# Collective oscillations of a one-dimensional trapped Bose-Einstein gas

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Starting from the hydrodynamic equations of superfluids, we calculate the frequencies of the collective oscillations of a harmonically trapped Bose-Einstein gas for various one-dimensional configurations at zero temperature. These include the mean-field regime described by Gross-Pitaevskii theory and the beyond-mean-field regime at small densities described by the Lieb-Liniger theory. The relevant combinations of the physical parameters governing the transition between the different regimes as well as the conditions of applicability of the hydrodynamic equations are discussed.

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## I. INTRODUCTION

Recent experiments on trapped Bose-Einstein gases at low temperature have pointed out the occurrence of characteristic one-dimensional (1D) features. These include deviations of the aspect ratio and of the release energy [1,2] from the 3D behavior as well as the appearence of thermal fluctuations of the phase, peculiar of 1D configurations [3]. Interest in 1D interacting Bose gases arises from the occurrence of quantum features which are not encountered in 2D and 3D. For example, in 1D the fluctuations of the phase of the order parameter rule out the occurrence of long-range order even at zero temperature [4]. Such systems cannot be, in general, described using traditional mean-field theories and require the development of a more advanced many-body approach. In the case of 1D Bose gases interacting with repulsive zero-range forces, this has been implemented by Lieb and Liniger [5] who studied both the equation of state and the spectrum of elementary excitations of a uniform gas. In the presence of harmonic trapping, 1D Bose gases exhibit interesting features. The corresponding equilibrium properties have been already discussed in a recent series of theoretical papers (see Refs. [6–8] and references therein). In the present work we investigate the consequences of harmonic trapping on the collective oscillations of an interacting 1D Bose gas at zero temperature. We will consider various configurations, ranging from the mean-field regime [9], where the healing length is larger than the average interparticle distance, to the Tonks-Girardeau limit [10] of an impenetrable gas of bosons where the system acquires Fermi-like properties. We will show that the frequency of the lowest compression mode provides a useful indicator of the different regimes.

We start our discussion from the hydrodynamic equations of superfluids in 1D,

$$\frac{\partial}{\partial t} n_1 + \Delta_z (n_1 v) = 0 \tag{1}$$

$$m\frac{\partial}{\partial t} v + \nabla_z \left( \mu_{\ell e}(n_1) + V_{ext} + \frac{1}{2} m v^2 \right) = 0, \tag{2}$$

which decribe the dynamic behavior of such systems at zero temperature. In these equations, $n_1(z,t)$ is the 1D density of the gas, $v(z,t)$ is the velocity field, while $V_{ext}(z)$ is the external trapping potential, which in the following will be assumed to be harmonic: $V_{ext}(z) = m\omega_z^2 z^2/2$. The hydrodynamic approach has been already successfully employed to predict the collective frequencies of 3D trapped Bose-Einstein condensates [11]. Its applicability is not, however, limited to the mean-field scenario. Actually in Ref. [5] it has been proven that in 1D Bose gases the velocity of sound, derived from the macroscopic compressibility, coincides with that derived from the microscopic calculation of the phonon excitation spectrum, confirming that Eqs. (1) and (2) are well suited to describe the collective oscillations also in these systems. A crucial ingredient of these equations is the local equilibrium ($\ell e$) chemical potential $\mu_{\ell e}$, which should be evaluated for a uniform 1D gas ($V_{ext}=0$) at the density $n_1$. Their applicability requires the validity of the local-density approximation along the $z$ direction. This is expected to be accurate for sufficiently large systems. Furthermore, Eqs. (1) and (2) should be limited to the study of macroscopic phenomena where variations in space take place over distances larger than the average distance between particles. We are also assuming that the motion in the radial direction is ''frozen.'' This corresponds to investigating the low-energy motions taking place along the $z$ direction and whose frequencies are much smaller than the radial trapping frequencies. From Eq. (2) one can easily calculate the ground-state profile through the equation

$$\mu_{\ell e}(n_1(z)) + V_{ext}(z) = \mu. \tag{3}$$

The collective oscillations are instead determined by writing the density in the form $n_1(z,t) = n_1(z) + e^{-i\omega t}\delta n_1(z)$, with the function $\delta n_1(z)$ obeying the linearized equation

$$\omega^2 \delta n_1(z) = \frac{1}{m}\nabla_z \left[ n_1(z) \nabla_z \left( \frac{\partial \mu_{\ell e}}{\partial n_1} \delta n_1(z) \right) \right], \tag{4}$$

which immediately follows from Eqs. (1) and (2).

