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# Input-output theory for fermions in an atom cavity

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We generalize the quantum optical input-output theory developed for optical cavities to ultracold fermionic atoms confined in a trapping potential, which forms an ''atom cavity.'' In order to account for the Pauli exclusion principle, quantum Langevin equations for all cavity modes are derived. The dissipative part of these multimode Langevin equations includes a coupling between cavity modes. We also derive a set of boundary conditions for the Fermi field that relate the output fields to the input fields and the field radiated by the cavity. Starting from a constant uniform current of fermions incident on one side of the cavity, we use the boundary conditions to calculate the occupation numbers and current density for the fermions that are reflected and transmitted by the cavity.

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## I. INTRODUCTION

In light of the remarkable achievement of Bose-Einstein condensation in 1995 [1], there has been a growing application of ideas from quantum optics to matter waves. This new field of atom optics [2] has included both theoretical and experimental investigations of matter wave coherence [3–7], atom lasers [8,9], nonlinear effects in matter waves including matter wave mixing [10,11], parametric amplification, and squeezing in coupled optical and matter waves [12]. However, the extension of these ideas to degenerate Fermi gases has proven difficult because of the Pauli exclusion principle that prohibits one from developing simple theoretical models based on only a few normal modes of the Schrödinger field. In addition to this, Fermi fields do not possess a classical limit analogous to the coherent state for Bose fields thereby making it impossible to develop semiclassical mean-field theories such as the Gross-Pitaevskii equation for Bose fields.

All in all, the physical intuition obtained from quantum optics cannot be directly applied to the theoretical investigations of fermions. Things which we take for granted in optics, such as what is meant by a beam of light or optical coherence, cannot be generalized in a straightforward manner to degenerate Fermi systems. It therefore seems necessary that in order to make progress in the theory of fermionic atom optics, fundamental model systems in quantum optics need to be reanalyzed from first principles. Recent work in this direction indicates that four-wave mixing and coherent amplification of matter waves can occur in fermionic systems as a result of cooperative many-particle quantum interference analogous to Dicke super-radiance [13–15]. However, a full treatment of nonlinear wave mixing among fermions that goes beyond lowest-order perturbation theory and a handful of fermions is still lacking due to the large number of modes needed.

The purpose of this paper is to consider another model system, the atomic analog of an optical cavity with two partially transmittive mirrors. A schematic of our system is illustrated in Fig. 1. It consists of an atom cavity formed by two potential barriers with a finite number of bound states. The cavity states are coupled to a continuum of free particle states on either side of the cavity via tunneling through the barriers. Our goal is to develop an input-output theory for fermions in the atom cavity that allows one to calculate the field radiated out of the cavity in terms of the field incident on the cavity. While we only treat the linear problem in this paper, we intend our model to serve as the basis for the treatment of nonlinear wave mixing processes among fermions. The intracavity nonlinear wave mixing among fermions should be significantly simpler than wave mixing in the continuum since in the continuum the number of modes needed is always greater than the number of fermions, but in the cavity the number of modes required is limited to the number of cavity bound states.

The input-output theory for a single mode of a lossy optical cavity was developed by Collett and Gardiner in the form of quantum Langevin equations for the cavity mode [16]. The great utility of this theory is that it allows one to incorporate the effects of quantum noise on the output field transmitted by the cavity as well as in the intracavity dynamics. Collett and Gardiner's formalism has been extended to bosonic matter fields in order to model the output coupling of atoms from a Bose-Einstein condensate in a single mode of an atom trap [17]. As we will show below, the necessity of treating all modes of the atom cavity for fermions leads to novel features not present in the single mode bosonic theories. Most significantly, the eigenstates of the cavity become coupled due to their mutual interaction with the same external continuum states. Second, the coupling of the reservoir modes to all cavity modes leads to the creation of coherences between fermions occupying the different single-particle

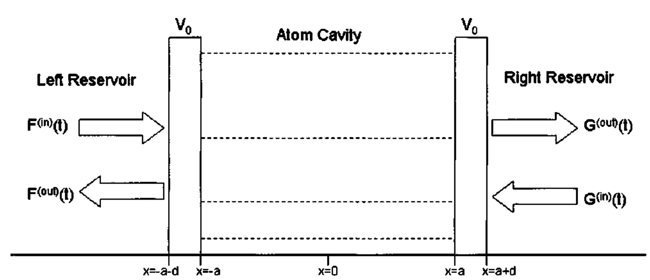


FIG. 1. Schematic diagram of the 1D atom cavity system.

 

modes in the radiated field even if the incident field is completely incoherent.

The outline of the paper is as follows: In Sec. II, we present our physical model for an atom cavity coupled to a continuum of reservoir states. In Sec. III, we derive a set of quantum Langevin equations for the fermionic annihilation and creation operators of the eigenstates of the cavity in terms of both the input and the output fields. These results are generalized to a two-sided cavity in Sec. IV. In Sec. V, we consider a constant current of fermions incident on one side of the atom cavity, and calculate the steady-state statistics of the fermions transmitted through the other side of the cavity. In Appendix A, we show that in a manner similar to the bosonic case, the presence of noise operators in the Langevin equations is necessary to preserve the anticommutation relations for the fermion operators. In Appendix B, we derive explicit forms for the coupling constants, which connect the intracavity modes to external continuum modes via a tunneling Hamiltonian.

## II. PHYSICAL MODEL

A physical schematic of our system is illustrated in Fig. 1. For simplicity, we restrict ourselves to one spatial dimension that allows space to be divided into five distinct regions. The region $-a \leqslant x \leqslant a$ between the two potential barriers of height $V_0$ represents the atom cavity. For thick barriers, the number of bound states of the cavity, $N+1$, is given by $N\pi/2 < \beta = \sqrt{2mV_0a^2/\hbar^2} \leqslant (N+1)\pi/2$, where $m$ is the mass of the atoms [19]. We will focus on the case where $N \gg 1$. The regions $-L-a-d<x<-a-d$ and $a+d<x<L+a+d$ represent the left and right reservoirs, respectively. $L$ is the length of the reservoir region. We let it go to infinity so that the fermions are described in terms of a continuum of free particle plane-wave states. The atoms located in the reservoir regions with energies less the $V_0$ couple to the cavity states by tunneling through the potential barriers located at $-a-d \leqslant x < -a$ and $a < x \leqslant a+d$.

Since the wave functions for atoms with energies greater than $V_0$ are not spatially localized in either the cavity or the reservoir regions, we restrict ourselves to single-particle states with energies less than $V_0$. In this case, we can meaningfully speak of left/right reservoir states and cavity states since the single-particle wave functions decay exponentially inside the potential barriers. Such a restriction is valid provided the initial state does not contain any occupied states with energies greater than $V_0$ and two-body collisional interactions between atoms, which can cause atoms to be scattered into higher-energy states, are negligible. The latter condition will indeed be satisfied for ultracold spin-polarized fermions since *s*-wave collisions are forbidden and *p*-wave collisions are negligible at these temperatures. Under these conditions, the states with energies larger than $V_0$ are not coupled to states with energies less than $V_0$.

