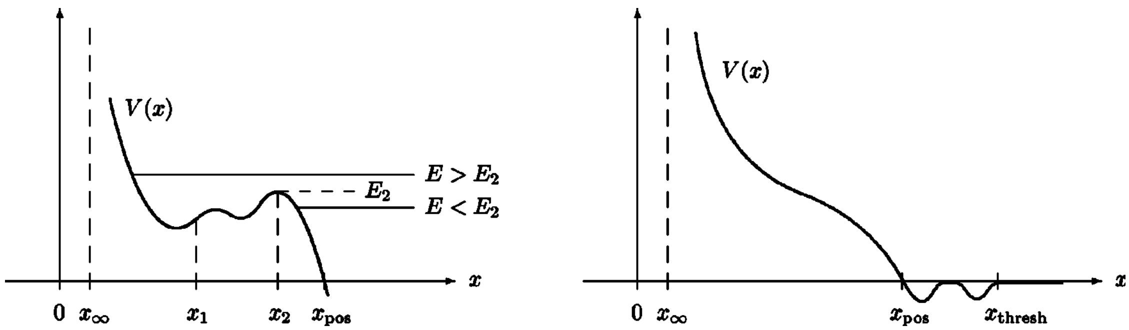

FIG. 3. (a) A nondecreasing part in the potential in $(x_\infty, x_{\rm pos}]$ causes a discontinuity in the time delay. (b) The obtained qualitative shape of the potential.

However, the required quantum time delay, Eq. (4.7), is a continuous function everywhere. This result tells us that on the region $(x_\infty, x_{\rm pos}]$ $x(E)$ is the inverse of $V(x)$ and is differentiable. We have also obtained the qualitative behavior of the candidate potential function [see Fig. 3(b)]. Coming from the right, it is zero above $x_{\rm thresh}$, nonpositive in $x_{\rm pos}<x<x_{\rm thresh}$, and is positive and increasing in $x_\infty<x\leqslant x_{\rm pos}$, diverging to $+\infty$ at x_∞.

Now we are ready to investigate the requirement $1/\sqrt{2m}\,\tau(E)=1/\sqrt{2m}\,\tau_{{\rm cl},x_0}(E)$:

$$\frac{|L|}{\sqrt{E}\left(1+\dfrac{2mL^2}{\hbar^2}E\right)}=\int_{x(E)}^{x_{\rm pos}}\frac{{\rm d}x}{\sqrt{E-V}}+\int_{x_{\rm pos}}^{x_0}\frac{{\rm d}x}{\sqrt{E-V}}-\frac{x_0}{\sqrt{E}}. \tag{4.14}$$

Observe that the second integral is bounded from above by $(x_0-x_{\rm pos})/\sqrt{E}$, since the potential is nonpositive on that interval. Employing again the "Landau trick" to the first integral (i.e., changing the variable from x to V, dividing by $\sqrt{W-E}$, and integrating between 0 and W), we find

$$\pi|L|/\sqrt{1+(2mL^2/\hbar^2)W}\leqslant-\pi x(W), \tag{4.15}$$

or

$$x(W)\leqslant-|L|/\sqrt{1+(2mL^2/\hbar^2)W}<0. \tag{4.16}$$

This, however, contradicts our previous result that all turning points have to be on the non-negative half line, showing that the requirement (4.14) cannot be fulfilled.

We can show that the preceding argument remains valid even if we allow discontinuity points in the potential—only slight modifications are necessary. The x_0 independence of the time delay implies $V=0$ at all continuity points, and hence everywhere, above $x_{\rm thresh}\cdot x_{\rm pos}$ is introduced in the same way and with the same properties as before. Equation (4.12) also remains valid: When assuming $x(E_*)<0$, the possible discontinuity points falling between $x(E_*)$ and x_0 can be covered by intervals of a total length less than, say, $\frac{1}{2}|x(E_*)|$. We omit even these covering intervals from the time delay, and on the remaining intervals the potential is continuous and has overall upper and lower bounds. Consequently, the high-energy asymptotics of the time delay is still at least $\sim 1/\sqrt{E}$.

