Coherent pulse output from Bose-Einstein condensates in Wannier-Stark systems

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(Received 28 May 2002; revised 22 August 2002; published 25 October 2002)

The pulsed output from a Bose-Einstein condensate can be described using ordinary one-particle quantum mechanics. The initial state is described in terms of Stark resonances truncated in momentum space. The states obtained in this way resemble the normalizable scattering states defined in terms of Moshinsky functions. The validity of this approach and the influence of the initial population on the pulse formation is discussed. Finally, we describe an experimental setup to manufacture and observe pulsed output in Wannier-Stark systems.

DOI: 10.1103/PhysRevA.66.043418 PACS number(s): 42.50.Vk, 03.65.−w

INTRODUCTION

Recently, experiments with Bose-Einstein condensates (BECs) in a Wannier-Stark system [1] attracted much attention. In these experiments, a Bose-Einstein condensate of rubidium atoms was created, which then was exposed to the optical potential of a standing laser wave. Additionally, the gravitational force accelerates the atoms in the x direction. In a good approximation, the system can be described by the one-dimensional Gross-Pitaevskii equation [2]

$$i\hbar\,\partial_t\Psi=\left[\frac{p^2}{2M}+\cos x+Mgx+U|\Psi|^2\right]\Psi, \tag{1}$$

where M is the atomic mass, U describes the interaction strength of the rubidium atoms, and g is the acceleration due to gravity. (A detailed introduction to the physics of BECs can be found in Ref. [3].) This nonlinear system shows basically all the features found in the analysis of the linear equation, such as Bloch oscillations of the condensate [4–6] or nonlinear Zener tunneling [7]. The system was also studied in a tight-binding approximation [8], neglecting the tunneling out of the potential, and thus the decay of the condensate.

Due to the gravitational force, the atoms in experiment [1] tunnel out of the traps, and a sequence of falling drops of atoms is observed. Neglecting the nonlinear term in Eq. (1), we explain the experimental results in this paper by the properties of a coherent superposition of Wannier-Stark resonances. In fact, tunneling of BECs is influenced by the nonlinear interaction [9], however, the physics leading to the experimental result [1] is already provided by single-particle quantum mechanics. Indeed, the numerical studies [2,10] show that for moderate densities of the condensate, the pulse formation is only slightly modified by the nonlinear term in the Gross-Pitaevskii equation (1).

The paper is organized as follows. In Sec. I we investigate the space-time decay of an initial state in a Stark system. We show that the wave function can be adequately described by a superposition of resonances that are truncated in momentum or coordinate space. In Sec. II we discuss the properties of a coherent superposition of Wannier-Stark resonances. It is shown that the essential information about the decay tail is contained in an amplitude modulation factor. After a brief general discussion of this factor, we present numerical results for the decay of an initial state which models the experimental setup of Ref. [1]. Based on results on the decay of Wannier-Stark resonances in combined dc-ac fields presented in Refs. [11,12], we finally propose a slightly modified setup that yields pulse output.

I. DECAY OF AN INITIAL WAVE FUNCTION

The aim of this section is to describe the space-time decay of an initial state that is localized around $x=0$ in a Stark system, i.e., a system that is influenced by an external dc field,

$$i\hbar\,\partial_t\Psi=\left[\frac{p^2}{2M}+V(x)+Fx\right]\Psi,\quad F\geqslant 0. \tag{2}$$

The potential $V(x)$ is assumed to be bounded, therefore each initial state will decay. A direct expansion of $\Psi(x)$ in terms of resonance wave functions is inappropriate because the resonance states are not normalizable. Therefore the description needs to be modified to take into account the finite extension of the initial state. Recently, this problem was analyzed for decaying quantum systems with a finite-range scattering potential [13,14]. (First steps in this direction can already be found in textbooks, e.g., Ref. [15].)

Let us adopt the approach of Ref. [14] to describe the decay of Stark resonances in momentum space. In this approach, the wave function $\Psi(k,t)$ is expressed in terms of stationary scattering states $\Psi_S(E)$. Let us fix these states by their coordinate space asymptotics: $\lim_{|x|\to\infty}\Psi_S(E,x)=Ai(a(x-E/F))$ with $a=\sqrt[3]{2F\hbar^2}$. Let us furthermore introduce two functions $g_\pm(E)$ by

$$\Psi_S(E,k)=g_\pm(E)\Psi_\pm(E,k), \tag{3}$$

where the states $\Psi_\pm(E,k)$ asymptotically coincide with the free solution,

$$\lim_{k\to\pm\infty}\Psi_\pm(E,k)=\exp\left(i\frac{\hbar^2k^3}{6F}-i\frac{Ek}{F}\right). \tag{4}$$

Note that the definition of the state $\Psi_+(E,k)$ directly reflects the way the resonance wave functions are calculated in a scattering approach to Stark systems presented in Refs. [16–18]. The functions $g_\pm(E)$ resemble the Jost function known from conventional scattering theory, i.e., they are related by $g_-(E^*)=g_+^*(E)$; the scattering matrix is given by their ratio $S(E)=g_-(E)/g_+(E)$ and the scattering states are normalized according to $\langle\Psi_S(E')|\Psi_S(E)\rangle =g_+(E)g_-(E)\ \delta(E-E')$. Expanding the initial state in the scattering states,

$$\Psi(k,t)=\int_{-\infty}^{\infty}dE\ \frac{f(E)}{g_+(E)g_-(E)}\Psi_S(E,k)\exp\left(-i\frac{Et}{\hbar}\right), \tag{5}$$

where $f(E)=\langle\Psi_S(E)|\Psi(t=0)\rangle$, and inserting definition (3) yields

$$\Psi(k,t)=\int_{-\infty}^{\infty}dE\ \frac{f(E)}{g_\mp(E)}\ \Psi_\pm(E,k)\exp\left(-i\ \frac{Et}{\hbar}\right). \tag{6}$$

Finally, we introduce the deviation $\varphi_\pm(E,k)$ from the asymptotic form by

$$\Psi_\pm(E,k)=\varphi_\pm(E,k)\exp\left(i\frac{\hbar^2k^3}{6F}-i\frac{Ek}{F}\right), \tag{7}$$

with $\lim_{k\to\pm\infty}\varphi_\pm(E,k)=1$, which directly follows from Eq. (4). We will use this property in the following. Then the integral (6) takes the form

$$\Psi(k,t)=\int_{-\infty}^{\infty}dE\ \frac{f(E)}{g_\mp(E)}\ \varphi_\pm(E,k)\times\exp\left(i\ \frac{\hbar^2k^3}{6F}-i\ \frac{Ek}{F}-i\ \frac{Et}{\hbar}\right). \tag{8}$$

We are mainly interested in the properties of the decay tail, where the wave functions of the resonances can be approximated by their asymptotic form. Therefore, let us assume that $|k|$ is sufficiently large and replace $\varphi_\pm(E,k)$ by its asymptotic value. Then the wave function can be approximated as

$$\Psi(k,t)=\exp\left(i\ \ \frac{\hbar^2k^3}{6F}\right)G_\pm\left(k+\frac{Ft}{\hbar}\right), \tag{9}$$

where $G_\pm(k)$ is the Fourier transform of $f(E)/g_\mp(E)$,

$$G_\pm(k)=\int_{-\infty}^{\infty}dE\ \frac{f(E)}{g_\mp(E)}\exp\left(-i\ \frac{Ek}{F}\right). \tag{10}$$

It follows that asymptotically the absolute square $|\Psi(k,t)|^2$ is a function of $k+Ft/\hbar$. Thus, the wave function moves linearly in time to smaller momenta $k(t)=k_0-Ft/\hbar$, i.e., it follows the free classical motion. Except the shift, the shape of the wave function remains constant (see also the discussion on Airy wave packets in Ref. [19]).

