

Entanglement properties of some fractional quantum Hall liquidsBei Zeng,¹ Hui Zhai,² and Zhan Xu^{1,2}¹*Department of Physics, Tsinghua University, Beijing, 100084, China*²*Center for Advanced Study, Tsinghua University, Beijing, 100084, China*

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We study the entanglement properties of some fractional quantum Hall liquids. We calculate the entanglement of the Laughlin wave function and the wave functions that are generated by the K matrix using the modified entanglement measure of indistinguishable fermions that is first proposed by Paškauskas and You [Phys. Rev. **A64**, 042310 (2001)].

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I. INTRODUCTION

Entanglement is no doubt an intriguing property of composite quantum systems. It is a correlation that is stronger than any classical correlation. It is found that entanglement is not only of interests in the interpretation of the foundation of quantum mechanics, but also a resource useful in quantum information processing and quantum computation. Although the theory of entanglement is widely developed in the systems of distinguishable particles, only very recently the entanglement properties in identical particle systems began to attract much attention [1–6] in the fields of quantum information and quantum computation.

Quite recently, it is realized that applying the theory of entanglement developed in quantum information science to the field of condensed matter physics may give us new insight in these problems, especially the abundant entanglement properties of ground-state wave function and the wave functions related to the quantum phase transitions [7,8]. Some systems have already been extensively studied, such as the models of Heisenberg spin chain [9] and harmonic chain [10]. It should be emphasized that these systems are all related to the crystal lattice, thus the entanglement can be calculated using the side entropy suggested by Zanardi [11] or just using the measure of entanglement suggested by the entanglement theory of distinguishable particle systems by taking account of the fact that each site can be viewed as one “party” in quantum information science.

It is noticed that the fractional quantum Hall liquid is a kind of strongly correlated quantum fluid, in which quantum correlation plays an essential role [12]. As the external magnetic field perpendicular to the two-dimensional electron-gas increases, the Hall conductance, as well as the filling factor jumps from one value to another. Correspondingly, a change of the wave function of the system takes place. It is well known that the fractional quantum Hall effect with the filling fractions $\nu = 1/(\text{odd integer})$ has been explained by Laughlin [13]. Besides these states, the quantum Hall liquid possesses an extremely rich internal structure, which is classified in terms of so-called K matrix [14]. So far it is clear that the fractional quantum Hall system experiences quantum phase transition, and some characters of one fractional quantum Hall state are essentially different from those of another state. Seeking some order or index to describe these essential characters can facilitate further understanding of the quan-

tum phase transition. Many work has been done along this direction [15]. Recently it is noticed that the states in fractional quantum Hall liquid have very sophisticated entanglement properties [16], and different states may have different entanglement properties. Thus it attracts our attention to apply the theory of entanglement developed in quantum information science to investigate the entanglement properties in this system.

However, since there is no longer any site in the liquid, at first sight it seems that we cannot use the entanglement measure of distinguishable particle systems to calculate the entanglement of this system. Surprisingly, it is found by Paškauskas and You [3] that the von Neumann entropy of the reduced single-particle density matrix remains to be a good entanglement measure for two identical particles, which is a natural extension of the entanglement measure of distinguishable particles. Thus, this entanglement measure of identical particles is sufficient to meet our need here.

In this paper, we study the entanglement properties of the Laughlin wave function and the wave functions that are generated by the K matrix of quantum Hall liquid. First in Sec. II we will explain the meaning of the entanglement measure of indistinguishable fermions that is proposed by Paškauskas and You, and further point out the relationship between this measure and the measure used in the distinguishable particle systems. We will find that a slight modification is needed in the measure of Paškauskas and You. Then we calculate the entanglement of the Laughlin wave function and the wave functions that are generated by the K matrix in the simplest case that the particle number N is 2 in Sec. III using the slightly modified entanglement measure of indistinguishable fermions. In Sec. IV the more sophisticated case with the particle number larger than 2 is considered. Some discussions about our results will be given in Sec. V.