The second quantized Hamiltonian for the cavity-reservoir system in the subspace of states with energies below the barrier is

$$H = H_S + H_L + H_R + H_{SR} + H_{SL}, \tag{1}$$

where $H_S$, $H_L$, and $H_R$ are the free Hamiltonians for the system (i.e., the atom cavity) and the left and right reservoir states, respectively,

$$H_S = \sum_{n=0}^{N} \hbar\Omega_n c_n^\dagger c_n, \tag{2}$$

$$H_L = \sum_k \hbar\omega_k a_k^\dagger a_k, \tag{3}$$

$$H_R = \sum_k \hbar\omega_k b_k^\dagger b_k. \tag{4}$$

Here, $c_n$ is a fermionic annihilation operator that destroys an atom in the cavity with wave function $\phi_n^{(s)}(x)$ and energy $\hbar\Omega_n = \hbar^2 K_n^2/2m$. Similarly, $a_k$ and $b_k$ are fermionic annihilation operators that destroy an atom in the left and right reservoirs, respectively, with the wave function

$$\phi_k^{(l,r)}(x) = \exp[ik(x \pm (a+d)]/L^{1/2}$$

and energy $\hbar\omega_k = \hbar^2 k^2/2m$ in the regions outside the barrier.

The coupling between the system and the reservoirs, $H_{SR}$ and $H_{SL}$, is given by effective tunneling Hamiltonians [20–23]

$$H_{SL} = i\hbar \sum_{n,k} [\kappa_{n,k} c_n^\dagger a_k - \kappa_{n,k}^* a_k^\dagger c_n], \tag{5}$$

$$H_{SR} = i\hbar \sum_{n,k} [\tilde{\kappa}_{n,k} c_n^\dagger b_k - \tilde{\kappa}_{n,k}^* b_k^\dagger c_n]. \tag{6}$$

In all cases the summation is restricted to those states with energies below the barrier. Explicit expressions for the tunneling matrix elements $\kappa_{n,k}$ and $\tilde{\kappa}_{n,k}$ are given in Appendix B. We note here that in one dimension (1D) the coupling constants depend only on the magnitude of $k$ and not on its sign.

In contrast to quantum optical systems, which are often approximated as a single cavity mode with a large occupation number, a full multimode treatment is required for fermions even if the number of fermions in the cavity is small $(\sim 1)$. This is because of the Pauli principle that forbids more than one atom from occupying the same cavity state, and thereby prevents one from singling out a particular state as being more important than the rest.

We conclude this section by noting that the general results presented below do not depend on the precise nature of our physical model. We use a stepwise constant potential because it leads to simple analytic results for the coupling between the reservoirs and cavity states. Our model system in Eq. (1) could be applied to any fermion system in which a finite number of discrete states are linearly coupled to a dense continuum of states via tunneling through a potential barrier. An atom cavity of this type could be created experimentally using a blue detuned optical dipole trap formed from hollow core Bessel beams [18].

### III. SINGLE-SIDED CAVITY

We now derive a set of integro-differential equations of motion for the cavity operators that only involve the initial or final state of the reservoir operators. We proceed by formally integrating the equations of motion for the reservoir operators and substituting these back into the equations for the cavity operators. In this section, we set $H_R = H_{SR} = 0$ so that the atom cavity is coupled to a single reservoir. This is analogous to an optical cavity with a single partially transmittive mirror. The inclusion of the right reservoir is summarized in the following section.

It is convenient to work with slowly varying operators in the interaction representation,

$$c_n(t) = e^{-i\Omega_n t}\hat{c}_n(t), \quad a_k(t) = e^{-i\omega_k t}\hat{a}_k(t). \tag{7}$$

By formally integrating the Heisenberg equations of motion for $\hat{a}_k(t)$ from the initial time $t_0$ to $t$,

$$\hat{a}_k(t) = \hat{a}_k(t_0) - \sum_m \kappa^*_{m,k}\int_{t_0}^{t} dt' e^{i(\omega_k-\Omega_m)t'}\hat{c}_m(t'), \tag{8}$$

where $\hat{a}_k(t_0)$ are the operators for the input field incident on the cavity barrier, and substituting this solution into the equations of motion for $\hat{c}_n(t)$, we obtain

$$\frac{d}{dt}\hat{c}_n(t) = \sum_k \kappa_{n,k}e^{-i(\omega_k-\Omega_n)t}\hat{a}_k(t_0) - \sum_m e^{i(\Omega_n-\Omega_m)t}\int_0^{t-t_0} d\tau\,\alpha_{n,m}(\tau)\hat{c}_m(t-\tau). \tag{9}$$

Here, the reservoir correlation function is given by

$$\alpha_{n,m}(\tau) = \sum_k \kappa_{n,k}\kappa^*_{m,k}e^{i(\Omega_m-\omega_k)\tau}, \tag{10}$$

which decays to zero in a characteristic time $\tau_c$ due to the destructive interference between the different oscillations. Note that $\tau_c$ depends, in general, on the cavity states $n$ and $m$ that are coupled by $\alpha_{n,m}(\tau)$, and $\tau_c^{-1}$ is of the order of the bandwidth of the reservoir, $V_0/\hbar$. Furthermore, if we assume that $\hat{c}_m(t)$ only changes significantly over a time scale $T_m \gg \tau_c$, then we can make the Markov approximation by setting $\hat{c}_m(t-\tau) = \hat{c}_m(t)$ in Eq. (9). For time intervals $t - t_0 \gg \tau_c$, we can then make the replacement

$$\int_0^{t-t_0} d\tau\,\alpha_{n,m}(\tau) \approx \int_0^{\infty} d\tau\,\alpha_{n,m}(\tau), \tag{11}$$

where the latter expression is given by

$$\int_0^{\infty} d\tau\,\alpha_{n,m}(\tau) = \gamma_{n,m} + i\Delta_{n,m}, \tag{12}$$

with

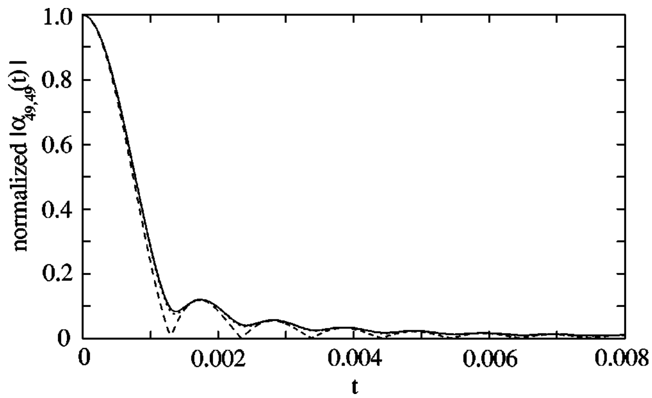


FIG. 2. Plot of $\alpha_{n,n}(\tau)$ for $n=49$ and $d/a=0.0001$ (solid line), 0.001 (dotted line), and 0.01 (dashed line). Times are measured in units of $\Omega_0^{-1}$. $V_0$ and $a$ were chosen so that the cavity contains 50 bound states.

$$\gamma_{n,m} = \pi\sum_k \kappa_{n,k}\kappa^*_{m,k}\delta(\Omega_m - \omega_k) \tag{13}$$

and

$$\Delta_{n,m} = P\sum_k \kappa_{n,k}\kappa^*_{m,k}\frac{1}{\Omega_m - \omega_k}. \tag{14}$$

These expressions are defined in the continuum limit such that $\Sigma_k \rightarrow (L/2\pi)\int dk$.