One can obtain Eq. (9) in a second way. Namely, we expand the initial state in plane waves instead of scattering states. If we then assume that the momenta are so large enough that the potential can be neglected in comparison with the dc field, every plane wave with initial momentum $\hbar k_0$ is accelerated to the plane wave with momentum $\hbar k_0 -Ft$. Representing the initial state by Eq. (9) with $t=0$, the equation correctly describes the evolution of the phases of the plane waves, too.

A. Relation to the resonances

Let us further evaluate Eq. (10). The integrand $f(E)/g_+(E)$ is an analytic function of the energy. It has poles at the zeros of $g_+(E)$ and therefore at the poles of the S matrix. If the initial wave function $\Psi(k,0)$ has a compact support, $f(E)$ is an entire function in the complex plane and does not provide additional poles. It follows that all poles of the integrand $f(E)/g_+(E)$ locate in the lower half of the complex plane.

These properties suggest to evaluate the integral (10) with the help of its residua, i.e., the poles of the S matrix. However, for this we need to know the asymptotic behavior of the integrand. Without knowing the analytical form of the function $f(E)/g_+(E)$, we are forced to make some assumptions on its asymptotic behavior in order to proceed further. In particular, if we assume that the function $f(E)/g_+(E)$ does not influence the behavior of the integrand at infinity, the integral yields the sum over the residua located within the appropriate contour. Explicitly, for $k>0$ the contour is closed in the lower half of the complex energy plane, for $k<0$ it is closed in the upper half. Therefore, we get

$$G_+(k)=\Theta(k)\sum_\nu\ A_\nu\ \exp\left(-i\frac{\mathcal{E}_\nu k}{F}\right), \tag{11}$$

where $\Theta(k)$ is the Heaviside function. The sum contains all poles $\mathcal{E}_\nu=E_\nu-i\Gamma_\nu/2$ of the S matrix (note that $\Gamma_\nu>0$), and the A_ν are the residua of $f(E)/g_+(E)$ at the poles. Inserting this result in Eq. (9) yields

$$\Psi(k,t)=\Theta(\hbar k+Ft)\sum_\nu\ A_\nu\ \exp\left(i\frac{\hbar^2k^3}{6F}-i\frac{\mathcal{E}_\nu k}{F}-i\frac{\mathcal{E}_\nu t}{\hbar}\right). \tag{12}$$

The terms of the sum are actually proportional to the asymptotic form of the resonance wave functions $\Psi_\nu(k,t)$. Thus, we can equivalently expand the wave function as

$$\Psi(k,t)=\Theta(\hbar k+Ft)\sum_\nu\ B_\nu\Psi_\nu(k,t), \tag{13}$$

with new coefficients B_ν. Therefore, in the Stark case we can describe the decay of an initial state by a superposition of resonances, where we take into account the space-time decay process in the prefactor $\Theta(\hbar k+Ft)$. This factor truncates the wave function at the momentum $\hbar k=-Ft$, i.e., only momenta with $\hbar k>-Ft$ contribute. With increasing time, the wave function extends to smaller momenta, where the edge moves according to the classical equation of motion.

Note that the location of the edge reflects the assumption we made on the behavior at infinity in order to explicitly evaluate the integral. For example, the function $f(E)/g_+(E)$ may contain an additional exponential $\exp(i\alpha E)$ (see, e.g., the example in Ref. [14]). Though this factor does not influence the poles, it nevertheless influences the argument of the Heaviside function. In fact, in a realistic situation the edge will be shifted because the truncation edge at $t=0$ has to reflect the extension of the initial state in momentum space. We take this into account by replacing $\hbar k$ in the argument of the Heaviside function by $\hbar(k+k_0)$, where k_0 describes the extension of the initial state in the negative k direction. Furthermore, if the initial state does not have a compact support but a tail in negative momentum direction, the edge will be smoothed and deformed. However, the qualitative behavior remains unchanged, the prefactor is approximately constant for positive arguments of the Heaviside function, and it approximately vanishes for negative arguments. Therefore, let us take the Heaviside description as a rough approximation to the real situation.

Let us finally note that the wave function constructed in this way is normalizable. In the positive momentum direction, the resonance wave functions decrease stronger than exponentially, and in the negative direction the wave function considered here is truncated. Note that in the Stark case the contributions of all resonances are truncated at the same momentum. This is the main difference to the result [14], where the contributions of different resonances extend with different velocities, according to their particular kinetic energies.

B. Basis of truncated resonances

Let us briefly present an independent derivation of formula (13) using truncated Stark resonance states, which are defined by the equation

$$\Psi_\nu^K(k)=\Theta(k+K)\Psi_\nu(k). \tag{14}$$

If $|K|$ is large enough that the resonances can be described by their asymptotic form, the evolution of the states $\Psi_\nu^K(k,0)\equiv\Psi_\nu^K(k)$ follows the law

$$\Psi_\nu^K(k,t)=\Theta(k+K+Ft/\hbar)\Psi_\nu(k,t), \tag{15}$$

with $\Psi_\nu(k,t)=\exp(-i\mathcal{E}_\nu t/\hbar)\Psi_\nu(k)$. Now, assume that the support of the initial state contains only momenta with $|k|<|K|$. If the truncated resonances $\Psi_\nu^K(k)$ provide an appropriate basis to expand such an initial state, we are directly led to Eq. (13).

In the following, we assume that we have a Wannier-Stark system, i.e., a system described by Eq. (2) with periodic potential $V(x)=V(x+a)$. Then we address this problem with the help of the Floquet-Bloch resonances $\Phi_{\nu,\kappa}(k)$, which are simultaneous eigenstates of the translation over one lattice period of the potential and the time evolution over one Bloch period $\tau_B=2\pi\hbar/(Fa)$. They provide a convenient basis set. In Ref. [20] it was shown that one can decompose these states into three parts, $\Phi_{\nu,\kappa}^{(\pm)}(k)$ and $\Phi_{\nu,\kappa}^{(0)}(k)$, where the $\Phi_{\nu,\kappa}^{(\pm)}(k)$ contain the asymptotes in the positive and negative momentum directions, and $\Phi_{\nu,\kappa}^{(0)}(k)$ contains the rest. In fact, for resonances the wave function has no asymptotes in the positive momentum direction due to the Siegert boundary conditions, thus $\Phi_{\nu,\kappa}^{(+)}(k)=0$.

Similar to the truncated Wannier-Stark states (14), we get the truncated Floquet-Bloch states by multiplication with the Heaviside function. Indeed, we assumed that for $|k|>|K|$ the asymptotic form is valid, therefore we can (without loss of generality) identify the truncated Floquet-Bloch states with the vectors $\Phi_{\nu,\kappa}^{(0)}(k)$. These states provide an appropriate basis for an expansion of the initial state. For each κ the Hilbert space spanned by the $\Phi_{\nu,\kappa}^{(0)}(k)$ is finite dimensional. If no resonance energies are degenerate, the $\Phi_{\nu,\kappa}^{(0)}(k)$ are linearly independent and we can indeed expand the initial state as

$$\Psi(k,0)=\int_{-1/2}^{1/2}d\kappa\sum_\nu a_\nu(\kappa)\Phi_{\nu,\kappa}^{(0)}(k). \tag{16}$$

The integral over the quasimomentum yields an expansion in terms of truncated Wannier-Stark resonances as in Eq. (13).

The validity of the expansion (16) depends on the validity of the decomposition, in particular on the property $\Phi_{\nu,\kappa}^{(+)}(k)=0$. Indeed, for finite $|K|$, the decomposition only approximates the real resonance wave functions, and there may be small deviations. However, as it was shown in Ref. [17], the method converges pretty fast for a reasonable potential $V(x)$ and, in particular, the resonance wave function decreases faster than exponentially in the positive momentum direction, and thus the error due to the finite $|K|$ is assumed to be small.