II. ENTANGLEMENT MEASURE OF INDISTINGUISHABLE FERMIONS

The first entanglement measure of indistinguishable fermions is introduced by Schliemann *et al.* [1,4]. In Ref. [1], they claimed that the concept of separability of a state in composite systems of fermions should be defined in terms of that the state can be expressed in the form of a single Slater determinant. In Ref. [4], they find that the wave function of two fermions can be written in the following standard form:

$$|\Psi\rangle = \frac{1}{\sqrt{\sum_{i=1}^k |z_i|^2}} \sum_{i=1}^k z_i f_{a_1(i)}^+ f_{a_2(i)}^+ |0\rangle, \quad (1)$$

where $f_{a_1(i)}^+ |0\rangle$ and $f_{a_2(i)}^+ |0\rangle$ represent the orthonormal basis of single-particle space. Following this idea, they defined an entanglement measure of two fermions as follows:

$$\eta(|\Psi\rangle) = |\langle \bar{\Psi} | \Psi \rangle|, \quad (2)$$

where $|\bar{\Psi}\rangle$ is the dual state of $|\Psi\rangle$. It can be verified that $\eta(|\Psi\rangle) = 0$ if and only if $|\Psi\rangle$ can be expressed in the form of a single Slater determinant and $\eta(|\Psi\rangle)$ is a smooth function ranging from $\eta(|\Psi\rangle) = 0$ for the separable state to $\eta(|\Psi\rangle) = 1$ for the maximum correlated state.

Later, Pařkauskas and You [3] suggested another entanglement measure for two fermions. They found that the von Neumann entropy of the reduced single-particle density-matrix remains to be a good entanglement measure for two identical particles. Thus, for the state (1), the entanglement can be calculated as

$$S_f^{PY} = -\text{tr}[\rho^f \ln \rho^f] = -\ln 2 - 4 \sum_{k=1}^{\leq n/2} |z_k/2|^2 \ln(|z_k/2|^2), \quad (3)$$

where n is the dimension of the single-particle space. S_f is also a smooth function ranging from $S_f = \ln 2$ for the separable state to $S_f = \ln n_e$ for the maximum correlated state, where n_e is the maximal even number less or equal to n .

However, a problem with this measure still remains, for the uncorrelated two-fermion state gives $S_f = \ln 2$ rather than 0. We point out here that this problem can be easily understood because it is noticed that the extra $\ln 2$ is just the entanglement contained in the antisymmetry of the wave function in the first quantized representation of two identical fermions. It is already known that this extra entanglement is of no use in quantum information processing. Since in identical particle systems it is important to find the entanglement beyond that involved in the (anti)symmetry induced by quantum statistics, we need to get rid of this extra $\ln 2$, i.e., this entanglement measure needs a slight modification

$$S_f = -\text{tr}[\rho^f \ln \rho^f] - \ln 2 = -2 \ln 2 - 4 \sum_{k=1}^{\leq n/2} |z_k/2|^2 \ln(|z_k/2|^2). \quad (4)$$

It must be emphasized that although this modification is quite slight, it is very important in the sense that this modification enables us to obtain the same value of entanglement measure when the system of identical particles is reduced to a system of distinguishable particles. To be more concrete, consider the following state of two-qubits written in the form of the Schmidt decomposition

$$|\psi\rangle = \alpha |00\rangle + \beta |11\rangle \quad (5)$$

whose second quantized counterpart is

$$|\psi\rangle = (\alpha a^\dagger b^\dagger + \beta c^\dagger d^\dagger) |0\rangle, \quad (6)$$

where $|\alpha|^2 + |\beta|^2 = 1$.

It is easy to get

$$S_f(|\psi\rangle) = -|\alpha|^2 \ln |\alpha|^2 - |\beta|^2 \ln |\beta|^2. \quad (7)$$

This is in accordance with the von Neumann entropy of two-qubit state (5). It is also a very easy task to show that when the two fermions are distinguishable, what the modified entanglement measure (4) give us is just the von Neumann entropy. In this sense we can say the modified entanglement measure (4) is the measure suitable for two-fermion systems, either identical or distinguishable.