In Sec. II, we evaluate the equation of state $\mu_{\ell e}(n_1)$ in the framework of the mean-field Gross-Pitaevskii theory. One can explore a rich variety of situations ranging from the

 

Thomas-Fermi regime in the radial direction to that of tight confinement where the motion in the radial direction is frozen [12]. In Sec. III we use the Lieb-Liniger theory to extend the analysis to regimes beyond mean field, including the limit of the Tonks-Girardeau gas. In both cases, we study the situation where the gas is trapped also along the $z$ direction. Then the density $n_1$ exhibits a $z$ dependence which is worth calculating as a function of the relevant parameters of the problem: the scattering length $a$, the number $N$ of atoms, and the radial and axial trapping frequencies $\omega_\perp$ and $\omega_z$, which are always assumed to satisfy the condition $\lambda \equiv \omega_z/\omega_\perp \ll 1$.

In Sec. IV we calculate the collective frequencies predicted by the hydrodynamic theory developing a sum-rule approach and explore, in particular, the behavior of the lowest compressional mode. Finally in Sec. V we summarize the main assumptions which are required in order to apply the hydrodynamic approach developed in the paper and draw our final conclusions.

## II. FROM THE 3D CIGAR TO THE 1D MEAN-FIELD REGIME

Let us consider a uniform system of length $L$ in the $z$ direction and confined by a harmonic potential $V(r_\perp) = m\omega_\perp^2 r_\perp^2/2$ in the radial direction. By writing the order parameter in the form $\Psi = \sqrt{n_1} f(\rho_\perp)/a_\perp$, where $n_1 = N/L$ is the 1D density, $a_\perp = \sqrt{\hbar/m\omega_\perp}$ is the oscillator length in the radial direction, and $\rho_\perp = r_\perp/a_\perp$ is the dimensionless radial coordinate, the 3D Gross-Pitaevskii equation takes the dimensionless form

$$\left( -\frac{1}{2}\frac{\partial^2}{\partial \rho_\perp^2} - \frac{1}{2\rho_\perp}\frac{\partial}{\partial \rho_\perp} + \frac{1}{2}\rho_\perp^2 + 4\pi a n_1 f^2 \right) f = \frac{\mu_{\ell e}}{\hbar\omega_\perp} f. \tag{5}$$

The function $f$ obeys the normalization condition $2\pi \int |f(\rho_\perp)|^2 \rho_\perp d\rho_\perp = 1$. In Eq. (5), $\mu_{\ell e}/\hbar\omega_\perp$ is the chemical potential in units of the radial quantum oscillator energy. Equation (5) shows that the relevant dimensionless parameter of the problem is $an_1$. It is worth considering two important limits. If $an_1 \gg 1$, one enters the radial Thomas-Fermi regime, where many configurations of the harmonic-oscillator Hamiltonian are excited in the radial direction and the equation of state takes the analytic form

$$\frac{\mu_{\ell e}}{\hbar\omega_\perp} = 2(an_1)^{1/2}. \tag{6}$$

Notice that in this limit the chemical potential is not linear in the density. This implies, in particular, that the sound velocity is related to the chemical potential by the law $c^2 = \mu_{\ell e}/2m$ [13] rather than by the Bogoliubov relation $c^2 = \mu_{\ell e}/m$. A second important case is the perturbative regime where $an_1 \ll 1$ (hereafter called the 1D mean field). In this case the solution of Eq. (5) approaches the Gaussian ground state of the radial harmonic oscillator and one finds the linear law

$$\frac{\mu_{\ell e}}{\hbar\omega_\perp} = 1 + 2an_1 \tag{7}$$

for the chemical potential.