Concluding, the approach using truncated Wannier-Stark resonances yields the same form as the approach described in Sec. I A. Therefore, in the following sections we describe the space-time decay of an initial state by a superposition of Stark resonances whose wave function is truncated in momentum space by the Heaviside function $\Theta(\hbar(k+k_0)+Ft)$, where k_0 is a free parameter which, together with the expansion coefficients, reflects the properties of the initial state.

C. Decay in coordinate space

The wave function in coordinate space is found by a Fourier transform of Eq. (12),

$$\Psi(x,t)=\int_{-\infty}^{\infty}dk\Theta(\hbar(k+k_0)+Ft)\times\sum_\nu A_\nu\exp\left(i\frac{\hbar^2k^3}{6F}-i\frac{\mathcal{E}_\nu k}{F}-i\frac{\mathcal{E}_\nu t}{\hbar}+ikx\right). \tag{17}$$

Let us evaluate the integral by means of the stationary phase approximation, which approximates the integral over the product of a rapidly oscillating and a slowly varying function by the formula [21]

$$\int dke^{i\lambda\phi(k)}f(k)\approx\sum_n\sqrt{\frac{2\pi}{\lambda\phi''(k_n)}}e^{i\lambda\phi(k_n)}f(k_n), \tag{18}$$

where the sum runs over the points k_n of stationary phase $\phi'(k_n)=0$. In the present case, we take into account only the real part of the energies $\mathcal{E}_\nu=E_\nu-i\Gamma_\nu/2$ and add the exponential of the imaginary part to the slowly varying function. Then the phase function of the different contributions is

$$\phi(k)=\frac{\hbar^2 k^3}{6F}-\frac{kE_\nu}{F}+kx. \tag{19}$$

The stationary phase condition is just the energy conservation, and the stationary points are the classical momenta

$$\hbar k_{\nu,\pm}=\pm\sqrt{2(E_\nu-Fx)}. \tag{20}$$

Furthermore, we have $\phi''(k_{\nu,\pm})=\hbar^2 k_{\nu,\pm}/F$. We can neglect the contributions from $k_{\nu,+}$ because for $x\to-\infty$, i.e., $k_{\nu,+}\to\infty$, they are exponentially small due to the weight factor $e^{-i\mathcal{E}_\nu k/F}$ in Eq. (17). To shorten the notation, let us introduce $p_\nu(x)=\hbar k_{\nu,-}$ [note that $p_\nu(x)\to-\infty$ for $x\to-\infty$]. If $p_\nu(x)\leqslant-\hbar k_0-Ft$, the prefactor is zero and the integral vanishes. On the other hand, if $p_\nu(x)\geqslant-\hbar k_0-Ft$, the integral of the contribution of the νth resonance yields approximately

$$I_\nu(x,t)=\exp\left(-i\frac{\mathcal{E}_\nu t}{\hbar}\right)\sqrt{\frac{2\pi F}{\hbar p_\nu(x)}}$$
$$\times\exp\left(-i\frac{p_\nu^3(x)}{3\hbar F}-\frac{\Gamma_\nu p_\nu(x)}{2\hbar F}\right). \tag{21}$$

The critical point is $p_\nu(x)=-\hbar k_0-Ft$, where the approximation breaks down because the Heaviside function is not a slowly varying function at this point. Actually, in the vicinity of this point, the integral interpolates between the other two possibilities. Let us skip a more detailed analysis and roughly describe the transition between both regimes by a Heaviside function of the argument $p_\nu(x)+\hbar k_0+Ft$, or, equivalently, of the argument $x+F(t+t_0)^2/2-E_\nu/F$, where the parameter $t_0=\hbar k_0/F$ is introduced.

The different contributions contain the asymptotic form of the resonance wave functions in coordinate space. We can therefore replace the contributions (21) by $\Psi_\nu(x,t)$. In analogy to Eq. (13), we then get a superposition of resonance wave functions that are truncated in coordinate space,

$$\Psi(x,t)=\sum_\nu B_\nu\Theta(x+F(t+t_0)^2/2-E_\nu/F)\Psi_\nu(x,t). \tag{22}$$

In comparison to Eq. (13), there are two differences. First, in coordinate space the truncation depends on the energy of the resonances. Furthermore, the edges of the different contributions move with a quadratic time dependence, which reflects the classical accelerated motion in a constant external field.

II. SUPERPOSITION OF WANNIER-STARK RESONANCES

Let us extend the results from the preceding section to analyze the decay of an initial state in a Wannier-Stark system with $H=p^2/2+\cos x+Fx$. Note again that this Hamiltonian describes the experiment [1], if we neglect the nonlinear term in Eq. (1). In the following discussion, we will mainly refer to the parameters of this setup.

As shown in the preceding section, we can describe a decaying state by a truncated superposition of resonance wave functions. In the following, we take into account only the resonances from the most stable Wannier-Stark ladder with the energies $\mathcal{E}_n=E_0+2\pi Fn-i\Gamma_0/2$. The index n can be interpreted as a site index. Note, however, that the equations can easily be modified to additionally take into account higher excited resonances. In the ground-ladder description, the initial state is given by a superposition

$$\Psi=\sum_n c_n\Psi_n \tag{23}$$

of ground-ladder resonances Ψ_n with energy $\mathcal{E}_n$. (To shorten the notation, we skip the truncation with the Heaviside function because the truncation does not influence the properties discussed in the following.) The time evolution of the wave function Ψ additionally contains the time-dependent phases,

$$\Psi(t)=\sum_n c_n\exp(-i\mathcal{E}_n t/\hbar)\Psi_n. \tag{24}$$

The states Ψ_n from the ground Wannier-Stark ladder are related by a coordinate shift by multiples of the lattice period 2π. In momentum space, this shift is a multiplication by an additional phase factor:

$$\Psi_n(k)=\exp(-i2\pi nk)\Psi_0(k). \tag{25}$$

Combining the phase relation in the momentum representation of the resonances with the different phases due to the time evolution, the time evolution of the superposition is given by

$$\Psi(k,t)=\Psi_0(k,t)\sum_n c_n\exp\left[-i2\pi n\left(\frac{Ft}{\hbar}+k\right)\right]$$
$$=\Psi_0(k,t)\tilde{C}\left(\frac{Ft}{\hbar}+k\right), \tag{26}$$

where $\Psi_0(k,t)=\exp(-i\mathcal{E}_0 t/\hbar)\Psi_0(k)$. Thus, the time evolution of the superposition is given by the time-evolved wave function at the mean energy, $\Psi_0(k,t)$, times the discrete Fourier transform $\tilde{C}(k)$ of the amplitudes c_n taken at the momenta $k+Ft/\hbar$. The function $\tilde{C}(k)$ is periodic in momentum space with $\tilde{C}(k+n)=\tilde{C}(k)$ and $n\in\mathbb{Z}$. Consequently, the amplitude modulation factor $\tilde{C}(k+Ft/\hbar)$ is also periodic in time with the period $\hbar/F=\tau_B$, the Bloch period.

The periodicity in momentum space suggests an alternative interpretation of the formula. Namely, if we do not expand the initial state in Wannier-Stark resonances but in Floquet-Bloch states, the corresponding expansion coefficients are just given by the function $\tilde{C}(\kappa)$, with $-1/2\leqslant\kappa\leqslant 1/2$. Due to the dc field, the time evolution of the quasimomentum follows the equation $\kappa(t)=\kappa-Ft/\hbar$. Then, the expansion coefficients at time t are given by the shifted function $\tilde{C}(\kappa+Ft/\hbar)$. The time evolution of the superposition is the integral over the entire Brillouin zone, which again leads to Eq. (26). We will come back to this interpretation in Sec. IV of this paper, where we propose a different method to generate pulse output in Wannier-Stark systems.