The Markov approximation assumes that the correlation function decays very rapidly, which requires that $\kappa_{n,k}$ vary slowly with $k$. Figure 2, shows the correlation function $|\alpha_{n,n}(\tau)|$ for the highest energy bound state in the potential well. This state has the longest correlation time since the $|\kappa_{n,k}|^2$ are the largest for this state and because the summation over reservoir states is restricted to states with energies less than $V_0$. For the three cases plotted in Fig. 2, the smallest value of $\gamma_{n,n}^{-1}$ among all the cavity states is $0.55\Omega_0^{-1}$ ($d/a=0.0001$), $0.62\Omega_0^{-1}$ ($d/a=0.001$), and $1.36\Omega_0^{-1}$ ($d/a=0.01$). In each case, we see that the correlation function goes to zero in a time much shorter than $T_n \approx \gamma_{n,n}^{-1}$. This shows that the Markov approximation is a very good approximation for our system with $T_n/\tau_C > 10^2$. By combining the above results, we obtain a quantum Langevin equation for each of the cavity modes,

$$\dot{c}_n(t) = -i\Omega_n c_n(t) - \sum_m (\gamma_{n,m} + i\Delta_{n,m})c_m(t) + F_n^{(in)}(t). \tag{15}$$

In Eq. (15), we have defined the *input noise operator* $F_n^{(in)}(t)$ as

$$F_n^{(in)}(t) = \sum_k \kappa_{n,k}e^{-i\omega_k(t-t_0)}a_k(t_0) \equiv \sum_k \kappa_{n,k}a_k^{(in)}(t), \tag{16}$$

where

$$a_k^{(in)}(t)=a_k(t_0)\exp[-i\omega_k(t-t_0)]$$

is the annihilation operator for mode $\phi_k^{(l)}(x)$ of the input Fermi field at time $t$, and the total input field operator is therefore

$$\Psi^{(in)}(x,t)=\sum_k a_k^{(in)}(t)\phi_k^{(l)}(x).$$

That is, $\Psi^{(in)}(x,t)$ is the free Fermi field that propagates from the initial time $t_0$ to $t$ in the Heisenberg picture.

Since the initial reservoir operators obey the anticommutation relations $\{a_k(t_0),a_{k'}^\dagger(t_0)\}=\delta_{k,k'}$ and $\{a_k(t_0),a_{k'}(t_0)\}=0$, it is easy to show that the noise operators obey the anticommutation relations

$$\{F_n^{(in)}(t),F_m^{(in)\dagger}(t-\tau)\}=e^{-i\Omega_m\tau}\alpha_{n,m}(\tau), \tag{17}$$

$$\{F_n^{(in)}(t),F_m^{(in)}(t-\tau)\}=0. \tag{18}$$

Furthermore, if we restrict ourselves to time scales much longer than the correlation time $\tau_c$, then we can approximate the correlation function in Eq. (17) by a $\delta$ function times the area under $\alpha_{n,m}(\tau)$, so that

$$\{F_n^{(in)}(t),F_m^{(in)\dagger}(t-\tau)\}\approx 2\gamma_{n,m}\delta(\tau). \tag{19}$$

Before proceeding, there are several features of Eq. (15) that are worth mentioning. First, the dissipative term $\Sigma_m(\gamma_{n,m}+i\Delta_{n,m})c_m(t)$ gives a damping term plus an energy shift for $n=m$, while for $n\neq m$ there is a nonzero coupling between cavity states. The coupling between cavity states is a result of all states coupling to the same reservoir, which leads to an indirect coupling between cavity states. Second, the noise operators couple all of the reservoir states to each of the cavity states. Moreover, the noise operators are different for each cavity state due to the $n$ dependence of the coupling constants.

Instead of solving for the reservoir operators in terms of the initial time $t_0$, one can instead solve for $\hat{a}_k(t)$ in terms of a final time $t_1>t$,

$$\hat{a}_k(t)=\hat{a}_k(t_1)+\sum_m \kappa_{m,k}^*\int_t^{t_1}dt' e^{i(\omega_k-\Omega_m)t'}\hat{c}_m(t'). \tag{20}$$

The operators $\hat{a}_k(t_1)$ represent the modes of the output field that contain the field radiated by the cavity at earlier times. By substituting this expression into the equations of motion for $\hat{c}_n(t)$, making the Markov approximation in the integrand, and transforming back to the Heisenberg representation, one obtains

$$\dot{c}_n(t)=-i\Omega_n c_n(t)+\sum_m (\gamma_{n,m}-i\Delta_{n,m})c_m(t)+F_n^{(out)}(t). \tag{21}$$

Here, $F_n^{(out)}(t)$ is the *output field noise operator* for state $n$,

$$F_n^{(out)}(t)=\sum_k \kappa_{n,k}e^{-i\omega_k(t-t_1)}a_k(t_1)\equiv\sum_k \kappa_{n,k}a_k^{(out)}(t), \tag{22}$$

where we have defined the output field annihilation operator $a_k^{(out)}(t)$ for the mode $\phi_k^{(l)}(x)$. The $a_k^{(out)}(t)$ are related to the output field of the reservoir by

$$\Psi^{(out)}(x,t)=\sum_k a_k^{(out)}(t)\phi_k^{(l)}(x).$$

It is clear that $\Psi^{(out)}(x,t)$ represents the free Fermi field for the reservoir that propagates from $t$ to the final time $t_1$. It is easy to see that if the $a_k(t_1)$ obey normal fermionic anticommutation relations, then the anticommutators for the output noise operators are the same as Eqs. (17) and (18).

The boundary condition for the barrier separating the cavity from the reservoir is obtained by subtracting Eq. (15) from Eq. (21),

$$F_n^{(in)}(t)-F_n^{(out)}(t)=2\sum_m \gamma_{n,m}c_m(t). \tag{23}$$

Equation (23) relates the noise operator for the output field to the input noise operator reflected by the barrier and the field radiated by the cavity. It is of the same form as the boundary condition for an optical cavity [16] except that in our case there are separate noise operators for each cavity modes since we cannot, in general, assume that the coupling constants are independent of the cavity state. In Appendix A, we use Eqs. (23) and (15) to derive the anticommutation relations between the cavity operators and the noise operators at arbitrary times.