Let us now add a harmonic confinement along the axial direction. In this case one has to solve Eq. (3) by imposing the normalization condition $\int n_1(z)dz = N$ to the 1D density. A useful quantity is the Thomas-Fermi radius $Z$ defined by the value of $z$ at which the equilibrium density $n_1(z)$ vanishes. According to Eq. (3), one has $\mu - \mu_{\ell e}(an_1 = 0) = (1/2)m\omega_z^2 Z^2$. In terms of $Z$, Eq. (3) can be rewritten as $\tilde{\mu}_{\ell e}(an_1(z)) = (m\omega_z^2 Z^2/2\hbar\omega_\perp)(1 - z^2/Z^2)$, where we have defined the dimensionless quantity $\tilde{\mu}_{\ell e}(an_1(z)) = [\mu_{\ell e}(an_1(z)) - \mu_{\ell e}(an_1 = 0)]/\hbar\omega_\perp$. This function is fixed by the solution of the Gross-Pitaevskii equation (5) [14]. Its inverse $\tilde{\mu}_{\ell e}^{-1}$ gives the value of $an_1$ as a function of $z$, and the normalization condition obeyed by the density can be written as

$$\sqrt{\lambda}\frac{Z}{a_z}\int_{-1}^{1} \tilde{\mu}_{\ell e}^{-1}\left[\frac{1}{2}\left(\sqrt{\lambda}\frac{Z}{a_z}\right)^2 (1-t^2)\right] dt = N\lambda\frac{a}{a_\perp}, \tag{8}$$

where $t = z/Z$, $a_z = \sqrt{\hbar/m\omega_z}$ is the oscillator length in the axial direction and $\lambda = \omega_z/\omega_\perp$ is the aspect ratio of the trap. Equation (8) explicitly points out the relevance of the dimensionless combination

$$N\lambda\frac{a}{a_\perp} = N\frac{aa_\perp}{a_z^2}. \tag{9}$$

From Eq. (8) one can calculate, for a given choice of the parameters, the radius $Z$ and hence the 1D density profile.

The radial Thomas-Fermi regime, hereafter called 3D cigar due to the elongated shape of the cloud, corresponds to $N\lambda(a/a_\perp) \gg 1$. In this case one has

$$Z = \frac{a_z}{\sqrt{\lambda}}\left(15N\lambda\frac{a}{a_\perp}\right)^{1/5} \tag{10}$$

and

$$n_1(z) = \frac{1}{16a}\left(15N\lambda\frac{a}{a_\perp}\right)^{4/5}\left(1 - \frac{z^2}{Z^2}\right)^2. \tag{11}$$

The 1D mean-field limit is instead reached if $N\lambda(a/a_\perp) \ll 1$, where one finds [7]

$$Z = \frac{a_z}{\sqrt{\lambda}}\left(3N\lambda\frac{a}{a_\perp}\right)^{1/3} \tag{12}$$

and

$$n_1(z) = \frac{1}{4a}\left(3N\lambda\frac{a}{a_\perp}\right)^{2/3}\left(1 - \frac{z^2}{Z^2}\right). \tag{13}$$

The density profiles are different in the two regimes, reflecting the different behavior of the equation of state. The con-

ditions of applicability of the local-density approximation employed above are determined by requiring that $Z \gg a_z$. In the 1D mean-field regime this implies the nontrivial condition (see also Refs. [6,7])

$$\left( \frac{N}{\sqrt{\lambda}} \frac{a}{a_\perp} \right)^{1/3} \gg 1. \tag{14}$$