The proof of the strict decreasing of V in $(x_\infty, x_{\rm pos}]$ holds, too. This also rules out discontinuity points $x_{\rm disc}$ in $(x_\infty, x_{\rm pos}]$ with $V(x_{\rm disc}-0)<V(x_{\rm disc}+0)$. Others are allowed but do not cause any trouble in the behavior of $x(E)$ because, for energies $E\in[V(x_{\rm disc}+0),V(x_{\rm disc}-0)]$, we then have $x(E)=x_{\rm disc}=\text{const}$ and $(d/dE)x(E)=0$. The transformation of the integration variable in the first integral in Eq. (4.14) remains applicable, while the second integral can also be estimated as before, in spite of any discontinuity points in $(x_{\rm pos}, x_{\rm thresh}]$. Therefore, we reach the same contradictory result (4.16) again.

Hence, interestingly enough, the walls with negative L do not admit a classical counterpart, i.e., they are genuinely quantum. Incidentally, we mention that if we demand only that the quantum time delay be reproduced in the $x_0\to\infty$ limit of $\tau_{{\rm cl},x_0}(E)$, then the required realization can be achieved (see the Appendix).

V. WKB EXACTNESS

The fact that for walls with $L=0$ and $L=\infty$ the transition kernel is almost WKB exact alludes us to examine whether this implies a complete exactness or not, and if so, whether such a feature persists to nonstandard walls as well. More precisely, we wish to see if the sum of amplitudes along the classical two paths, the direct world line from $(x,t)=(a,0)$ to (b,T) and the bouncing path that hits the wall $x=0$ before arriving at (b,T), give the exact result [see Fig. 4(a)]. The question, therefore, is whether or not the kernels (2.8), (2.9), and (2.10) can be rewritten in the form of a sum of the corresponding two terms as

$$\begin{aligned}K(b,T;a,0)=&\sqrt{(m/2\pi i\hbar T)}e^{im/2\hbar T(b-a)^2}\\&+\sqrt{(1/2\pi i\hbar)(\partial^2 S_{\rm bounce}/\partial a\,\partial b)}\\&\times e^{(i/\hbar)S_{\rm bounce}(b,T;a,0)},\end{aligned} \tag{5.1}$$

where $S_{\rm bounce}(b,T;a,0)$ is the classical action for the bounce path, and the factor before the second exponential term comprises the van Vleck determinant and the Maslov phase factor corresponding to the turning point (see Ref. [21] for the details). In the spirit of the sections herebefore, here again the wall is considered not necessarily to be simply the infinitely high vertical potential wall at the origin but to be

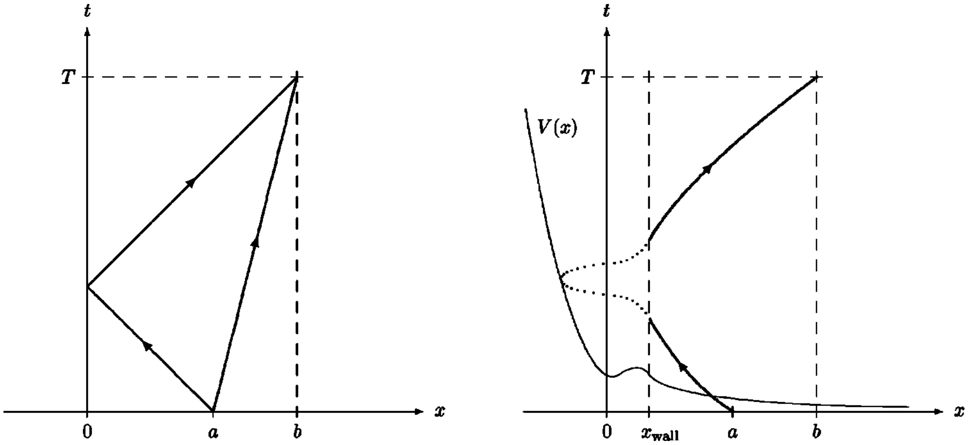

FIG. 4. (a) The direct and the bounce paths. (b) The bounce path under a wall-realizing potential.