It is also noticed that to calculate entanglement of two fermions with Eq. (4) has another three advantages. First, we can see from Eq. (4) the close relationship between this measure and the measure used in the distinguishable particle systems. Second, we can either calculate this measure directly from the first quantized representation using the skills developed to calculate the entanglement measure of distinguishable particles or from the second quantized representation by just calculating the one-particle density matrix and diagnosing it. Third, this measure can be extended to multiparticle case naturally to calculate the entanglement between one particle and the other particles in the system with a slight modification

$$S'_f = -\text{tr}[\rho^f \ln \rho^f] - \ln N, \quad (8)$$

where N is the particle number. It is noticed that the entanglement measure (2) cannot be directly extended to multiparticle case [6].

Although in multiparticle case this measure cannot give us all the entanglement properties of the state, we can get at least some information about the entanglement properties of the state. In fact it is well known that in both the distinguishable and identical particle case the question of quantify the multiparticle entanglement is still open.

Considered all in all, we will use Eqs. (4) and (5) to explore the entanglement properties in quantum Hall liquid.

III. ENTANGLEMENT IN QUANTUM HALL LIQUID ($N=2$ CASE)

The Laughlin wave function for a quantum Hall liquid with filling factor $1/m$ reads

$$\psi_m(z_1, \dots, z_N) = \prod_{j < k} (z_j - z_k)^m \exp\left(-\frac{1}{4} \sum_i |z_i|^2\right), \quad (9)$$

where z_i s are complex coordinates of the electrons and m is a positive odd number. It is already known that the case $m = 1$ corresponds to the integer quantum Hall effect, where the Laughlin wave function is just a single Slater determinant, i.e.,

$$\psi_1(z_1, \dots, z_N) = \exp\left(-\frac{1}{4} \sum_i |z_i|^2\right) \times \det \begin{pmatrix} 1 & z_1 & \dots & z_1^N \\ \dots & \dots & \dots & \dots \\ 1 & z_N & \dots & z_N^N \end{pmatrix}, \quad (10)$$

thus ψ_1 is separable.

However, when $m \neq 1$, ψ_m cannot be expressed in the form of a single Slater determinant, i.e., it is entangled. In order to explore the entanglement properties of Laughlin wave function in the case $m > 1$, we need to calculate the amount of entanglement contained in these functions. First, we consider the simplest case that the particle number $N = 2$, then the Laughlin wave function will be

$$\begin{aligned} \psi_m(z_1, z_2) &= (z_1 - z_2)^m \exp\left(-\frac{1}{4}(|z_1|^2 + |z_2|^2)\right) \\ &= \sum_{k=0}^m (-1)^k C_m^k z_1^{m-k} z_2^k \exp\left[-\frac{1}{4}(|z_1|^2 + |z_2|^2)\right] \\ &= \sum_{k=0}^{(m-1)/2} (-1)^k C_m^k (z_1^{m-k} z_2^k - z_1^k z_2^{m-k}) \\ &\quad \times \exp\left[-\frac{1}{4}(|z_1|^2 + |z_2|^2)\right] \\ &= \left[\sum_{k=0}^{(m-1)/2} (-1)^k C_m^k \det \begin{pmatrix} z_1^{m-k} & z_1^k \\ z_2^{m-k} & z_2^k \end{pmatrix} \right] \\ &\quad \times \exp\left[-\frac{1}{4}(|z_1|^2 + |z_2|^2)\right]. \end{aligned} \quad (11)$$