### III. FROM THE 1D MEAN-FIELD TO THE TONKS-GIRARDEAU REGIME

Deviations from the mean-field regime become important when the healing length $\xi = (8\pi a n)^{-1/2}$ is comparable to the average distance $d$ between particles. In the presence of tight radial confinement one can use the relationship $n = n_1/\pi a_\perp^2$ between the 3D density evaluated at $r_\perp = 0$ and the 1D density $n_1 = \int n(r_\perp) d\vec{r}_\perp$. When $a_\perp$ becomes smaller than $d$, one can write $d = 1/n_1$. One then obtains the result $\xi/d = \sqrt{a_\perp^2 n_1/8a}$, which becomes smaller and smaller as the 1D density decreases, thereby suggesting the occurrence of important deviations from the mean-field behavior for very dilute 1D samples. This should be contrasted with the 3D case, where the mean-field condition ($\xi > d$) is better and better satisfied as the density decreases. The combination $a_\perp^2 n_1/a$ can then be used as an indicator of the applicability of the mean-field approach. When its value becomes of the order of 1 or smaller, one enters a new regime characterized by important quantum correlations. The corresponding many-body problem was investigated by Lieb and Liniger [5], who considered 1D repulsive zero-range potentials of the form $g_{1D}\delta(z)$. The coupling constant $g_{1D}$ is conveniently written in the form $g_{1D} = 2\hbar^2/(m a_{1D})$, where $a_{1D}$ is the relevant interaction length of the problem [21]. Under suitable conditions, the 1D interaction parameter $g_{1D}$ can be related to the 3D coupling constant characterizing the effective interaction $4\pi\hbar^2 a\,\delta(\mathbf{r})/m$ of the Gross-Pitaevskii theory [21]. This relation is particularly simple if the 3D scattering length $a$ is much smaller than the radial confinement, fixed by the radial oscillator length $a_\perp$. In this case, by averaging the 3D force over the radial density profile, one finds [6]

$$g_{1D} = \frac{2\hbar^2}{m} \frac{a}{a_\perp^2} \quad \text{and} \quad a_{1D} = \frac{a_\perp^2}{a}. \tag{15}$$

The comparison with the expression for $\xi/d$ derived above shows that the deviations from the mean field increase by decreasing $a_{1D} n_1$.

In the Lieb-Liniger theory, the energy per particle can be written as

$$\epsilon(n_1) = \frac{\hbar^2}{2m} n^2 e(\gamma(n_1)) \tag{16}$$

and can be obtained by solving the system of equations

$$g_\lambda(x) = \frac{1}{2\pi} + \frac{1}{\pi} \int_{-1}^{1} \frac{\lambda}{\lambda^2 + (y-x)^2} g_\lambda(y) dy, \tag{17}$$

$$\gamma(\lambda) = \lambda \left( \int_{-1}^{1} g_\lambda(x) dx \right)^{-1}, \tag{18}$$

$$e(\gamma) = \frac{\gamma^3}{\lambda^3} \int_{-1}^{1} g_\lambda(x) x^2 dx, \tag{19}$$

where $\gamma = 2/a_{1D} n_1$. The chemical potential can then calculated using the thermodynamic relation

$$\mu_{\ell e} = \frac{\partial [n_1 \epsilon(n_1)]}{\partial n_1}. \tag{20}$$

The above equations show that in the Lieb-Liniger scenario, the energy per particle $\epsilon$ and the chemical potential $\mu_{le}$, when expressed in units of the energy $\hbar^2/2m a_{1D}^2$, are universal functions of the dimensionless parameter $a_{1D} n_1$ [14].

An important limit is the high-density regime $a_{1D} n_1 \gg 1$, where one has $\epsilon(n_1) = \hbar^2 n_1/m a_{1D}$. Using Eq. (15), one finds that this coincides with the 1D mean-field result (7) for the chemical potential (a part from the constant term arising from the radial external force). Note that only if $a_\perp \gg a$, will the condition $a_{1D} n_1 \gg 1$, required to realize the mean-field regime in the framework of the Lieb-Liniger theory, be compatible with the condition $a n_1 \ll 1$ needed to rule out 3D effects (see Sec. II) [22].

The other important limit is the low-density Tonks-Girardeau limit $a_{1D} n_1 \ll 1$, where the chemical potential takes the value

$$\mu_{\ell e} = \pi^2 \hbar^2 n_1^2/2m. \tag{21}$$

Here the chemical potential no longer depends on the interaction coupling constant and reveals a typical Fermi-like behavior [10].