Equation (23) is not very useful since it is the mode operators of the output field, $a_k^{(out)}(t)$, which are needed to calculate properties of the output field such as mode occupation statistics, current density, etc. Therefore, we must extract from Eq. (23) a boundary condition for the annihilation operators of the modes of the input and output fields.

First, we note that in the limit of an infinite potential barrier the system-reservoir coupling vanishes, $\kappa_{n,k}\equiv 0$. In this limit, a fermion incident from the left is perfectly reflected by the barrier. Since $\Psi^{(in)}(x,t)$ and $\Psi^{(out)}(x,t)$ are the free Schrödinger fields that propagate forward in time from $t_0\rightarrow-\infty$ to $t$ and from $t$ to $t_1\rightarrow\infty$, respectively, it follows that

$$a_k(t_0)e^{i\omega_k t_0}=-a_{-k}(t_1)e^{i\omega_k t_1}. \tag{24}$$

The total field is the sum of incident and reflected fields,

$$\Psi(x,t)=\Psi^{(in)}(x,t)+\Psi^{(out)}(x,t). \tag{25}$$

It follows from Eqs. (24) and (25) that $\Psi(-a-d,t)=0$ and that the eigenmodes of the reservoir are standing waves for $V_0\rightarrow\infty$. Note that Eqs. (24) and (25) are second quantized versions of the relations for the incident and reflected wave functions from an infinite potential barrier [19].

For the reservoir, the displacement operator is given by

$$D(x)=e^{-iPx/\hbar},$$

where $P=\Sigma_k \hbar k a_k^\dagger a_k$ is the momentum operator for the reservoir [24]. Multiplying Eq. (23) by $D(x)$ on the left and $D^\dagger(x)$ on the right gives after some manipulation [25],

$$\sum_k \kappa_{n,k}(a_k^{(in)}(t)-a_{-k}^{(out)}(t))e^{ikx}=2\pi\sum_{m,k}\kappa_{n,k}\kappa_{m,k}^* \times\delta(\Omega_m-\omega_k)c_m(t)e^{ikx}. \tag{26}$$

Multiplying Eq. (26) by $\int_{-L-a-d}^{-a-d}dxe^{-ik'x}$ and taking the limit $L\to\infty$ gives the boundary condition for the modes of the input and output fields,

$$a_k^{(in)}(t)-a_{-k}^{(out)}(t)=2\pi\sum_m \kappa_{m,k}^*\delta(\Omega_m-\omega_k)c_m(t). \tag{27}$$

Physically, Eq. (27) says that the difference between the mode of the output field propagating away from the barrier with momentum $-\hbar k$ is the input field with momentum $\hbar k$ reflected by the barrier plus the field radiated by the cavity. Furthermore, only those states of the cavity that have the same energy as the reservoir mode can radiate into that mode. For a one-dimensional system, there will only be a single cavity mode that contributes to the right-hand side (rhs) of Eq. (27). The $\delta(\Omega_m-\omega_k)$ comes from Eq. (11) where we assumed times much longer than width of the reservoir correlation function. Hence, it follows from the uncertainty relation $\Delta E\Delta t\sim\hbar$ that the range of reservoir energies that couple to each cavity mode goes to zero as $t-t_0\to\infty$.

## IV. TWO-SIDED CAVITY

The generalization of the preceding to a two-sided cavity, $H_{SR},H_R\neq 0$, is straightforward since the reservoirs couple independently to the cavity. For the right reservoir, we define the input and output noise operators

$$G_n^{(in)}(t)=\sum_k \tilde{\kappa}_{n,k}b_k^{(in)}(t), \tag{28}$$

$$G_n^{(out)}(t)=\sum_k \tilde{\kappa}_{n,k}b_k^{(out)}(t), \tag{29}$$

where

$$b_k^{(in)}(t)=b_k(t_0)e^{-i\omega_k(t-t_0)}, \tag{30}$$

$$b_k^{(out)}(t)=b_k(t_1)e^{-i\omega_k(t-t_1)} \tag{31}$$

represent the input and output annihilation operators for the free field modes of the right reservoir at time $t$. Associated with the right reservoir noise operators are damping constants,

$$\tilde{\gamma}_{n,m}=\pi\sum_k \tilde{\kappa}_{n,k}\tilde{\kappa}_{m,k}^*\delta(\Omega_m-\omega_k), \tag{32}$$

and radiative energy shifts

$$\tilde{\Delta}_{n,m}=P\sum_k \tilde{\kappa}_{n,k}\tilde{\kappa}_{m,k}^*\frac{1}{\Omega_m-\omega_k}. \tag{33}$$

The quantum Langevin equations for the cavity mode operators expressed in terms of the input fields from the left and right are then

$$\dot{c}_n(t)=-i\Omega_n c_n(t)-\sum_m [(\gamma_{n,m}+i\Delta_{n,m})c_m(t)+(\tilde{\gamma}_{n,m}+i\tilde{\Delta}_{n,m})c_m(t)]+F_n^{(in)}(t)+G_n^{(in)}(t). \tag{34}$$

One may also derive Langevin equations analogous to Eq. (21) involving the output noise operators for the two reservoirs, which can be used along with Eq. (34) to derive the boundary conditions for the noise operators in the left and right reservoirs,

$$G_n^{(in)}(t)-G_n^{(out)}(t)=2\sum_m \tilde{\gamma}_{n,m}c_m(t), \tag{35}$$

$$F_n^{(in)}(t)-F_n^{(out)}(t)=2\sum_m \gamma_{n,m}c_m(t). \tag{36}$$

Using the boundary condition $b_k(t_0)e^{i\omega_k t_0}=-b_{-k}(t_1)e^{i\omega_k t_1}$ that corresponds to Eq. (24) for the free fields in the right reservoir, the boundary condition for $b_k^{(in)}(t)$ and $b_k^{(out)}(t)$ can be derived in the same manner as Eq. (27). One finds

$$b_k^{(in)}(t)-b_{-k}^{(out)}(t)=2\pi\sum_m \tilde{\kappa}_{m,k}^*\delta(\Omega_m-\omega_k)c_m(t), \tag{37}$$

$$a_k^{(in)}(t)-a_{-k}^{(out)}(t)=2\pi\sum_m \kappa_{m,k}^*\delta(\Omega_m-\omega_k)c_m(t). \tag{38}$$

Equations (34), (37), and (38) are the central result of this section. Given an initial state for the two reservoirs at $t_0$, Eq. (34) can be used to calculate the state of the cavity at some later time. The mode operators for the output field of the left and right reservoirs can then be determined from the boundary conditions (37) and (38).