Obviously, to calculate the measure of entanglement (4) from the first quantization point of view is a very difficult task, for this is a problem concerning continuous variable entanglement which always cannot be obtained analytically. Then we need to tackle this problem from the second quantization point of view. Fortunately, if the one-particle space is chosen appropriately, the calculation can be done within a finite dimensional space. It is noticed that these functions in the lowest Landau level with different angular momentum are written as

$$\{f_i(z) = A_i z^i \exp(-\frac{1}{4}|z|^2)\}_{i=0}^m. \quad (12)$$

The family of functions $f_i(z), i=0, \dots, m$ form an orthogonal basis of one-particle space, where A_i is the normalization factor, i.e.,

$$\begin{aligned} A_i &= \frac{1}{\sqrt{\int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} (x^2 + y^2)^i \exp\left[-\frac{1}{2}(x^2 + y^2)\right] dx dy}} \\ &= \frac{1}{\sqrt{\pi 2^{i+1} i!}} \end{aligned} \quad (13)$$

and $z = x + iy$.

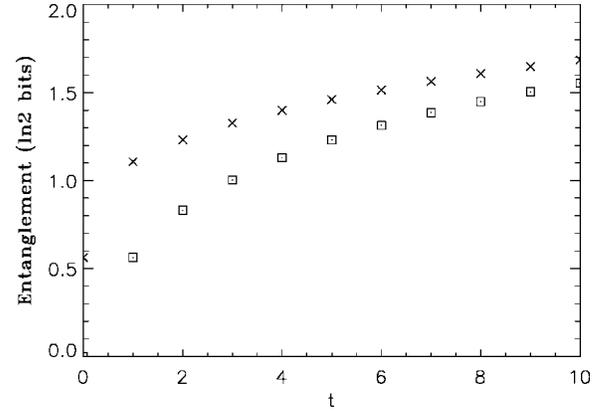


FIG. 1. The $N=2$ case: The variations of $S_f(|\psi_m\rangle)$ and $S_f(|\phi_m\rangle)$ with $t=(m-1)/2$ in the units of $\ln 2$ bits. The boxes represent the value of $S_f(|\psi_m\rangle)$ of the Laughlin wave functions and the crosses represent the value of $S_f(|\phi_m\rangle)$ of the wave functions generated by the K matrix.

Define a set of creation operators $\{a_i^\dagger\}_{i=0}^m$ corresponding to $f_i(z)$ by

$$\langle z | a_i^\dagger | 0 \rangle = f_i(z). \quad (14)$$

Thus we can rewrite ψ_m in the second quantized form as

$$\begin{aligned} |\psi_m\rangle &= \left(\frac{1}{\sqrt{\sum_{k=0}^{(m-1)/2} \frac{(C_m^k)^2}{A_k^2 A_{m-k}^2}}} \right) \\ &\quad \times \sum_{k=0}^{(m-1)/2} (-1)^k \left[\frac{C_m^k}{A_k A_{m-k}} \right] a_{m-k}^\dagger a_k^\dagger | 0 \rangle \\ &= 2^{-[(m-1)/2]} \sum_{k=0}^{(m-1)/2} \sqrt{C_m^k} a_{m-k}^\dagger a_k^\dagger | 0 \rangle. \end{aligned} \quad (15)$$

It is easy to find that $|\psi_m\rangle$ in Eq. (12) has the form of the Slater decomposition (1), thus we can calculate the entanglement of $|\psi_m\rangle$ directly using formula (4)

$$\begin{aligned} S_f(|\psi_m\rangle) &= -2 \ln 2 - 2^{(m-1)} \sum_{k=0}^{(m-1)/2} C_m^k \ln(2^{-(m+1)} C_m^k) \\ &= -2 \ln 2 - 2^{-m} \sum_{k=0}^m \ln(2^{-(m+1)} C_m^k). \end{aligned} \quad (16)$$

The variation of $S_f(|\psi_m\rangle)$ with $t=(m-1)/2$ is shown in Fig. 1. We can see from Fig. 1 that the entanglement increases with m increasing.