In the presence of axial harmonic trapping, the ground state density profile has been evaluated in Ref. [7] using the local-density approximation (3). In this case the normalization condition $\int n_1(z) dz = N$ takes the form

$$\frac{Z a_{1D}}{a_z^2} \int_{-1}^{1} \tilde{\mu}_{\ell e}^{-1} \left[ \left( \frac{Z a_{1D}}{a_z^2} \right)^2 (1 - t^2) \right] dt = \frac{N a_{1D}^2}{a_z^2}. \tag{22}$$

Similarly to Eq. (8), we have introduced the radius $Z$ at which the density vanishes and the inverse of the function $\tilde{\mu}_{\ell e}(n_1 a_{1D})$, where $\tilde{\mu}_{\ell e}$ is now the chemical potential expressed in units of $\hbar^2/2m a_{1D}^2$. Equation (22) shows that the relevant combination of parameters to describe the transition between the mean-field and Tonks-Girardeau regime is given by $N a_{1D}^2/a_z^2$, as already pointed out in Refs. [6,7]. Using Eq. (15), one finds

$$N \frac{a_{1D}^2}{a_z^2} = N\lambda \frac{a_\perp^2}{a^2}, \tag{23}$$

which differs by the factor $(a_\perp/a)^3$ from the combination (9) characterizing the transition between the mean-field regimes discussed in Sec. II.

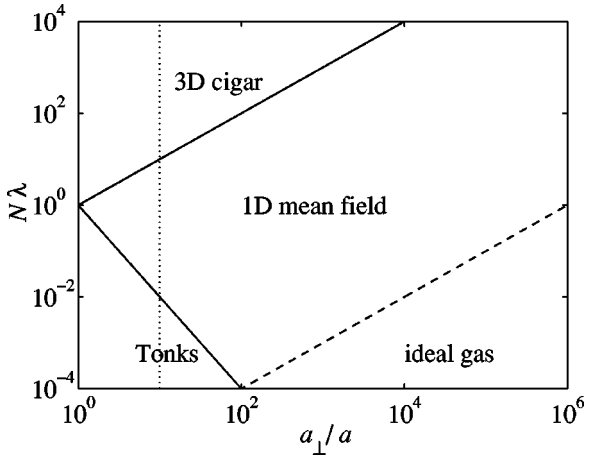


FIG. 1. Phase diagram in the plane $N\lambda$ vs $a_\perp/a$; the dashed line indicates $N\lambda=\lambda^{3/2}a_\perp/a$ for $\lambda=10^{-4}$.

Analytic solutions are obtained in the two limits $N\lambda a_\perp^2/a^2 \gg 1$ and $N\lambda a_\perp^2/a^2 \ll 1$. In the first case, one recovers the 1D mean-field result (13). In the second one, one finds the profile [8,20]

$$n_1(z)=\frac{\sqrt{2N}}{\pi a_z}\left(1-\frac{z^2}{Z^2}\right)^{1/2}, \tag{24}$$

with $Z=\sqrt{2N}a_z$. In this case (Tonks-Girardeau regime) the applicability of the local-density approximation simply requires $N\gg 1$.

The transition between the different regimes (3D cigar, 1D mean-field, and Tonks-Girardeau regime) is schematically illustrated in Fig. 1, where we describe the phase diagram in the plane $N\lambda$ vs $a_\perp/a$. According to the results presented in Secs. II and III, the line $N\lambda=a_\perp/a$ separates the 3D cigar from the 1D mean-field regimes, while the line $N\lambda=(a/a_\perp)^2$ separates the 1D mean-field regime from the Tonks-Girardeau regime. The 3D and the Tonks-Girardeau regions move more and more far apart when $a_\perp/a$ becomes larger and larger, leaving place for the 1D mean-field configuration.

In Fig. 1 we also plot the line $N\lambda=\lambda^{3/2}a_\perp/a$ for the particular choice $\lambda=10^{-4}$. According to Eq. (14), below this line the system tends to behave like an ideal gas. Notice that the Tonks-Girardeau regime always requires that the condition

$$\frac{a_\perp}{a} \ll \frac{1}{\sqrt{\lambda}} \tag{25}$$

be satisfied. At the same time, this condition rules out the reachability of the ideal gas.