## V. OUTPUT FIELD STATISTICS

We illustrate how to utilize these results for a particular initial state of the cavity plus reservoirs. Specifically, we assume that the atom cavity initially contains no atoms and that the right reservoir is likewise in the vacuum state. Furthermore, the left reservoir contains a ‘‘beam’’ of fermions incident on the barrier at $x=-a-d$ with a spatially uniform current density equal to $\rho\hbar q/m$, where $\rho=\mathcal{N}/L$ is the linear atomic density and $\mathcal{N}$ is the total number of fermions. This physical configuration is represented by the initial state vector

$$|\Psi(t_0)\rangle=\prod_{|k-q|\leq k_F} a_k^\dagger(t_0)|0\rangle, \tag{39}$$

where $k_F=2\pi\rho$ is the Fermi momentum. $|\Psi(t_0)\rangle$ represents a zero-temperature Fermi distribution that has been given a Galilean boost that displaces the gas by $q$ in $k$ space. This is analogous to the optical case where an incoherent white light source is used to drive an optical cavity.

Even though $|\Psi(t_0)\rangle$ is a state with a fixed number of atoms, it acts like a constant input flux of fermions on the cavity. This is because of the implicit use of the Born approximation in the derivation of the Langevin equations, i.e., the reservoir is assumed to be so large that the backaction of the system is negligible.

Using the results of the preceding section, we can calculate how the state of the left and right reservoirs are modified due to their coupling to the cavity. In particular, we focus on the occupation numbers

$$n_k^{(L)}(t)=a_k^{(out)\dagger}(t)a_k^{(out)}(t), \tag{40}$$

$$n_k^{(R)}(t)=b_k^{(out)\dagger}(t)b_k^{(out)}(t) \tag{41}$$

as well as on the current density operators for the reservoirs

$$J^{(L,R)}(x,t)=\sum_{\bar{q}} j_{\bar{q}}^{(L,R)}(t)e^{i\bar{q}x},$$

where

$$j_{\bar{q}}^{(L)}(t)=\frac{\hbar}{2mL}\sum_k (2k+\bar{q})a_k^{(out)\dagger}(t)a_{k+\bar{q}}^{(out)}(t), \tag{42}$$

$$j_{\bar{q}}^{(R)}(t)=\frac{\hbar}{2mL}\sum_k (2k+\bar{q})b_k^{(out)\dagger}(t)b_{k+\bar{q}}^{(out)}(t) \tag{43}$$

are the spatial Fourier components of the current. Note that $mL\langle j_0^{(L,R)}(t)\rangle$ is the average momentum in the output fields.

Equation (34) for $c_n(t)$ can be numerically integrated but we note that because of the exponential dependence of the reservoir-cavity coupling constants on the barrier height and thickness, the off-diagonal coupling is much smaller than the energy difference between the cavity modes,

$$|\Omega_n-\Omega_m|\gg|\gamma_{n,m}+\tilde{\gamma}_{n,m}|,|\Delta_{n,m}+\tilde{\Delta}_{n,m}|$$

for $n\neq m$. In fact, a numerical evaluation of $|\gamma_{n,m}+\tilde{\gamma}_{n,m}|$ and $|\Delta_{n,m}+\tilde{\Delta}_{n,m}|$ using the coupling constants of Appendix B indicates that these terms are at least two orders of magnitude smaller than the energies of the cavity modes. It is therefore an excellent approximation to neglect all off-diagonal terms in the equations of motion. Furthermore, we can perform a renormalization of the cavity mode energies by absorbing the radiative energy shifts into them, $\Omega_m+\Delta_{m,m}+\tilde{\Delta}_{m,m}\rightarrow\Omega_m$.

For times much longer than the lifetimes of the cavity modes, $(t-t_0)(\gamma_{n,n}+\tilde{\gamma}_{n,n})\gg 1$, the system reaches a steady state with the solution

$$c_m(t)=\sum_k \frac{\kappa_{m,k}a_k^{(in)}(t)+\tilde{\kappa}_{m,k}b_k^{(in)}(t)}{i(\Omega_m-\omega_k)+\Gamma_m}, \tag{44}$$

where $\Gamma_m=\gamma_{m,m}+\tilde{\gamma}_{m,m}$. It is easy to see from Eq. (44) that the occupation numbers for the cavity modes have a Lorentzian profiles,

$$\langle c_m^\dagger c_m\rangle=\sum_k \frac{|\kappa_{m,k}|^2\langle n_k(t_0)\rangle}{(\Omega_m-\omega_k)^2+\Gamma_m^2}, \tag{45}$$

where

$$\langle n_k(t_0)\rangle=\Theta(k_F-|k-q|)$$

are the occupation numbers for the input field. Due to the broadband nature of the input field all cavity modes with energies in the range $\omega_{k_F+q}-\omega_{k_F-q}$ will have a significant population with higher-energy cavity states having larger populations due to the $|\kappa_{m,k}|$'s exponential dependence on energy.

Using Eq. (44) and the boundary conditions (37) and (38), we obtain steady-state expectation values of the occupation of the output field modes,

$$\langle n_{-k}^{(L)}\rangle=\left(1-4\pi\sum_m \frac{\Gamma_m\delta(\Omega_m-\omega_k)|\kappa_{m,k}|^2}{(\Omega_m-\omega_k)^2+\Gamma_m^2}\right)\langle n_k(t_0)\rangle+4\pi^2\sum_{m,n,k'}\frac{|\kappa_{m,k}|^2|\kappa_{m,k'}|^2\delta(\Omega_m-\Omega_n)\delta(\Omega_m-\omega_k)}{(\Omega_m-\omega_{k'})^2+\Gamma_m^2}\langle n_{k'}(t_0)\rangle, \tag{46}$$

$$\langle n_k^{(R)}\rangle=4\pi^2\sum_{m,n,k'}\frac{|\tilde{\kappa}_{m,k}|^2|\kappa_{m,k'}|^2\delta(\Omega_m-\Omega_n)\delta(\Omega_m-\omega_k)}{(\Omega_m-\omega_{k'})^2+\Gamma_m^2}\langle n_{k'}(t_0)\rangle. \tag{47}$$

Equations (46) and (47) represent the changes in occupation numbers due to reflection and transmission through the cavity. The most significant feature of Eq. (46) is that fermions in the input beam are perfectly reflected by the barrier, $\langle n_{-k}^{(L)}\rangle=\langle n_k(t_0)\rangle$, unless there exists a cavity mode that is degenerate in energy with state $k$ of the reservoir. The second term represents the interference term between the fermions reflected by the barrier and the fermions that have tunneled into the barrier and subsequently tunneled back out into the left reservoir. The last term in Eq. (46) represents the tunneling of atoms into mode $-k$ as a result of atoms from mode $k'$ that have tunneled into the cavity and then tunneled out of the cavity.