realized by some sequence of more general reflecting potentials. What we require is that the potential sequence must converge uniformly to zero for all $x \geqslant x_{\rm wall}$ with some $x_{\rm wall}$ which may be positive, and that, for any a, $b > x_{\rm wall}$, the bounce path tends to the standard bounce world line depicted on Fig. 4(a), at least on the space-time region $x > x_{\rm wall}$. Otherwise we let the reflecting potential sequence be arbitrary to the left of $x_{\rm wall}$ and, therefore, at the limit of the sequence, the resultant action $S_{\rm bounce}$ can differ from the action $S^{(0)}_{\rm bounce} = m(a+b)^2/2T$ that corresponds to the simplest case of the infinitely high vertical potential wall with no extra action contribution caused by the wall.

Even these very mild assumptions allow us to observe some important, generally valid, properties. The first one is that, although the direct path is also influenced by a nonvanishing potential, its WKB contribution $\sqrt{(i/2\pi\hbar)(\partial^2 S_{\rm direct}/\partial a \partial b)}e^{i/\hbar S_{\rm direct}}$ will still reduce to the first term of Eq. (5.1). Indeed, since in the limit we have $V \to 0$ and hence the velocity of the particle tends uniformly to $(b-a)/T$, we trivially find $E \to E^{(0)}_{\rm direct} = m/2(b-a)^2/T^2$ and $S_{\rm direct} \to S^{(0)}_{\rm direct} = m/2(b-a)^2/T$. The nontrivial question that remains to be shown concerns with the property of the derivative, $\partial^2 S_{\rm direct}/\partial a \partial b \to \partial^2 S^{(0)}_{\rm direct}/\partial a \partial b$, but this can be seen by writing the action as

$$S_{\rm direct} = \int_0^T {\rm d}t(E-2V) = -TE + 2\int_0^T {\rm d}t(E-V)$$
$$= -TE + \sqrt{2m}\int_a^b {\rm d}x\sqrt{E-V}, \qquad (5.2)$$

which is valid for a, $b > x_{\rm wall}$, and evaluating

$$\frac{\partial^2 S_{\rm direct}}{\partial a \partial b} = -\frac{\sqrt{m/2}}{\sqrt{E-V(a)}}\frac{\partial E}{\partial b}$$
$$= -\sqrt{2m}\left[\sqrt{E-V(a)}\sqrt{E-V(b)}\int_a^b \frac{{\rm d}x}{\sqrt{E-V}^3}\right]^{-1}. \qquad (5.3)$$

Here, the energy E of the direct path is determined by the condition

$$\sqrt{m/2}\int_a^b \frac{{\rm d}x}{\sqrt{E-V}} = T, \qquad (5.4)$$

which is used to evaluate $\partial^2 S_{\rm direct}/\partial a \partial b$ in Eq. (5.3). Plugging the limiting values for the energies and the potential in Eq. (5.3), we find the required property.