## IV. COLLECTIVE OSCILLATIONS

Let us now discuss the behavior of the collective oscillations. The hydrodynamic equation (4) has simple analytic solutions if the density derivative of the chemical potential is a power-law function: $\partial\mu_{\ell e}/\partial n_1 \propto n_1^{\gamma-1}$. The Thomas-Fermi, the 1D mean-field, and the Tonks-Girardeau regimes belong to this class of solutions with $\gamma=1/2$, $\gamma=1$, and $\gamma=2$, respectively. Hence in these three relevant limits the dispersion relation for the collective frequencies can be obtained analytically. By looking for solutions of the form $n_1^{\gamma-1}\delta n_1(z)=z^k+az^{k-2}\ldots$, where $k\geqslant 1$ and only positive powers of $z$ are included in the polynomial, Eq. (4) yields the result

$$\omega^2=\omega_z^2\frac{k}{2}[2+\gamma(k-1)]. \tag{26}$$

The hydrodynamic solutions $\delta n_1(z)$ exhibit a nonanalytic behavior at the Thomas-Fermi radius. This reflects the drawback of the theory near the classical boundary, which does not, however, affect the correctness of the dispersion relation (26). The case $k=1$ corresponds to the center-of-mass motion whose frequency is given by $\omega=\omega_z$ independent of the value of $\gamma$. The most interesting $k=2$ case (lowest compressional mode) is instead sensitive to the regime considered. One finds $\omega^2=(5/2)\omega_z^2$, $\omega^2=3\omega_z^2$, and $\omega^2=4\omega_z^2$ for the 3D cigar, 1D mean-field, and Tonks-Girardeau regimes, respectively. The result $\omega^2=(5/2)\omega_z^2$ was first derived in Ref. [11] by solving the hydrodynamic equations for a trapped 3D system in the limit of a highly elongated trap ($\omega_z \ll \omega_\perp$). This prediction has been confirmed experimentally with high precision [15]. The result $\omega^2=3\omega_z^2$ was derived in Refs. [16–18], while the result $\omega^2=4\omega_z^2$ for the Tonks-Girardeau gas is simply understood by recalling that, in this limit, there is an exact mapping with the 1D ideal Fermi gas [10] where the excitation spectrum, in the presence of harmonic confinement, is $\omega=k\omega_z$. The same result has been recently derived in Ref. [19] using the mean-field equations of Kolomeisky *et al.* [20].

In order to evaluate the collective frequencies in the intermediate regimes where the hydrodynamic equations are not analytically soluble, we have developed a sum-rule approach [11], based on the evaluation of the ratio

$$\hbar^2\omega^2=\frac{m_1}{m_{-1}} \tag{27}$$

between the energy weighted and inverse energy weighted sum rules. In the following we will limit the discussion to the lowest compression mode which is naturally excited by the operator $\Sigma_{i=1}^N z_i^2$. The energy weighted moment is given by $m_1=(1/2)\langle[\Sigma_{i=1}^N z_i^2,[H,\Sigma_{i=1}^N z_i^2]]\rangle=(2N\hbar^2/m)\langle z^2\rangle$, where $\langle z^2\rangle=\int n_1(z)z^2dz/N$ is the average square radius fixed by the ground-state solution $n_1(z)$. The inverse energy weighted moment $m_{-1}$ is related to the static polarizability $\alpha$ by $m_{-1}=(1/2)\alpha$. This is evaluated by adding the pertubation $-\epsilon z^2$ to the Hamiltonian and calculating the corresponding changes $\delta\langle z^2\rangle$ of the expectation value of the square radius: $\alpha=N\delta\langle z^2\rangle/\epsilon$. Adding the perturbation $-\epsilon z^2$ is equivalent to changing the frequency $\omega_z$ of the harmonic confinement, so that the result for the collective frequency takes the compact form

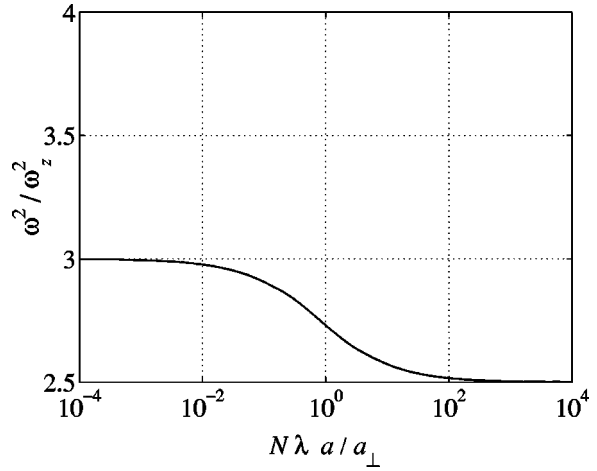


FIG. 2. Transition between the 1D mean-field and the 3D cigar regimes: $\omega^2/\omega_z^2$ as a function of the parameter $N\lambda a/a_\perp$.