A. Asymptotic expansion

Let us analyze the wave function in coordinate space. It is given by the Fourier transform of Eq. (26), i.e., by

$$\Psi(x,t)=\int_{-\infty}^{\infty}dk\,\exp(ikx)\Psi_0(k,t)\tilde{C}\left(\frac{t}{\tau_B}+k\right). \tag{27}$$

Using the convolution property of the Fourier transform, we can express the wave function in coordinate space as the convolution of $\Psi_0(x,t)$ and the Fourier transform of $\tilde{C}(k)$. Proceeding further, we find it more instructive to explicitly perform the transformation. We are mainly interested in the behavior of the decay tail, therefore let us assume that $|k|$ is large enough, so that we can approximate the Wannier-Stark resonance by an Airy function, which up to a constant factor yields

$$\Psi(x,t)=\int_{-\infty}^{\infty}dk\,\exp(ikx)\times\exp\left(i\frac{\hbar^2k^3}{6F}-ik\frac{\mathcal{E}_0}{F}-i\frac{\mathcal{E}_0t}{\hbar}\right)\tilde{C}\left(\frac{t}{\tau_B}+k\right). \tag{28}$$

Similar to the preceding section, we concentrate on the decay tail, where $x\to-\infty$, and solve the integral using the stationary phase approximation. Explicitly, we assume that the function $\tilde{C}(k+t/\tau_B)$ is slowly varying and apply formula (18). The phase function (19) yields two stationary points $\hbar k_{\pm}=\pm\sqrt{2(E_0-Fx)}$. Again, we can neglect the positive one because its contribution is exponentially small. Then, setting $p(x)=\hbar k_-$, we get

$$\Psi(x,t)=\exp\left(-i\frac{\mathcal{E}_0t}{\hbar}\right)\sqrt{\frac{2\pi F}{\hbar p(x)}}\times\exp\left(-i\frac{p^3(x)}{3\hbar F}-\frac{\Gamma_0p(x)}{2\hbar F}\right)\tilde{C}\left(\frac{t}{\tau_B}+\frac{p(x)}{\hbar}\right). \tag{29}$$

Again, we can express the result as the product of two factors, one of which is the asymptotic form of the particular Wannier-Stark resonance $\Psi_0(x,t)$. The other one is a time-dependent amplitude modulation factor. Therefore, replacing the asymptotic form of the resonance by $\Psi_0(x,t)=\exp(-i\mathcal{E}_0t/\hbar)\Psi_0(x)$, we can write Eq. (29) in a form similar to Eq. (26),

$$\Psi(x,t)=\Psi_0(x,t)\;\tilde{C}\left(\frac{t}{\tau_B}+\frac{p(x)}{\hbar}\right). \tag{30}$$

Indeed, up to an overall exponential decay due to the imaginary part of $\mathcal{E}_0$, the absolute value of the first factor is time independent. Furthermore, as can be seen from the asymptotic form, it is a slowly varying function of the coordinate. The whole information about the time-dependent tunneling process through the sequence of barriers imposed by the periodic potential is contained in the second factor that will be analyzed further in the following section.

B. Amplitude modulation factor

Let us first calculate the amplitude modulation factor $\tilde{C}(k)$ for some exemplifying initial distributions. If only a single Wannier-Stark resonance is occupied in the beginning, $c_n=\delta_{0,n}$, the function $\tilde{C}(k)$ is constant and the decay is not modified. If two arbitrary resonances are equally populated initially,

$$c_0=c_R=\frac{1}{\sqrt{2}},\quad c_n=0\quad\text{otherwise}, \tag{31}$$

with $R\in\mathbb{Z}$, then the square of the amplitude is modulated by the factor

$$|\tilde{C}(k)|^2=2\cos^2(\pi Rk). \tag{32}$$

The position of the maxima of this function differs by $\Delta k=1/R$, which corresponds to a time period of τ_B/R. Another simple case is when every Rth resonance is populated, and the first N of these resonances are populated equally,

$$c_{nR}=\frac{1}{\sqrt{N}}\quad\text{for}\quad 0\leqslant n<N,\quad c_n=0\quad\text{otherwise}, \tag{33}$$

which yields the modulation factor

$$|\tilde{C}(k)|^2=\frac{1}{N}\;\frac{\sin^2(NR\pi k)}{\sin^2(R\pi k)}, \tag{34}$$

which is well known from the analysis of diffraction gratings. For the particular case $R=1$ with N subsequent resonances populated, we get

$$|\tilde{C}(k)|^2=\frac{1}{N}\;\frac{\sin^2(N\pi k)}{\sin^2(\pi k)}. \tag{35}$$

Finally, if the initial condition is a Gaussian distribution $c_n\sim\exp(-\beta n^2)$ as assumed in Ref. [1], the amplitude modulation factor consists of a sequence of Gaussians [22],

$$|\tilde{C}(k)|^2=\frac{\pi}{\beta}\left(\sum_{n\in\mathbb{Z}}\exp\left[-\frac{\pi^2}{\beta}\;(k-n)^2\right]\right)^2\approx\frac{\pi}{\beta}\sum_{n\in\mathbb{Z}}\exp\left[-\frac{2\pi^2}{\beta}\;(k-n)^2\right]. \tag{36}$$

The last approximation holds for small β.

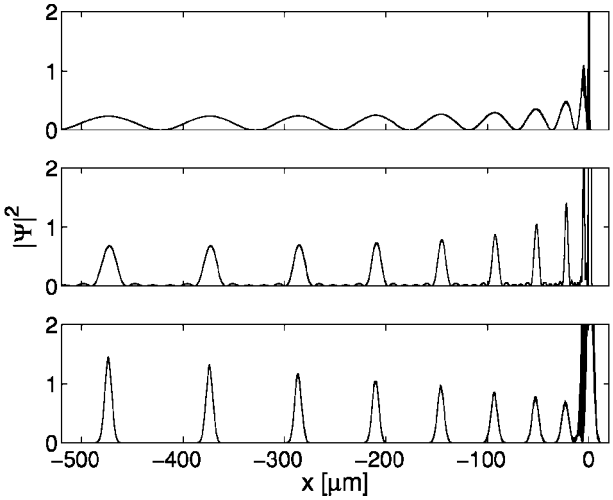

FIG. 1. Absolute square of the wave function of a superposition of Wannier-Stark resonances. The first figure depicts the wave function for the distribution (31) with $R=1$, the second one the distribution (35) with $N=6$, and the third one the Gaussian distribution (36) with $\beta=1/15^2$.

1. Modulation in coordinate space

The last two functions consist of a series of equidistant peaks at integer values of k. For large N or small β, the amplitude is concentrated at these peaks while it approximately vanishes between them. In coordinate space the peaks appear at the points with

$$p(x)=-F(t+j\tau_B),\quad j\in\mathbb{Z},\tag{37}$$

i.e., at the coordinates

$$x=x_0-\frac{F}{2}\ (t+j\tau_B)^2,\tag{38}$$

where $x_0=E_0/F$ is the classical turning point. Thus, as a function of time, the peaks accelerate according to the classical equation of motion of a free particle subject to a constant electric field. Additionally, the peaks broaden linearly with increasing time (or with increasing j). For example, the full width at half maximum Δk of the function (35) for large N is approximately $\Delta k=1/N$. The x coordinates of the corresponding points are given by

$$x_\pm=x_0-\frac{F}{2}\left(t+j\tau_B\pm\frac{\tau_B}{2N}\right)^2,\tag{39}$$

which yields a peak width

$$\Delta x=x_--x_+=F\Delta k\,\tau_B(t+j\tau_B)=\frac{F\tau_B}{N}(t+j\tau_B).\tag{40}$$

A similar relation holds for the width of the sequence of Gaussians (36), with $\Delta k=\sqrt{2\beta\ln 2}/\pi$.