We know that a quasihole excitation above $|\psi_m\rangle$ is described by $\psi(\xi) = \sqrt{N(\xi, \xi^*)} \prod_i (\xi - z_i) \psi_m$. These quasiholes can form new quantum fluid in the second level, leading to more complicated filling fraction, such as $\nu = \frac{7}{2}$. It is well known that there are hierarchical fractional quantum Hall states. The wave functions of such states can be constructed

with the help of K -matrix [14]. As an example, the state characterized by using $K = \begin{pmatrix} m & 1 \\ 1 & -2 \end{pmatrix}$ has a filling fraction

$$\nu = \frac{1}{m - (-\frac{1}{2})} = \frac{2}{2m+1},$$

and is described by the generalized Laughlin wave function:

$$\begin{aligned} \phi_m(z_1, z_2) = & (z_1 - z_2)^m \int \int d\xi_1 d\xi_2 (\xi_1 - z_1)(\xi_1 - z_2)(\xi_2 \\ & - z_1)(\xi_2 - z_2)(\xi_1^* - \xi_2^*)^2 \exp[-\frac{1}{3}(|\xi_1|^2 \\ & + |\xi_2|^2)] \exp[-\frac{1}{4}(|z_1|^2 + |z_2|^2)]. \end{aligned} \quad (17)$$

Physically, these states have two quasipoles above the $|\psi_m\rangle$ state, and then these quasipoles form quantum Hall liquid with $\nu = \frac{1}{2}$. Integrating over the coordinates of the quasipoles, one obtains a new dancing pattern of these electrons which is different from that of $|\psi_m\rangle$. Naturally, it leads to a change of the entanglement property. It is found that

$$\begin{aligned} \int \int d\xi_1 d\xi_2 (\xi_1 - z_1)(\xi_1 - z_2)(\xi_2 - z_1)(\xi_2 - z_2)(\xi_1^* \\ - \xi_2^*)^2 \exp(-\frac{1}{4}(|\xi_1|^2 + |\xi_2|^2)) = -162\pi(z_1^2 + z_2^2). \end{aligned} \quad (18)$$

Thus $\phi_m(z_1, z_2)$ can be rewritten as

$$\phi_m(z_1, z_2) = (z_1 - z_2)^m (z_1^2 + z_2^2) \exp[-\frac{1}{4}(|z_1|^2 + |z_2|^2)]. \quad (19)$$

In the second quantized form the normalized state vector of $\phi_m(z_1, z_2)$ reads

$$\begin{aligned} |\phi_m\rangle = & \left(\frac{1}{\sqrt{\sum_{k=0}^{(m+1)/2} \frac{(C_m^k + C_m^{k-2})^2}{A_k^2 A_{m-k+2}^2}}} \right) \\ & \times \sum_{k=0}^{(m+1)/2} \left[\frac{C_m^k + C_m^{k-2}}{A_k A_{m-k+2}} \right] a_{m-k+2}^\dagger a_k^\dagger |0\rangle. \end{aligned} \quad (20)$$

Obviously, $|\phi_m\rangle$ in Eq. (17) also has the form of the Slater decomposition (1), thus we can calculate the entanglement of $|\phi_m\rangle$ directly using formula (4),

$$\begin{aligned} S_f(|\phi_m\rangle) = & -4 \sum_{k=0}^{(m+1)/2} \left(\frac{\frac{(C_m^k + C_m^{k-2})^2}{A_k^2 A_{m-k+2}^2}}{2 \left(\sum_{k=0}^{(m+1)/2} \frac{(C_m^k + C_m^{k-2})^2}{A_k^2 A_{m-k+2}^2} \right)} \right) \\ & \times \ln \left(\frac{\frac{(C_m^k + C_m^{k-2})^2}{A_k^2 A_{m-k+2}^2}}{2 \left(\sum_{k=0}^{(m+1)/2} \frac{(C_m^k + C_m^{k-2})^2}{A_k^2 A_{m-k+2}^2} \right)} \right) - 2 \ln 2. \end{aligned} \quad (21)$$