Second, we make the observation that the energy of the bounce path converges to $E^{(0)}_{\rm bounce} = m/2(a+b)^2/T^2$. This follows from our requirement that the bounce path must tend to the standard bounce world line outside $x_{\rm wall}$, because, then the velocity of the particle tends uniformly to $(a+b)/T$ under the vanishing potential. In addition, we find that, although $\Delta S_{\rm bounce} = S_{\rm bounce} - S^{(0)}_{\rm bounce}$ does not necessarily tend to zero, in the limit it becomes independent of a and b. This can be seen as follows:

$$S_{\rm bounce} = -TE + \sqrt{2m}\int_{x(E)}^a {\rm d}x\sqrt{E-V} + \sqrt{2m}\int_{x(E)}^b {\rm d}x\sqrt{E-V} \qquad (5.5)$$

and

$$\partial S_{\rm bounce}/\partial a = \sqrt{2m}\sqrt{E-V(a)}, \qquad (5.6)$$

where now the energy of the bounce path is determined by

$$\sqrt{\frac{m}{2}}\int_{x(E)}^a \frac{{\rm d}x}{\sqrt{E-V}} + \sqrt{\frac{m}{2}}\int_{x(E)}^b \frac{{\rm d}x}{\sqrt{E-V}} = T \qquad (5.7)$$

[again, Eq. (5.7) is used also for the result (5.6)]. Since now $E \to E^{(0)}_{\rm bounce}$, it follows that

$$\partial S_{\rm bounce}/\partial a \to m(a+b)/T = \partial S^{(0)}_{\rm bounce}/\partial a, \qquad (5.8)$$

so $\partial \Delta S_{\rm bounce}/\partial a \to 0$. The b independence of $\Delta S_{\rm bounce}$ is proven analogously.

Third, if we restrict ourselves to the potential sequences of the type (3.1) then Eqs. (5.5) and (5.7) read

$$S_{\rm bounce} = -TE + \sqrt{2m}(a+b)\sqrt{E} + 2\sqrt{2m}d\sqrt{E+|V_2|} \qquad (5.9)$$

and

$$\sqrt{m/2}\frac{a+b}{\sqrt{E}}+\sqrt{m/2}\frac{d}{\sqrt{E+|V_2|}}=T. \tag{5.10}$$

From Eq. (5.9) we have that

$$\lim_{d\to 0}\Delta S_{\rm bounce}=\lim_{d\to 0}(2\sqrt{2m}d\sqrt{|V_2|}). \tag{5.11}$$

Furthermore, $\partial^2 S_{\rm bounce}/(\partial a\partial b)$ can be computed by differentiating Eq. (5.6), and using $\partial E/\partial b$, the latter is obtained by expressing $b=b(E)$ from Eq. (5.10) and applying $\partial E/\partial b=1/[\partial b/\partial E]$. Taking the limit of the result gives m/T, so we find that, in the limit, the square-root factors in the two terms of Eq. (5.1) equal each other for these steplike potential sequences.

By virtue of these properties, we are able to discuss the question of complete WKB exactness. In the cases $L=0$ and $L=\infty$, it is possible to reproduce the required action contribution $\Delta S_{\rm bounce}=\pi\hbar$ and $\Delta S_{\rm bounce}=0$, respectively, for example, with the steplike potential sequences (3.1). In fact, choosing for $L=0$,

$$V_1(d)={\rm const}\times d^{-1},\quad V_2(d)=-\frac{\hbar^2}{2m}\left(\frac{\pi}{2}\right)^2 d^{-2} \tag{5.12}$$

(a potential sequence with $\alpha_0=0$ and $\beta_0=\pi/2$), and for $L=\infty$,

$$V_1(d)=\frac{\hbar^2}{2m}c^2 d^{-1},\quad V_2(d)=-\frac{\hbar^2}{2m}cd^{-3/2}, \tag{5.13}$$

which is the case $\nu=-\frac{1}{2}$ of Eq. (3.11), provides just these needed action contributions [cf. Eq. (5.11)]. Note that these potential sequences are, at the same time, correct realizations of the quantum boundary condition with $L=0$, respectively $L=\infty$, as well. Nevertheless, they are not unique even among the steplike realizations with these properties, and presumably other potential shapes can also serve as examples for even both the complete WKB exactness and realizing the quantum boundary condition.