Figure 1 shows the coordinate space wave function of a superposition of Wannier-Stark resonances at $t=0$ for the distributions discussed in the beginning of this section, namely, for two equally populated neighbored resonances, $N=6$ equally populated resonances, and a Gaussian distribution, respectively. In the following we will use scaled units in which the potential is 2π periodic with a well depth of 1 and $M=1$. This can be achieved by replacing $\hbar\to 2\pi\hbar/(a\sqrt{M\gamma})$ and $F\to aF/(2\pi\gamma)$, where γ is half the depth of the periodic potential. The parameters of the Hamiltonian were chosen to meet the experimental setup in Ref. [1], which yields for the trap depth of 1.4 recoil energies a scaled Planck constant $\hbar=3.3806$ and a scaled field strength $F=0.0661$. The most stable resonance has a scaled width of $\Gamma_0=7.4726\times10^{-3}$ corresponding to a lifetime of 9.66 ms, which is approximately nine Bloch times $\tau_B=1.09$ ms (A potential depth of 2.1 recoil energies yields a lifetime of 73 ms, in reasonable agreement with the 50 ms reported in Ref. [1]. For a potential depth of one recoil energy, the lifetime is 2.59 ms).

The wave functions were calculated the following way. First, a particular ground-state resonance $\Psi_0(k)$ was computed [17,23]. Following Eq. (26), this function was modulated with the amplitude modulation factors of the initial distributions presented at the beginning of this section. Finally, the result was Fourier transformed to coordinate space.

The behavior described above can clearly be seen in Fig. 1. The decay tail of the wave function consists of a number of peaks that broaden with decreasing coordinate. [For Eq. (36) the broadening can only be observed for β not too small.] As a function of time, these maxima accelerate toward the negative x direction, i.e., a sequence of pulses arises. The positions of the maxima are in good agreement with the formula (39), which reads $x=-F(n\tau_B)^2/2$. For example, the last three peaks correspond to $n=7$, 8, 9, which yields the positions $x=-286\ \mu$m, $-374\ \mu$m, $-474\ \mu$m, respectively. The relative amount of probability stored in the peaks increases with decreasing coordinate, which just reflects that in every pulse a certain amount of the probability stored in the main body of the resonance drops out and is accelerated by the external field. Note that the pulse formation can be interpreted as the result of Bloch oscillations [24,25]. During each Bloch period one peak tunnels out as the oscillating state arrives at the left turning point.

2. Stability against noise

Naturally the question arises if the behavior found is stable against noise. This topic is briefly illustrated by the numerical results shown in Fig. 2. In all cases the initial distribution was assumed to be given by the Gaussian $c_n\sim\exp(-\beta n^2)$ with $\beta=1/15^2$, as assumed in Ref. [1]. In the first part, the amplitudes c_n are multiplied with real random coefficients taken from the interval $[0.5,2]$, then the distribution is renormalized. The form of the peaks is slightly modified, however, the overall behavior is pretty stable against amplitude noise.

The second panel in Fig. 2 shows the effect of phase noise. Explicitly, the coefficients were modified according to $c_n\sim\exp(-\beta n^2)\exp(i\phi_n)$, where ϕ_n was chosen randomly in the interval $[-0.4\pi,0.4\pi]$. Now the effect of the noise is stronger. The shape of the pulses is modified, and the wave function does not vanish any more in between the pulses.

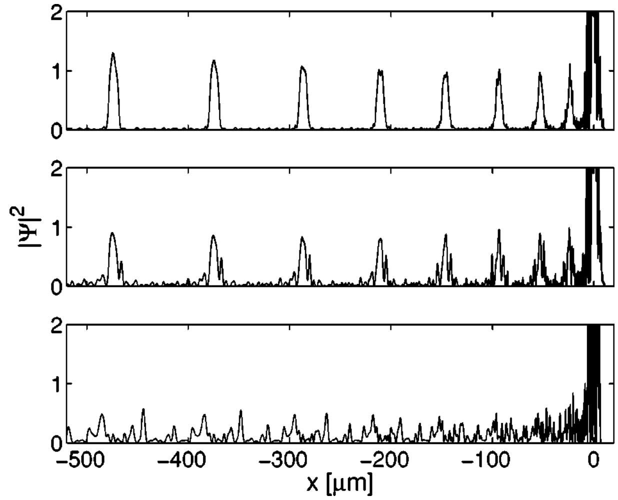

FIG. 2. Superposition of Wannier-Stark resonances with noisy Gaussian initial distribution $c_n \sim \exp(-\beta n^2)$ with $\beta = 1/15^2$. The first panel shows the effect of amplitude noise, the second panel of phase noise, and the last one assumes random phases of the initial distribution.

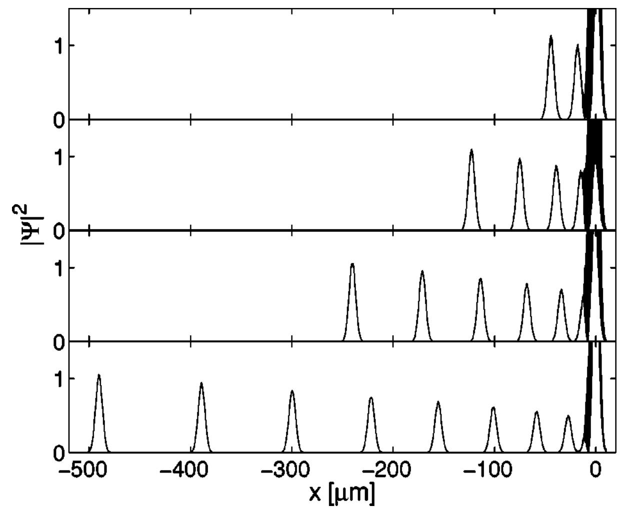

FIG. 3. Space-time decay of the wave function. The initial state is assumed to be a Gaussian distribution of the field-free ground Wannier states. From top to bottom, the panels correspond to $t = 3$ ms, 5 ms, 7 ms, and 10 ms, respectively.

However, qualitatively the pulse output still survives. If we take into account that the phases vary within 40% of the possible values, i.e., we have strong noise, we can resume that the behavior is also stable against phase noise. Finally, if we randomly choose the phases from the interval $[-\pi, \pi]$, as in the lowest panel of Fig. 2, the pulse structure disappears, and the wave function periodically repeats a random structure, which is only stretched due to the acceleration.

C. Decay in atomic tunnel arrays

It is straightforward to combine the results of the two previous sections. Generally, we have to truncate the wave front approximately at the coordinate $x = -F(t+t_0)^2/2 + E_0/F$. Furthermore, for the initial distributions in Wannier-Stark systems considered above, a series of pulses with constant shape is formed, which move with the same time dependence, i.e., their maxima are located at $x = -F(t + m\tau_B)^2/2 + E_0/F$. Consequently, to take into account the truncation, we have to remove all pulses with $m > t_0/\tau_B$. Actually, we have a superposition of resonances with energies $\mathcal{E}_n = \mathcal{E}_0 + 2\pi F n$, therefore we have to truncate the contribution of each resonance at a different coordinate. However, one can easily surmount this problem by truncating in momentum space.

Let us, in this way, describe the space-time decay in the experiment [1] where pulse output from a Wannier-Stark system was found. We already addressed the setup above. The scaled parameters of the Hamiltonian are $\hbar = 3.3806$ and $F = 0.0661$. In the experiment, the initial state was a Bose-Einstein condensate that extends over approximately 30 space periods. In our description, we assume a Gaussian distribution of the ground resonances according to the formula $c_n \sim \exp(-\beta n^2)$ with $\beta = 1/15^2$.