The variation of $S_f(|\phi_m\rangle)$ with $t = (m-1)/2$ is also shown in Fig. 1. We can see from Fig. 1 that the entanglement also increases when m is increasing. Comparing the two curves in Fig. 1 we can find that for each m , $S_f(|\phi_m\rangle) > S_f(|\psi_m\rangle)$. This fact can be understood by noting that adding a hierarchical structure will result in the increase of quantum entanglement. It is interesting to notice that $S_f(|\psi_3\rangle) = S_f(|\phi_1\rangle)$. This can also be found from their explicit expression, i.e.,

$$\psi_3(z_1, z_2) = (z_1 - z_2)^3 \sim (a_0^\dagger a_3^\dagger + \sqrt{3} a_1^\dagger a_2^\dagger) |0\rangle, \quad (22)$$

$$\phi_1(z_1, z_2) = (z_1 - z_2)(z_1^2 + z_2^2) \sim (\sqrt{3} a_0^\dagger a_3^\dagger + a_1^\dagger a_2^\dagger) |0\rangle. \quad (23)$$

IV. ENTANGLEMENT IN QUANTUM HALL LIQUID ($N > 2$ CASE)

Now we turn to the much more difficult case $N > 2$. As mentioned above, we can only get some information of the entanglement between one particle and other particles using Eq. (5). We again use second quantized representation to calculate the entanglement of this special kind of multiparticle states.

Take $N = 3$ Laughlin wave function, for example,

$$\begin{aligned} \psi'_m(z_1, z_2, z_3) = & (z_1 - z_2)^m (z_1 - z_3)^m (z_2 - z_3)^m \\ & \times \exp[-\frac{1}{4}(|z_1|^2 + |z_2|^2 + |z_3|^2)]. \end{aligned} \quad (24)$$

We have

$$S'_f(|\psi'_m\rangle) = -tr[\rho^f \ln \rho^f] - \ln 3. \quad (25)$$

The second quantized form of $|\psi'_3\rangle$ is

$$\begin{aligned} |\psi'_3\rangle = & \left(\frac{1}{A_0 A_3 A_6} a_0^\dagger a_3^\dagger a_6^\dagger + \frac{3}{A_0 A_4 A_5} a_0^\dagger a_4^\dagger a_5^\dagger \right. \\ & + \frac{6}{A_1 A_3 A_5} a_1^\dagger a_3^\dagger a_5^\dagger + \frac{3}{A_1 A_2 A_6} a_1^\dagger a_2^\dagger a_6^\dagger \\ & \left. + \frac{15}{A_2 A_3 A_4} a_2^\dagger a_3^\dagger a_4^\dagger \right) |0\rangle. \end{aligned} \quad (26)$$

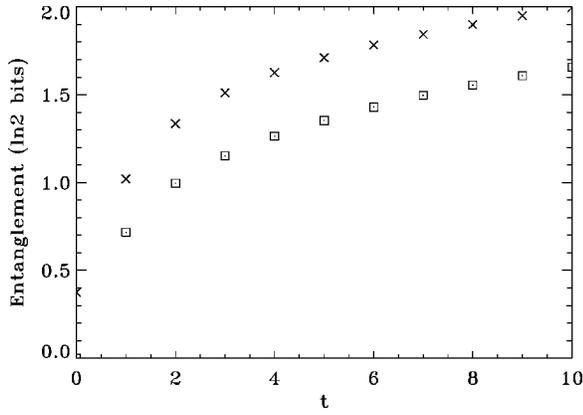


FIG. 2. The $N=3$ case: The variations of $S'_f(|\psi'_m\rangle)$ and $S_f(|\phi'_m\rangle)$ with $t=(m-1)/2$ in the units of $\ln 2$ bits. The boxes represent the value of $S'_f(|\psi'_m\rangle)$ of the Laughlin wave functions and the crosses represent the value of $S'_f(|\phi'_m\rangle)$ of the wave functions generated by the K matrix.

It is noted for $|\psi'_3\rangle$, the single-particle density matrix (up to a normalization factor) $\rho_{\mu\nu}^f = \langle \psi'_3 | a_{\mu}^{\dagger} a_{\nu} | \psi'_3 \rangle$ has no off-diagonal elements, thus $S'_f(|\psi'_3\rangle)$ can easily be calculated from the form of $|\psi'_3\rangle$.