In order to gain additional physical insight, we simplify these expressions by noting that the denominator in Eq. (47) as well as the last term in Eq. (46) are sharply peaked around $\omega_{k'}=\Omega_m=\omega_k$. Therefore we can replace $k'$ with $k$ in this term and drop the summation over $k'$. Using $\Gamma_m=2\gamma_{m,m}\approx 2\pi|\kappa_{m,k}|^2\delta(\Omega_m-\omega_k)$, we have

$$\langle n_{-k}^{(L)}\rangle \approx \langle n_k(t_0)\rangle - \mathcal{L}(\omega_k)(\langle n_k(t_0)\rangle - \langle n_{-k}(t_0)\rangle), \quad (48)$$

$$\langle n_k^{(R)}\rangle \approx \mathcal{L}(\omega_k)(\langle n_k(t_0)\rangle + \langle n_{-k}(t_0)\rangle), \quad (49)$$

where

$$\mathcal{L}(\omega_k)=\sum_m \frac{\Gamma_m^2}{(\Omega_m-\omega_k)^2+\Gamma_m^2}. \quad (50)$$

When $\langle n_{-k}(t_0)\rangle=0$, Eq. (48) indicates that $\langle n_{-k}^{(L)}\rangle\approx 0$, a result of the complete destructive interference between the fermions that are directly reflected by the barrier and the fermions that tunnel out of the cavity into the left reservoir. At the same time, Eq. (49) indicates that fermions resonant with a cavity mode tunnel through the right side with unit probability, $\langle n_{\pm k}^{(R)}\rangle \approx \langle n_k(t_0)\rangle$. These transmission resonances are similar to the situation in an optical Fabry-Perot cavity.

Figure 3 shows a plot of $\langle n_k^{(L)}\rangle$ and $\langle n_k^{(R)}\rangle$ using Eqs. (46) and (47) for an incident beam with fermions occupying the states $k=2\pi/L,\ldots,(2\pi/L)7501$. Note that we have taken the reservoir to consist of discrete $k$ states, $k=2\pi n/L$ with $n=0,\pm 1,\ldots,\pm n_{max}$ and $n_{max}=L\sqrt{2mV_0}/2\pi\hbar=10^4$. $V_0$ and $a$ were chosen so that the cavity contained 50 bound states. Each line in Fig. 3 corresponds to a *single* reservoir $k$ state and as such the width of the lines are greatly exaggerated. The plots give good qualitative agreement with the above discussion with each line located at the $k$ state that is closest in energy to a particular cavity state. The amplitude of the transmission resonances is close to 1 for the higher-energy $k$ states, while for the lowest-energy resonances, the amplitudes are about 0.25. The reduction in the amplitude for the low-energy states in comparison to Eqs. (48) and (49) comes from $\Gamma_m \equiv 2\pi|\kappa_{m,k}|^2\eta(\Omega_m) \gtrsim 2\pi|\kappa_{m,k}|^2\delta(\Omega_m-\omega_k)$, where $\eta(\omega)\sim\omega^{-1/2}$ is the continuum density of states in the reservoir [see Eq. (13)].

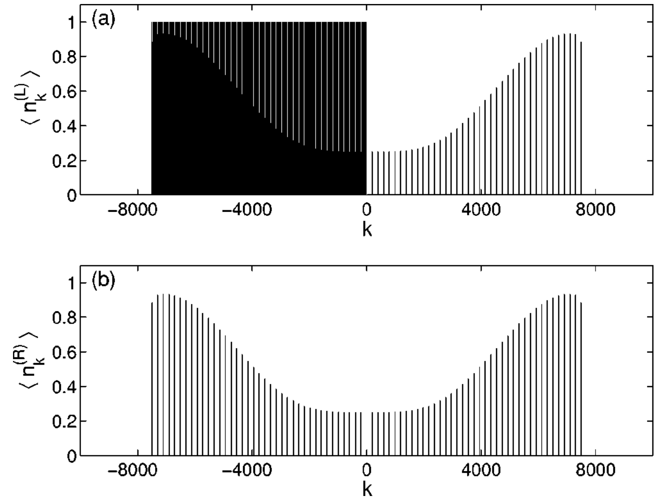


FIG. 3. Plot of (a) $\langle n_k^{(L)}(t)\rangle$ and (b) $\langle n_k^{(R)}(t)\rangle$ for $d/a=10^{-4}$ and $a/L=1.2\times 10^{-5}$. $k$ is in units of $2\pi/L$.

The steady-state current density in the output field of the left and right reservoirs are given by

$$\langle j_{-\bar{q}}^{(L)}\rangle = -\frac{\rho\hbar q}{m}\delta_{\bar{q},0} - \frac{\hbar}{2mL}\sum_k (2k+\bar{q})\Bigg[-2\pi\sum_m \kappa^*_{m,k+\bar{q}}\kappa_{m,k}\left(\frac{\delta(\Omega_m-\omega_k)\langle n_{k+\bar{q}}(t_0)\rangle}{-i(\Omega_m-\omega_{k+\bar{q}})+\Gamma_m}+\frac{\delta(\Omega_m-\omega_{k+\bar{q}})\langle n_k(t_0)\rangle}{i(\Omega_m-\omega_k)+\Gamma_m}\right)$$
$$+4\pi^2\sum_{m,n,k'}\frac{\kappa_{m,k}\kappa^*_{n,k+\bar{q}}\kappa^*_{m,k'}\kappa_{n,k'}\delta(\Omega_m-\omega_k)\delta(\Omega_n-\omega_{k+\bar{q}})}{[-i(\Omega_m-\omega_{k'})+\Gamma_m][i(\Omega_n-\omega_{k'})+\Gamma_n]}\langle n_{k'}(t_0)\rangle\Bigg] \quad (51)$$

and

$$\langle j_{\bar{q}}^{(R)}\rangle = \frac{\hbar}{2mL}\sum_k (2k+\bar{q})\left(4\pi^2\sum_{m,n,k'}\frac{\tilde{\kappa}_{m,k}\tilde{\kappa}^*_{n,k+\bar{q}}\kappa^*_{m,k'}\kappa_{n,k'}\delta(\Omega_m-\omega_k)\delta(\Omega_n-\omega_{k+\bar{q}})}{[-i(\Omega_m-\omega_{k'})+\Gamma_m][i(\Omega_n-\omega_{k'})+\Gamma_n]}\langle n_{k'}(t_0)\rangle\right), \quad (52)$$

respectively.