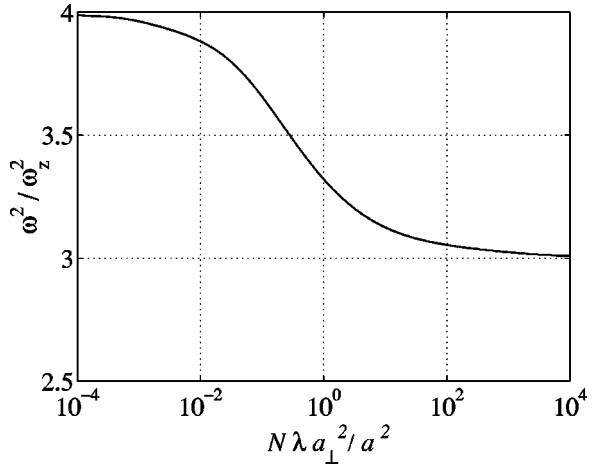


FIG. 3. Transition between the Tonks-Girardeau and the 1D mean-field regimes: $\omega^2/\omega_z^2$ as a function of the parameter $N\lambda(a_\perp/a)^2$.

$$\omega^2 = -2\frac{\langle z^2\rangle}{d\langle z^2\rangle/d\omega_z^2}. \tag{28}$$

The result (28) provides, in general, an upper bound to the frequency of the lowest state excited by $z^2$. It is immediate to verify that this bound coincides with the exact frequency in the three relevant limits discussed in the paper (3D cigar, 1D mean field, and Tonks-Girardeau).

By determining numerically the density profiles in the intermediate regimes we can now calculate the frequency of the lowest compressional mode using the sum-rule formula (28) for any intermediate regime. It is worth noticing that result (28) for the collective frequency, being based on general sum-rule arguments, applies also to regimes beyond the mean-field, where the density profile $n_1(z)$, and hence the value of $\langle z^2\rangle$, cannot be evaluated starting from the Gross-Pitaevskii equation of state.

We have first calculated the ratio $\omega^2/\omega_z^2$ as a function of the dimensionless parameter $N\lambda a/a_\perp$, thereby exploring the transition between the 3D cigar and the 1D mean-field regimes. The results are reported in Fig. 2. For the experimental conditions of Refs. [1,2], where $N\lambda a/a_\perp = 0.24$ and 0.08, we predict $\omega^2/\omega_z^2 = 2.85$ and 2.91, respectively, confirming that those experiments are actually touching the transition between the two mean-field regimes.

For the transition from the 1D mean-field regime to the Tonks-Girardeau gas, the results are reported in Fig. 3 as a function of the dimensionless parameter $N\lambda(a_\perp/a)^2$. For example, using $N=10$, $\lambda = 10^{-3}$, and $a_\perp/a = 10$, we predict $\omega^2/\omega_z^2 = 3.3$. To enter more deeply into the Tonks-Girardeau regime, even more extreme experimental conditions are required. In the asymptotic Tonks-Girardeau regime, one finds the value $\omega = 2\omega_z$, which coincides with the oscillation frequency for an ideal gas. With respect to an ideal gas, the system is, however, characterized by a very different density profile reflecting its fermionic nature.

By fixing the ratio $a_\perp/a$, both transitions can be represented on a single plot as a function of $N\lambda$. This corresponds to moving on a vertical path in the phase diagram of Fig. 1. Of course, the accessible states of the systems depend of the chosen geometry and in particular on the value of $\lambda$: Figure 4 shows the evolution of the collective frequency as a function of the parameter $N\lambda$ for two different choices of the ratio $a_\perp/a$. The corresponding curves reveal the transition between the 3D cigar, 1D mean-field, and Tonks-Giradeau regimes.