Figure 3 shows the space-time decay of this initial state at the times $t = 3$ ms, 5 ms, 7 ms, and 10 ms, respectively. The figure was calculated the following way. First the momentum space representation of one particular resonance from the ground Wannier-Stark ladder was calculated. Then the amplitude of the resonance was modulated with the amplitude modulation factor (36) taken at the specified times. This wave function was truncated in momentum space according to Eq. (13). Actually, we shifted the truncation edge by $k_0 = 1/2$ in order to avoid a truncation directly at the maxima. However, as discussed above, the location of the edge may be shifted from the result (13). As can be seen in Fig. 1, the wave function approximately vanishes between the maxima, and therefore the result does not change if we slightly modify the size of the shift. Finally, the resulting function was Fourier transformed into coordinate space.

The figure closely resembles the findings of the experiment [1]. A series of pulses is formed, which accelerate according to the free motion. At a fixed value of the coordinate, the sequence is periodic in time after the first pulse passed, up to an overall exponential decay, which reflects the fact that every drop takes away a certain amount of probability.

III. WANNIER-STARK MODE LOCKING

Let us finally discuss how to prepare an initial state which decays by a regular pulse formation. If the initial state populates only one Wannier-Stark resonance, the decay is continuous and no pulses develop. An arbitrary initial state will contain contributions from several resonances and thus yield a modified output. An initial distribution with random phases, however, leads to random output. Thus, the question arises as to how to prepare initial states that populate several Wannier-Stark resonances with well-defined phases.

In the experiment [1], the fixed phase relation was achieved by the self-interaction of the Bose-Einstein condensate. In the following, we show that one can easily prepare an appropriate initial state within single-particle quantum mechanics by temporarily adding an ac field with frequency matching the Bloch frequency $\omega_B = 2\pi/\tau_B$. Explicitly, the technique is as follows: Take an arbitrary initial state in the Wannier-Stark system and expose it for a finite time T_{int} to an additional ac field $F_\omega x \cos(\omega_B t)$, which is then switched off. If the field strength is sufficiently large and the interaction time T_{int} is long enough, the initial state decays with a pulse output.

The reason is the dependence of the width (or the lifetime) of the Floquet-Bloch states of the dc-ac system on the quasimomentum [12,20,26]. Namely, let $\phi_{\alpha,\kappa}(k)$ denote the Floquet-Bloch states of the dc-ac Hamiltonian

$$H = \frac{p^2}{2} + \cos x + Fx + F_\omega x \cos(\omega_B t). \tag{41}$$

Then we can expand the initial state $\psi_0(k)$ in the Floquet-Bloch states,

$$\psi_0(k) = \Theta(k + k_0) \sum_\nu \int_{-1/2}^{1/2} d\kappa\, a_\nu(\kappa)\, \phi_{\nu,\kappa}(k), \tag{42}$$

where the $a_\nu(\kappa)$ are periodic functions of the quasimomentum and the index ν sums over the different bands. In analogy to Eq. (13), the Heaviside function is taken as an approximation to the real situation and ensures the cutoff in momentum space. Let us assume that mainly the ground band is populated and skip the sum over the band indices in the following. In fact, higher excited states rapidly decay and therefore mainly influence the edge of the decay tail of the wave function. Let us disregard the actual form of the edge in the following. Then, after N periods of driving, the wave function reads

$$\psi_0(k, N\tau_B) = \Theta(\hbar(k + k_0) + FN\tau_B) \times \int_{-1/2}^{1/2} d\kappa\, a(\kappa)\, e^{-i\mathcal{E}_0(\kappa)N\tau_B/\hbar}\, \phi_{0,\kappa}(k). \tag{43}$$

Now the ac field is switched off, and we take the final state $\psi_0(k, N\tau_B)$ as the initial state for the pure dc dynamics. We then expand in the basis of the Floquet states of the new dc Hamiltonian

$$H = \frac{p^2}{2} + \cos x + Fx, \tag{44}$$

which we denote by $\Phi_{\nu,\kappa}(k)$. If we again restrict ourselves to the ground band, the expansion in the new basis reads

$$\psi_0(k, N\tau_B) = \Theta(\hbar(k + k_0) + FN\tau_B) \times \int_{-1/2}^{1/2} d\kappa\, b(\kappa)\, e^{-i\mathcal{E}_0(\kappa)N\tau_B/\hbar}\, \Phi_{0,\kappa}(k), \tag{45}$$

with the prefactor $b(\kappa) = a(\kappa)\langle \Phi_{0,\kappa} | \phi_{0,\kappa} \rangle$. We can equivalently treat the functions $b(\kappa)$ and $\mathcal{E}(\kappa)$ as periodic functions of the momentum instead of the quasimomentum, and then shift them in front of the integral. It remains the integral of the states $\Phi_{0,\kappa}(k)$ over the entire Brillouin zone, which yields one particular Wannier-Stark resonance $\Psi_0(k)$ from the ground Wannier-Stark ladder. Then we get

$$\psi_0(k, N\tau_B) = \Theta(\hbar(k + k_0) + FN\tau_B)\, b(k) \times \exp\left(-i\frac{\mathcal{E}_0(k)N\tau_B}{\hbar}\right) \Psi_0(k). \tag{46}$$

Thus, the prefactor $b(k)\exp[-i\mathcal{E}_0(k)N\tau_B/\hbar]$ takes the role of the amplitude modulation factor $\tilde{C}(k)$ of the new initial state. Let us briefly discuss its form in the following section.

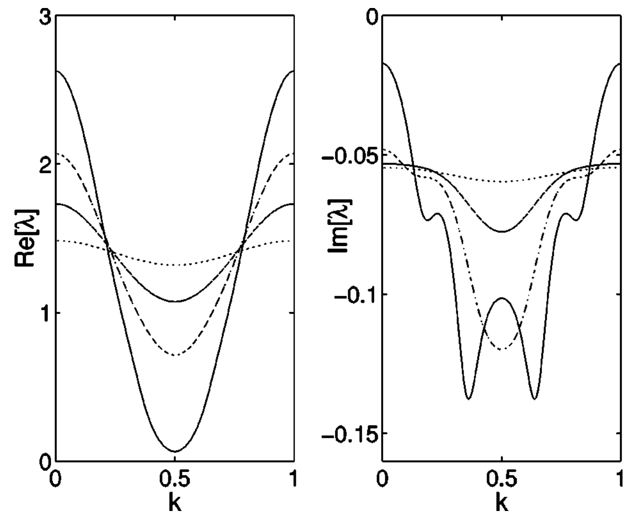

FIG. 4. Real and imaginary parts of the quasiangles $\lambda = \mathcal{E}\tau_B/\hbar$ of the system (41) with parameters $\hbar = 3.3806$, $F = 0.0661$, and different $\epsilon = F_\omega/\omega^2$. The dotted line corresponds to $\epsilon = 0.1$, the dashed line to $\epsilon = 0.4$, the dashed-dotted line to $\epsilon = 0.8$, and the solid line to $\epsilon = 1.5$.

A. Mode-locked amplitude modulation factor

Figure 4 shows the real and the imaginary parts of the quasiangles $\lambda = \mathcal{E}\tau_B/\hbar$ for the system (41) with parameters $\hbar = 3.3806$ and $F = 0.0661$ taken from the experiment [1] and different values of the ac field strength $F_\omega = \epsilon\omega_B^2$, which cover the range from weak to strong driving. In all cases the real parts approximately follow a cosine,

$$E(k) = E_0 + \Delta_E \cos(2\pi k). \tag{47}$$

The imaginary parts do so only for small ϵ, whereas for larger values strong modulations of the width as a function of the quasimomentum appear, which are due to the interaction with higher excited Wannier-Stark ladders. For small ϵ, we can approximate the width $\Gamma_0(k)$ of the ground band states by a cosine dispersion relation [12,26],

$$\Gamma(k) = \Gamma_0 + \Delta_\Gamma \cos(2\pi k), \tag{48}$$

and the absolute value of the amplitude modulation factor is given by

$$|\tilde{C}(k)|^2 = |b(k)|^2 \exp\left(-\frac{\Gamma_0 N \tau_B}{\hbar} - \frac{\Delta_\Gamma N \tau_B}{\hbar} \cos(2\pi k) \right).$$

If the interaction time $T = N\tau_B$ is large enough [and if $b(k)$ is sufficiently smooth], the strong modulation of the exponential dominates the form of the amplitude modulation factor. Then the wave function is periodically peaked in momentum space. Of course, such a periodically peaked structure is also found for larger values of ϵ where formula (48) is no longer valid. In fact, due to the stronger modulation of the width, it appears even for short interaction times.