The first term on the rhs of Eq. (51) is the incident current reflected by the barrier. The average momenta in the output fields are proportional to $\langle j_0^{(L)}\rangle = -\hbar/mL\Sigma_k k\langle n_{-k}^{(L)}\rangle$ and $\langle j_0^{(R)}\rangle=\hbar/mL\Sigma_k k\langle n_k^{(R)}\rangle$. In the left reservoir, $\langle j_0^{(L)}\rangle\approx -\rho\hbar q/m$ since most of the fermions are perfectly reflected by the barrier. In the right reservoir, the transmitted current $\langle j_0^{(R)}\rangle\approx 0$ since for those states that are resonant with the cavity $\langle n_k^{(R)}\rangle=\langle n_{-k}^{(R)}\rangle$, while for nonresonant reservoir states $\langle n_k^{(R)}\rangle\approx 0$. Physically, this is due to the fact that atoms tunnel

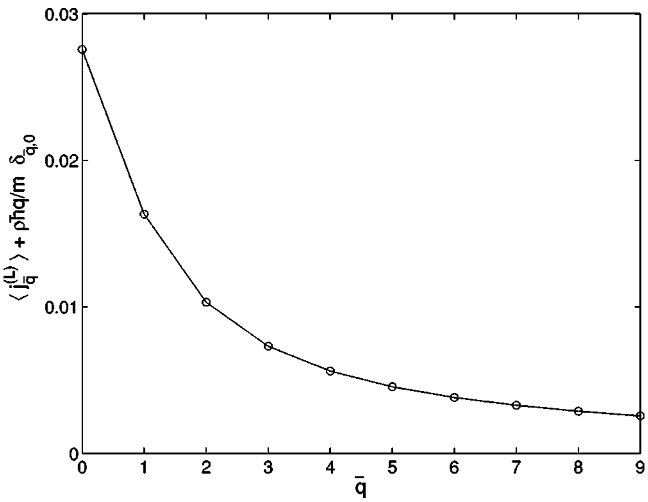


FIG. 4. Plot of $\langle j_{\bar{q}}^{(L)}\rangle$ for the same parameters as Fig. 3. $\bar{q}$ are in units of $2\pi/L$ and the current is in units of $\hbar/2ma^2$. $\rho\hbar q/m = 5.27$ in these units.

into the right reservoir from a standing-wave cavity mode, and therefore they have equal probability to tunnel into states with positive and negative momentum.

The $\bar{q}\neq 0$ terms are the spatial modulations in the current density that build up in the reservoirs as a result of the reservoir-cavity mode coupling. This can generate a coherence between the $k$ and $k+\bar{q}$ modes when there is a finite amplitude for an atom initially in state $k$ to tunnel into the cavity and then tunnel back out of the cavity into state $k+\bar{q}$. The change in $\langle j_{\bar{q}}^{(L)}\rangle$ is of order $\kappa^2$ rather than $\kappa^4$ as was the case for Eq. (46) since the current only involves generating a coherence between $k$ and $k+\bar{q}$ rather than the transfer of population. Furthermore, the $\kappa^2$ terms in Eq. (51) are only finite for $|\omega_{k+\bar{q}}-\omega_k|<\Gamma_m$, which implies that the coherence is only generated between reservoir states whose energies lie within the linewidth of a particular cavity mode. Consequently, decreasing the thickness and the height of the barriers will make the linewidths of the cavity states larger and thereby increase the magnitude of $\bar{q}\neq 0$ components of the current.

Figure 4 shows a plot of $\langle j_{\bar{q}}^{(L)}\rangle$ for several values of $\bar{q}$. The $\bar{q}\neq 0$ components of the current are about two orders of magnitude smaller than $\langle j_0^{(L)}\rangle$ and they decay away with increasing $\bar{q}$. We do not plot $\langle j_{\bar{q}}^{(R)}\rangle$ since the current is equal to 0 to within our numerical accuracy. This is because the cavity linewidths are so narrow that it becomes nearly impossible to satisfy both the $\delta$ functions in the numerator and the Lorentzian denominators of Eq. (52) for $\bar{q}\neq 0$ .

## VI. CONCLUSION

In this paper, we have extended the quantum optical input-output theory to atom cavities containing fermions. This formalism can easily be applied to intracavity nonlinear atom optical processes, such as four-wave mixing between fermions [13] or coherent photoassociation of fermions into molecular bosons [27].

In our work, we have dealt with a system-reservoir coupling that results from tunneling of atoms through a potential barrier, in which case the Markov approximation was shown to be justified. However, of equal importance to possible experiments is the situation in which atoms may be coupled into or out of the trap via induced Raman transitions to trapped states. In this case the Markov may break down [28]. In a future work, we plan to extend the treatment given here to broadband coupling in which the system may become non-Markovian.

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## APPENDIX A: ANTICOMMUTATION RELATIONS BETWEEN CAVITY AND NOISE OPERATORS

In this appendix, we examine some of the consequences of Eqs. (15) and (23). The correlations between the input and output fields and the cavity modes may be expressed in terms of the anticommutators for the cavity and noise operators, i.e., nonvanishing equal-time anticommutators imply that the operators are not independent. From Eq. (15), one sees that the solution for $c_n(t)$ can be expressed in terms of the initial conditions for the cavity operators, $c_n(t_0)$ and the $a_k(t_0)$. It follows immediately that $\{c_n(t),F_m^{(in)}(t')\}=0$ for all $t$ and $t'$ since the only nonvanishing anticommutators at $t_0$ are between creation and annihilation operators. In a similar manner, it immediately follows from Eq. (21) that $\{c_n(t),F_m^{(out)}(t')\}=0$. Formally integrating Eq. (15) from $t_0$ to $t$ shows that $c_n(t)$ depends on $F_m^{(in)}(t')$ for $t'<t$ [note that $c_n(t)$ will, in general, depend on all of the noise operators, $F_m^{(in)}(t')$, not just $n=m$, due to the intracavity mode coupling]. Using Eq. (19) it then follows that

$$\{c_n(t-\tau),F_m^{(in)\dagger}(t)\}=0 \tag{A1}$$

for $\tau>0$. This is nothing more than a statement of causality, the system can only depend on the past input. In a similar manner, formal integration of Eq. (21) from $t+\tau$ to $t_1$ implies that

$$\{c_n(t+\tau),F_m^{(out)\dagger}(t)\}=0 \tag{A2}$$

for $\tau>0$. Again, this is nothing more than the statement that the output of the system can only depend on the state of the system in the past. Using Eqs. (A2) and (23), one obtains

$$\{c_n(t+\tau),F_m^{(in)\dagger}(t)\}=2\sum_p \gamma_{m,p}^{*}\{c_n(t+\tau),c_p^{\dagger}(t)\},$$

while using Eq. (A1) along with Eq. (23) gives

$$\{c_n(t-\tau),F_m^{(out)\dagger}(t)\}=-2\sum_p \gamma_{m,p}^*\{c_n(t-\tau),c_p^\dagger(t)\}$$

for $\tau>0$. It follows from Eq. (23) that the equal-time anticommutators can be written as $\{c_n(t),F_m^{(in/out)\dagger}(t)\}=\pm\Sigma_p\gamma_{m,p}^*\{c_n(t),c_p^\dagger(t)\}+A_{n,m}$. The operator $A_{n,m}$ is determined from the equations of motion (15) and (21) to be $-i\Sigma_p\Delta_{m,p}^*\{c_n(t),c_p^\dagger(t)\}$. If we define the step function $u(\tau)$ as

$$u(\tau)=\begin{cases}1, & \tau>0,\\ 1/2, & \tau=0,\\ 0, & \tau<0,\end{cases}$$

then the anticommutators for arbitrary $\tau$ are given by

$$\{c_n(t+\tau),F_m^{(in)\dagger}(t)\}=2u(\tau)\sum_p(\gamma_{m,p}^*-i\Delta_{m,p}^*\delta_{\tau,0})\{c_n(t+\tau),c_p^\dagger(t)\}, \qquad (A3)$$

$$\{c_n(t+\tau),F_m^{(out)\dagger}(t)\}=-2u(-\tau)\sum_p(\gamma_{m,p}^*+i\Delta_{m,p}^*\delta_{\tau,0})\{c_n(t+\tau),c_p^\dagger(t)\}. \qquad (A4)$$