On the other side, for the other walls $L\neq 0$, ∞, one can prove that no potential sequence can account for the kernels (2.9) and (2.10) irrespective of whether the potential sequence reproduces the correct quantum boundary condition or not. To see this, let us write these kernels in the form

$$\sqrt{(m/2\pi i\hbar T)}[e^{im/2\hbar T(b-a)^2}+A_L(a,b,T)e^{(i/\hbar)S^{(0)}_{\rm bounce}}]. \tag{5.14}$$

If the complete WKB exactness holds, then $\arg A_L(a,b,T)$ should correspond to the limit of $\Delta S_{\rm bounce}/\hbar$, which we know is unavoidably independent of a and b. However, actually $\arg A_L(a,b,T)$ does depend on a and b, as can be checked simply, for example, on the large-T asymptotics of $A_L(a,b,T)$,

$$A_L(a,b,T)\approx\begin{cases}-\sqrt{(m/2\pi i\hbar T)}e^{-(2imL/\hbar T)(a+b-L)}, & L<0,\\ \dfrac{2}{L}e^{-(a+b)/L}e^{-im/2\hbar T[(a+b)^2-(\hbar T/mL)^2]}, & L>0,\end{cases} \tag{5.15}$$

as one finds from Eqs. (2.9) and (2.10).

We thus learn that the quantum walls with $L=0$ and $L=\infty$, which correspond to the Dirichlet $\psi(0)=0$ and the Neumann $\psi'(0)=0$ boundary condition, respectively, are distinguished in the U(1) family of walls with respect to the WKB exactness. These two cases are distinguished also by their scale invariance, which arises due to the absence of the scale parameter. The relationship between the two, the WKB exactness and scale invariance, is however unclear.

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APPENDIX: WEAK CLASSICAL REALIZATION OF THE TIME DELAY FOR $L<0$

Here, we outline how a weaker classical realization of the quantum time delay, namely, the $x_0\to\infty$ limit of the classical time delay $\tau_{{\rm cl},x_0}(E)$, can be determined for the walls $L<0$. Let us assume that we have a strictly decreasing positive potential such that, for a fixed finite x_0 and all energies E above $V(x_0)$, $\tau_{{\rm cl},x_0}(E)=\tau(E)$. We use the Landau trick again, dividing this equation by $\sqrt{W-E}$, integrating now between $V(x_0)$ and W, and evaluating the left-hand side by changing the variable to V. From the result we can express $x(W)$ to find

$$x(W)=\frac{x_0}{\pi}\arccos\left[1-\frac{2V(x_0)}{W}\right]-\frac{2|L|/\pi}{\sqrt{1+(2mL^2/\hbar^2)W}}\times\arccos\left(\frac{1+\dfrac{2mL^2}{\hbar^2}W}{1+\dfrac{2mL^2}{\hbar^2}V(x_0)}\frac{V(x_0)}{W}\right)^{1/2}. \tag{A1}$$

Now we perform the limit $x_0\to\infty$, with a fixed W. The second term on the rhs of Eq. (A1) remains finite no matter how $V(x_0)$ changes correspondingly. Consequently, to have a finite $x(W)$ in the limit, $\arccos[1-2V(x_0)/W]$ has to tend to zero. This means $V(x_0)\to 0$, and from $\cos\varepsilon\approx 1-\varepsilon^2/2$ ($\varepsilon\approx 0$) we have the asymptotics $\arccos[1-2V(x_0)/W]\approx 2\sqrt{V(x_0)/W}$ so to reach a finite limit of Eq. (A1) $x_0\sqrt{V(x_0)}$ has to converge to a constant. Introducing

$$c:=\lim_{x_0\to\infty}\frac{2\sqrt{2m}}{\pi\hbar}x_0\sqrt{V(x_0)}, \tag{A2}$$

which will be a free parameter in the realizing potential, the limit of Eq. (A1) is

$$x(W)=\frac{\hbar}{\sqrt{2m}}(c/\sqrt{W}-1/\sqrt{\hbar^2/2mL^2+W}). \qquad \text{(A3)}$$