The behavior in coordinate space is additionally modified by the dispersion due to the real parts of the quasienergies. If we approximate the real parts by Eq. (47) and again apply the stationary phase approximation in the Fourier transform of Eq. (46), the stationary points k_s are solutions of a slightly modified equation:

$$\frac{\hbar^2 k_s^2}{2} + 2\pi\Delta_E N \sin(2\pi k_s) = E_0 - Fx. \tag{49}$$

The implications are as follows. In coordinate space, the form of the peaks is changed compared to the dispersion-free case, in particular, the peaks can be broadened or narrowed. Which of both possibilities occurs can most easily be seen from the function

$$\frac{dx}{dk_s} = -\frac{\hbar^2 k_s}{F} - \frac{4\pi^2 \Delta_E N}{F} \cos(2\pi k_s). \tag{50}$$

Its absolute value relates the width of the peaks in momentum space to the width in coordinate space. Actually, for resonances only the stationary points with $k_s < 0$ are important, the others can be neglected. In this region, the absolute value of the function dx/dk_s takes minima at integer k_s, and maxima at half integer values. As can be seen from Fig. 4, the peaks in momentum space appear at integer k_s, and thus the dispersion additionally narrows the peaks.

Note that for small $|k_s|$ there may be three instead of one stationary point on each branch of the square root. Then the wave function shows additional interferences due to the interaction of the three different contributions. However, for $|k_s| > 4\pi^2 \Delta_E N/\hbar^2$ these interferences disappear. Furthermore, for large $|k_s|$, the dispersion only perturbatively influences the shape of the peaks because its contribution remains constant while the width increases proportionally to $|k_s|$. Thus, for large $|k_s|$, i.e., for large negative values of x, the shape of the peaks of the decay tail reflects the function $\Gamma(k)$.

B. Decay in driven tunnel arrays

Let us support the analysis by some numerical results that describe the following setup. An initial state is driven during a time $T = N\tau_B$ by an ac field, which is then switched off. The shape of the pulses that are formed are described by the amplitude modulation factor $b(k)\exp[-i\mathcal{E}(k)N\tau_B/\hbar]$. The figures show the wave function after additional $12\tau_B$ of undriven decay in order to make sure that the pulses shown are

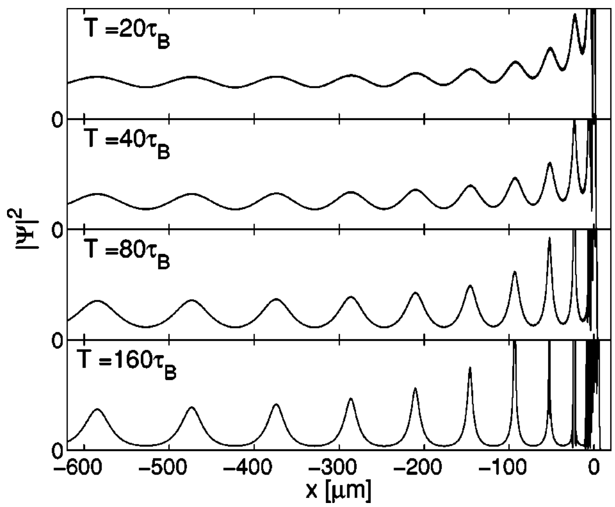

FIG. 5. Tail of the wave function after the system was driven by an ac field for different periods. The parameters are $\hbar = 3.3806$, $F = 0.0661$, and $\epsilon = 0.1$.

built after the ac field was switched off. Numerically, we started with Eq. (46) and assumed the function $b(k)$ to be constant. Explicitly, we calculated the resonances wave function $\Psi_0(k)$, multiplied by the amplitude modulation factor $\exp[-i\mathcal{E}(k)N\tau_B/\hbar]$, where the dispersion relation was calculated independently, and Fourier transformed the result into coordinate space.

Figure 5 shows the decay tails that develop for a weak ac field with $\epsilon = 0.1$. After short interaction times, the tail is slightly modulated. For longer interaction times, the modulation depth increases and pulses develop, which finally are clearly separated. Note that, apart from effects due to the dispersion, we can decrease the width of the pulses by further increasing the interaction time, which provides a simple way to tune the width experimentally.

A crucial point of the weak-field regime is the long ac driving time, which is needed to generate well-separated pulses. The relevant time scale is set by the most long-lived state from the ground band. For the case $\epsilon = 0.1$, the minimum width is $\Gamma_{min} = 7.214 \times 10^{-3}$, which corresponds to a lifetime of 10.0 ms, i.e., approximately 10 τ_B. Thus, the interaction time is much longer than the lifetime of the most stable state. Consequently, a predominant part of the initial wave function has already decayed before pulses are being formed. One can, however, surmount this problem by increasing the field strength of the ac field.

Figure 6 shows the decay tail for a strong ac field with $\epsilon = 1.5$. Now the pulses develop after much shorter interaction times. For $\epsilon = 1.5$, the function $\Gamma(k)$ has four minima (see Fig. 4), which are due to two crossings with higher excited Wannier-Stark ladders. Note that one can directly read off this property from the substructure of the pulses on the decay tail.

The interaction times necessary to generate separated pulses are much shorter for strong ac fields. In fact, now the necessary duration of the driving is even shorter than the lifetime of the most stable state. The minimum width for the case $\epsilon = 1.5$ is $\Gamma_{min} = 2.27 \times 10^{-3}$, which is less than one-

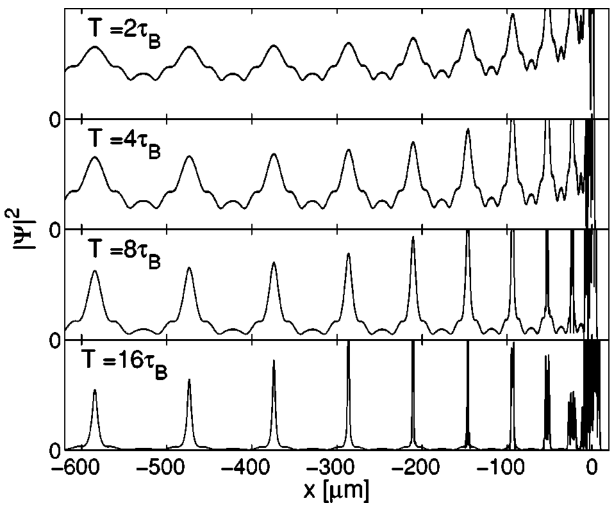

FIG. 6. Tail of the wave function after the system was driven by an ac field for different periods with strong field strength $\epsilon = 1.5$. The other parameters are as in Fig. 5.