Finally, we show that Eq. (15) preserve the equal-time anticommutators for the system operators. Since Eq. (15) constitute an $N\times N$ system of equations, an explicit solution will be nontrivial for $N>2$. However, it is already apparent that $\{c_n(t),c_m(t)\}=0$ since, as we stated before, $c_n(t)$ can be expressed in terms of the initial states for the operators at $t_0$ and $\{c_n(t_0),c_m(t_0)\}=\{c_n(t_0),a_k(t_0)\}=0$. We are therefore left with showing that $\{c_n(t),c_m^\dagger(t)\}=\delta_{n,m}$. Calculating the derivative of the anticommutator using Eqs. (15) and (A3), one obtains

$$\frac{d}{dt}\{c_n(t),c_m^\dagger(t)\}=-i(\Omega_n-\Omega_m)\{c_n(t),c_m^\dagger(t)\}.$$

Integrating this expression and using the initial condition $\{c_n(t_0),c_m^\dagger(t_0)\}=\delta_{n,m}$, we obtain the desired result.

## APPENDIX B: TUNNELING COUPLING CONSTANTS

In this appendix, we derive explicit expressions for the tunneling coupling constants $\kappa_{n,k}$ and $\tilde{\kappa}_{n,k}$. Tunneling in many-body systems was first treated by Bardeen [22] and later elaborated by Prange [21] and Harrison [26] in the context of electron tunneling across the insulator junctions. Bardeen showed that if there exists a potential barrier separating two regions of space, then the tunneling matrix element $T_{a,b}$ for the tunneling of a particle from state $\phi_b(x)$ on one side of the barrier into the state $\phi_a(x)$ on the opposite side of the barrier is given by the off-diagonal current density in the barrier,

$$T_{a,b}=\frac{-\hbar^2}{2m}\left[\phi_a^*(x)\frac{d}{dx}\phi_b(x)-\phi_b(x)\frac{d}{dx}\phi_a^*(x)\right]_{x=x_1}. \qquad (B1)$$

Here $x_1$ is a point inside the barrier and $\phi_a(x)$ and $\phi_b(x)$ are the continuations of the single-particle wave functions into the barrier where they decay exponentially. $T_{a,b}$ is independent of the choice of $x_1$ provided the energy difference between the two states $\phi_a$ and $\phi_b$ is much less than the height of the barrier. The relationship between Eq. (B1) and the overlap of the Hamiltonian between states localized on either side of the barrier is discussed in Ref. [21].

For the system illustrated in Fig. 1, we identify $-i\hbar\kappa_{n,k}^*$ with $T_{a,b}$ when $\phi_a(x)=\phi_k^{(l)}(x)$ and $\phi_b(x)=\phi_n^{(s)}(x)$. Similarly, $-i\hbar\tilde{\kappa}_{n,k}^*$ is equal to $T_{a,b}$ when $\phi_a(x)=\phi_k^{(r)}(x)$ and $\phi_b(x)=\phi_n^{(s)}(x)$.

For the purpose of calculating the coupling constants, we take the $\phi_n^{(s)}(x)$ to be the eigenstates of the finite potential well corresponding to $d\rightarrow\infty$. In the classically allowed region, $-a\leqslant x\leqslant a$, $\phi_n^{(s)}(x)$ is proportional to $\cos(K_nx)$ for even $n$ and $\sin(K_nx)$ for odd $n$, while inside the barrier $\phi_n^{(s)}(x)\sim\exp(-\sqrt{2m(V_0-\hbar\Omega_n)/\hbar^2}|x|)$. The cavity wave numbers $K_n$ are determined by the solutions to

$$K_na\tan K_na=\sqrt{\beta^2-(K_na)^2} \qquad (B2)$$

for even $n$, and

$$K_na\cot K_na=-\sqrt{\beta^2-(K_na)^2} \qquad (B3)$$

for $n$ odd [19].

For even $n$, we find

$$\kappa_{n,k}=\frac{-i\hbar\sigma}{m}\sqrt{\frac{2K_n}{L(1+(\sigma/k)^2)(K_na+\cot K_na)}}e^{-\sigma d}\cos K_na \qquad (B4)$$

with

$$\tilde{\kappa}_{n,k}=\kappa_{n,k}. \qquad (B5)$$

In a similar manner, we find for odd $n$,

$$\kappa_{n,k}=\frac{i\hbar\sigma}{m}\sqrt{\frac{2K_n}{L(1+(\sigma/k)^2)(K_na-\tan K_na)}}e^{-\sigma d}\sin K_na \qquad (B6)$$

with

$$\tilde{\kappa}_{n,k}=-\kappa_{n,k}. \qquad (B7)$$

In both cases, $\sigma=\sqrt{2mV_0/\hbar^2-k^2}$ is the inverse of the penetration depth of the reservoir state into the barrier and $\beta^2=2ma^2V_0/\hbar^2$ is the dimensionless barrier height.

An interesting consequence of Eqs. (B5) and (B7) is that if $n$ is even and $m$ is odd or vice versa, then

$$(\gamma_{n,m}+i\Delta_{n,m})+(\tilde{\gamma}_{n,m}+i\tilde{\Delta}_{n,m})\equiv 0. \tag{B8}$$

On the other hand, if $n$ and $m$ are both even or both odd, then

$$(\gamma_{n,m}+i\Delta_{n,m})+(\tilde{\gamma}_{n,m}+i\tilde{\Delta}_{n,m})=2(\gamma_{n,m}+i\Delta_{n,m}). \tag{B9}$$

Since $\phi_n^{(s)}(x)$ is an eigenstate of the parity operator with parity $(-1)^n$, it follows from Eq. (34) that for a two-sided cavity only states of the same parity are coupled. This is a direct consequence of the our model system in Fig. 1 being invariant under spatial reflections, which implies that even when the coupling of the cavity states to the reservoirs is taken into account, parity will still be a good quantum number for the cavity states. No such result holds for the single-sided cavity since the cavity-plus-reservoir system is no longer invariant with respect to reflections.

Finally, we note that the approximation $\omega_k\approx\Omega_n$ used in deriving $\kappa_{n,k}$ becomes exact for the evaluation of the $\gamma_{n,n}$ and for the cavity boundary conditions (27) and (37), since in these expressions $\kappa_{n,k}$ is always multiplied by $\delta(\Omega_n-\omega_k)$.


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