The figure clearly shows that while for large values of $a_\perp/a$ the three regimes are clearly visible, for $a_\perp/a = 10$ the 1D mean field cannot be identified. In fact, in this case (dashed line), by increasing $N\lambda$, the Lieb-Liniger theory predicts the achievement of the 1D mean-field regime when the Gross-Pitaevskii theory already exhibits significant 3D effects and the two curves cannot be matched. We point out that this effect is independent of the separated values of $N$ and $\lambda$, provided that the condition of applicability of the local-density approximation (14) is satisfied. When the transition cannot be accounted for using separately the Lieb-Liniger and the Gross-Pitaevskii theories or when the local-

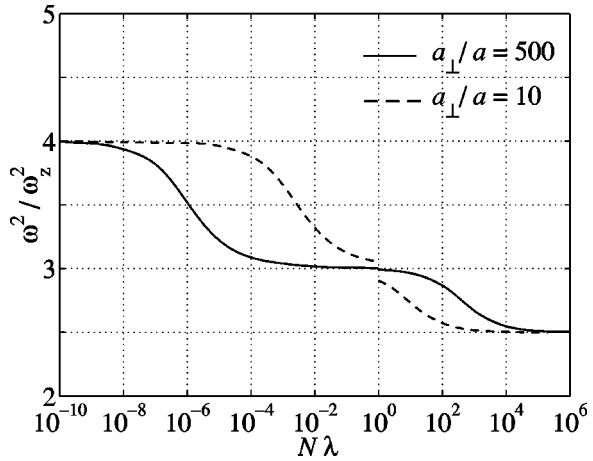


FIG. 4. Dispersion law $\omega^2/\omega_z^2$ as a function of the parameter $N\lambda$ for two choices of the ratio $a_\perp/a$ (full line $\lambda = 500$, dashed line $\lambda = 10$).

density approximation is no longer applicable, more complete many-body approaches are required [23,24].

## V. CONCLUSIONS

According to the discussions presented in this paper, the transition between the different regimes (3D cigar, 1D mean field, Tonks-Girardeau) is specified once the two combinations $N\lambda$ and $a_\perp/a$ of parameters are known [see Eqs. (9) and (23)]. However, these results have been derived by making some important assumptions that are worth recalling.

(1) The aspect ratio $\lambda=\omega_z/\omega_\perp$ should be significantly smaller than 1 in order to ignore the dynamic coupling with the radial excitations whose motion is not taken into account in the 1D formulation in Eqs. (1) and (2) of the hydrodynamic equations. In practice, values of $\lambda$ smaller than $10^{-1}$ are small enough to neglect such a coupling.

(2) The local-density approximation should be guaranteed along the axial direction in order to apply the hydrodynamic approach to the collective modes. This condition, which is equivalent to requiring $Z \gg a_z$, takes a different form depending on whether we are in the Tonks-Girardeau or in the 1D mean-field regime. In the first case the condition implies $N \gg 1$. In the second case the condition instead takes the form $N\lambda \gg \lambda^{3/2} a_\perp/a$ [see Eq. (14)]. If this condition is not satisfied, the system, instead of being in the 1D mean-field Thomas-Fermi regime, will approach the ideal gas configuration.

(3) The scattering length $a$ has been assumed to be sufficiently smaller than $a_\perp$. In fact only if $a_\perp > 10a$, the transition between the regimes investigated in this paper can be described using the Gross-Pitaevskii theory (from 3D cigar to 1D mean field) and the Lieb-Liniger theory (from 1D mean field to Tonks-Girardeau).

(4) One should finally recall that the results of this paper have been derived at zero temperature. They are expected to be applicable at temperatures significantly smaller than the 1D quantum degeneracy energy $\hbar^2 n_{1D}^2/2m$.

In conclusion, in this paper we have exploited the dependence of the frequencies of the collective oscillations of a harmonically trapped 1D Bose gas on the equation of state. This provides an efficient tool to explore the transition between the different regimes exhibited by such systems, pointing out the crucial interplay between the effects of quantum correlations and dimensionality.

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