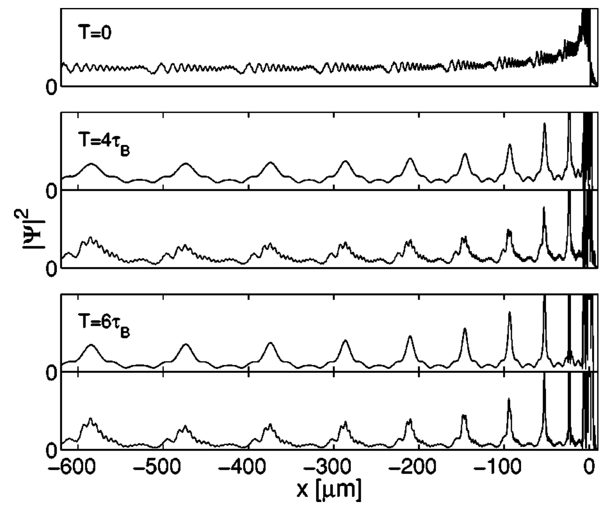

FIG. 8. Time evolution of wave packet (52) for $\sigma = 2$ and $x_0 = \pi$ after different periods of driving; the duration of the driving changes from $T=0$ (top) to $T=6\tau_B$ (bottom). In the last two cases, the numerical wave packet results (bottom subplot) are compared with the corresponding results from Eq. (46) (top subplot).

third of the minimum width for $\epsilon = 0.1$. The corresponding lifetime is 31.7 ms, which is approximately twice as large as the maximum interaction time in Fig. 6. Thus, in the strong-field regime, pulses develop before a substantial part of the wave function has decayed.

In the two lower panels of Fig. 6, one can clearly see the narrowing caused by the dispersion. In particular, the first peaks (counted from the right) strongly oscillate, which reflects the existence of three stationary points of the phase function in this region. However, the last peaks have approximately the same shape, i.e., here the narrowing affects the shape only perturbatively.

Thus, if the field strength is sufficiently large, the proposed setup seems to provide a tool to explore the experimental dependence of the lifetime on the quasimomentum. Figure 7 shows some possible shapes that develop if the field strength is increased from the perturbative to the strong-field regime. The first three cases correspond to the field strengths, the dispersion relations $\mathcal{E}(k)$ of which are shown in Fig. 4. With increasing field strength, the decay tail develops additional substructures. In particular, additional minima occur, which correspond to the avoided crossings with excited Wannier-Stark states. For $\epsilon = 0.8$, the effect mainly shows up in a strong deformation of the shape compared to the case $\epsilon = 0.4$. For $\epsilon = 1.5$, already four minima exist, i.e., two crossings with excited ladders. Finally, for $\epsilon = 2.2$ even a fifth minimum appears, which signalizes the occurrence of the next crossing. By comparing Figs. 4 and 6 we see that the relation $\Gamma(k)$ is mirrored in the shape of the decay tail.

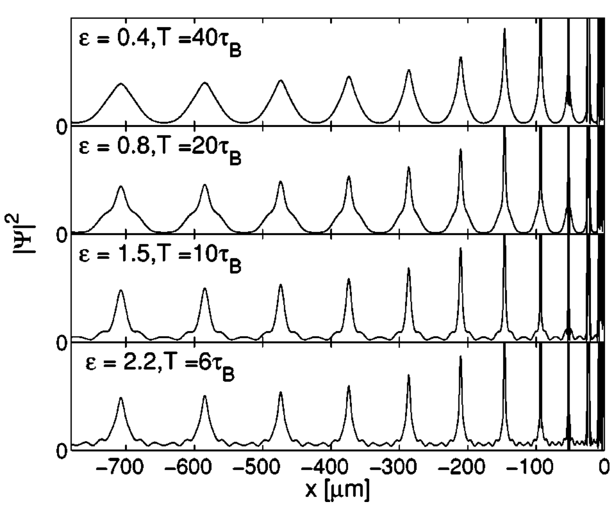

FIG. 7. Tail of the wave function after several periods of driving by ac fields with different strengths. While the field strength is increased, the interaction time is decreased such that the width of the peaks is comparable.

Up until now we have discussed results obtained directly from Eq. (46) where we set $b(k) = 1$. Let us compare these results with an exact time evolution of an initial state, i.e., a wave-packet propagation. For this we propagated an initial Gaussian wave packet,

$$\psi_\sigma(x) = \frac{1}{\sqrt[4]{\pi\sigma}} \exp\left(-\frac{(x-x_0)^2}{2\sigma} \right), \tag{51}$$

in the Wannier-Stark system after driving it for different periods with $\epsilon = 1.5$. Figure 8 compares the results obtained in this way for $\sigma = 2$ and $x_0 = \pi$, with the predictions of Eq. (46). Shown is the output from the BEC without driving ($T = 0$) and for an additional driving during times $T = 4\tau_B$ and $T = 6\tau_B$. In the last two cases, the numerical wave-packet results (bottom subplot) are compared with the corresponding results from Eq. (46) (top subplot).

Without driving ($T = 0$), the initial state decays without pulse formation. This changes if we turn on the driving ($T = 4\tau_B$ and $T = 6\tau_B$) after which a pulsed output develops. The positions, the widths, and the relative sizes of the different peaks are in perfect agreement with the predictions of Eq. (46). The difference in the structure reflects the contribution

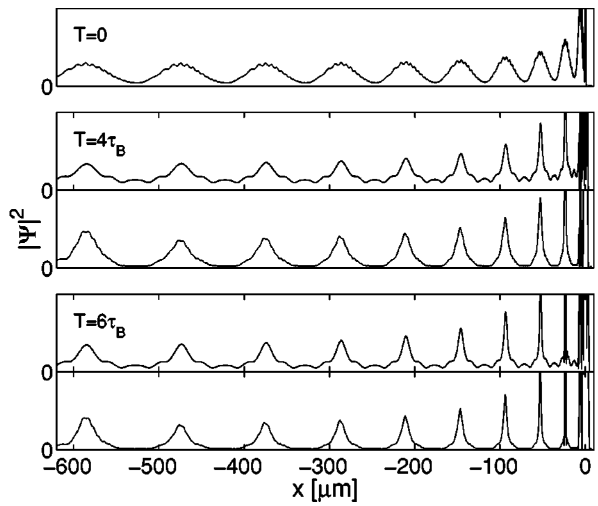

FIG. 9. Same as Fig. 8, however, for a wave packet more widely spread in coordinate space ($\sigma = 10$).

of $b(k)$ in the real quantum propagation. As it can be seen from the lower part of Fig. 8, this influence decreases with longer driving, as expected from Eq. (46).

Figure 9 shows a similar discussion for a Gaussian wave packet $\sigma = 10$, $x_0 = \pi$. Initially, this wave packet is more widely spread in coordinate space and has contributions from more than one state of the ground Wannier-Stark ladder. Therefore it shows an output in the form of broad pulses even without an initial driving ($T = 0$), as discussed in Sec. II. After driving the system for $T = 4\tau_B$ or $T = 6\tau_B$, we see that the shape of the pulses resembles the one predicted by Eq. (46). Minor differences can again be attributed to the role of the prefactor $b(k)$.

CONCLUSION

The wave function of a decaying state in a Stark system can be written as a superposition of resonance states truncated in momentum space. The movement of the edge of the truncation obeys a classical equation of motion. The states constructed in this way are—in contrast to the usual resonant or decaying states—normalizable. This result resembles the one obtained in Ref. [14] for scattering wave functions described in terms of Moshinsky functions.

By applying this general result to a Wannier-Stark system, we analyzed the decay in both momentum and coordinate space of a state that populates primarily the first Wannier-Stark ladder. In both representations the time evolution can be written as a product of the time evolution of a resonance state and the Fourier transform of the amplitudes of the initial population. The whole information about the time-dependent tunneling process is contained in the second factor. Using an appropriate initial distribution we were able to describe the experimental results of Ref. [1]. Finally, an experimental setup for preparing an initial state that shows regular pulse formation during decay using the interaction with an ac field is proposed. In addition, an analysis of the pulse shape may provide a method to experimentally access the function $\Gamma(k)$. We conclude that it should be possible to use the setup described to observe the pulsed decay in Wannier-Stark systems.

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