One can check that the inverse of this $x(W)$ is really a strictly decreasing potential tending to zero if $c\geqslant 1$, and that the time delay corresponding to it is

$$\tau_{\mathrm{cl},x_0}(E)=\hbar\left[c\,\frac{\sqrt{1-V(x_0)/E}-1}{\sqrt{V(x_0)E}} + \frac{1}{\sqrt{\hbar^2/2mL^2+V(x_0)}}\left(\frac{1}{\sqrt{E}}-\frac{\sqrt{E-V(x_0)}}{\frac{\hbar^2}{2mL^2}+E}\right)\right], \qquad \text{(A4)}$$

whose $x_0\to\infty$ limit is really the desired quantum time delay (4.7) (independently of c). The potential itself is obtained by solving the biquadratic equation that follows from Eq. (A3), and reads, for example, for $c=1$,

$$V(x)=\frac{2\hbar^2}{mL^2}\left(\frac{x}{|L|}\right)^{-2/3}\left[\left(\frac{x}{|L|}\right)^{2/3}+\eta(x)^{-1} + 2\sqrt{\eta(x)-\eta(x)^4}\right]^{-2} \qquad \text{(A5)}$$

with

$$\eta(x)=\frac{1}{\sqrt{2}}\{[\sqrt{1+\tfrac{1}{27}(x/|L|)^4}+1]^{1/3} - [\sqrt{1+\tfrac{1}{27}(x/|L|)^4}-1]^{1/3}\}^{1/2}. \qquad \text{(A6)}$$

[1] C. Manuel and R. Tarrach, Phys. Lett. B **328**, 113 (1994).
[2] T. Cheon, T. Shigehara, and K. Takayanagi, J. Phys. Soc. Jpn. **69**, 345 (2000).
[3] T. Cheon and T. Shigehara, Phys. Lett. A **243**, 111 (1988).
[4] T. Shigehara, H. Mizoguchi, T. Mishima, and T. Cheon, IEICE Trans. Fundamentals **E82-A**, 1708 (1999).
[5] R. Jackiw, *Diverse Topics in Theoretical and Mathematical Physics*, (World Scientific, Singapore, 1995), paper I.3.
[6] R. Tarrach, e-print hep-th/9502020.
[7] T. Cheon and T. Shigehara, Phys. Rev. Lett. **82**, 2536 (1999).
[8] I. Tsutsui, T. Fülöp, and T. Cheon, J. Phys. Soc. Jpn. **69**, 3473 (2000).
[9] T. Cheon, T. Fülöp, and I. Tsutsui, Ann. Phys. (N.Y.) **294**, 1 (2001).
[10] T. Cheon, Phys. Lett. A **248**, 285 (1998).
[11] I. Tsutsui, T. Fülöp, and T. Cheon, J. Math. Phys. **42**, 5687 (2001).
[12] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics* (Springer, New York, 1988).
[13] F. Shimizu, Phys. Rev. Lett. **86**, 987 (2001).
[14] T. Fülöp and I. Tsutsui, Phys. Lett. A **264**, 366 (2000).
[15] E. Farhi and S. Gutmann, Int. J. Mod. Phys. A **5**, 3029 (1990).
[16] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, Heidelberg, 1991).
[17] T. E. Clark, R. Menikoff, and D. H. Sharp, Phys. Rev. D **22**, 3012 (1980). The possible bound state is not treated in this paper.
[18] A. Galindo and P. Pascual, *Quantum Mechanics I* (Springer-Verlag, Heidelberg, 1990), p. 154.
[19] D. Bollé, and T. A. Osborn, Phys. Rev. D **13**, 299 (1976).
[20] L. D. Landau and E. M. Lifshitz, *Mechanics, Course of Theoretical Physics* Vol. 1 (Butterworth-Heinemann, Oxford, 1976).
[21] L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981).