In fact, since $\psi'_m(z_1, z_2, z_3)$ is a homogeneous polynomial of z_1, z_2 , and z_3 apart from an exponential factor, there will be no off-diagonal elements of the single-particle density matrix, for arbitrary positive odd value of m . This is also true for any particle number N . This fact makes the calculation of S'_f much easier. Our result for the $N=3$ case is shown in Fig. 2. It can be seen from Fig. 2 that S'_f also increases when m is increasing.

Similar things happen in the calculation of S'_f of a state that is generated by the same kind of K -matrix mentioned above, since in these wave functions the polynomial is also homogeneous. Take $N=3$, for example,

$$\begin{aligned} \phi'_m(z_1, z_2, z_3) &= (z_1 - z_2)^m (z_1 - z_3)^m (z_2 - z_3)^m \\ &\times \exp\left[-\frac{1}{4}(|z_1|^2 + |z_2|^2 + |z_3|^2)\right] \\ &\times \int \int d\xi_1 d\xi_2 (\xi_1 - z_1)(\xi_1 - z_2)(\xi_1 - z_3) \\ &\times (\xi_2 - z_1)(\xi_2 - z_2)(\xi_2 - z_3)(\xi_1^* - \xi_2^*)^2 \\ &\times \exp\left[-\frac{1}{3}(|\xi_1|^2 + |\xi_2|^2)\right]. \end{aligned} \quad (27)$$

Since

$$\begin{aligned} &\int \int d\xi_1 d\xi_2 (\xi_1 - z_1)(\xi_1 - z_2)(\xi_1 - z_3)(\xi_2 - z_1)(\xi_2 - z_2) \\ &\times (\xi_2 - z_3)(\xi_1^* - \xi_2^*)^2 \exp\left[-\frac{1}{3}(|\xi_1|^2 + |\xi_2|^2)\right] \\ &= -162\pi(z_1^2 z_2^2 + z_1^2 z_3^2 + z_2^2 z_3^2). \end{aligned} \quad (28)$$

We get

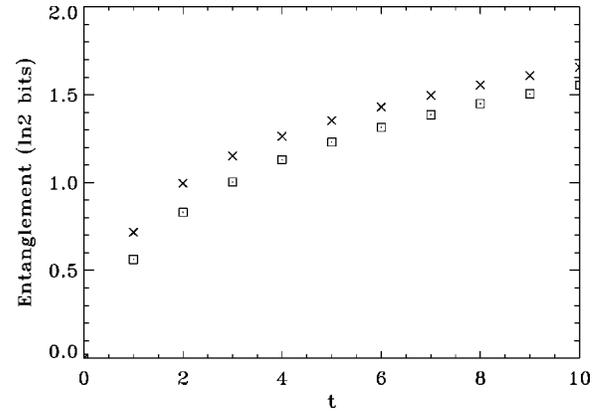


FIG. 3. The variation of $S_f(|\psi_m\rangle)$ and $S'_f(|\psi'_m\rangle)$ of the Laughlin wave functions with $t=(m-1)/2$ in the units of $\ln 2$ bits. The boxes represent the value of $S_f(|\psi_m\rangle)$ of the $N=2$ case and the crosses represent the value of $S'_f(|\psi'_m\rangle)$ of the $N=3$ case.

$$\begin{aligned} \phi'_m(z_1, z_2, z_3) &= (z_1 - z_2)^m (z_1 - z_3)^m (z_2 - z_3)^m (z_1^2 z_2^2 + z_1^2 z_3^2 \\ &+ z_2^2 z_3^2) \exp\left[-\frac{1}{4}(|z_1|^2 + |z_2|^2)\right], \end{aligned} \quad (29)$$

which is also a homogeneous polynomial of z_1, z_2 , and z_3 apart from an exponential factor.

Thus the calculation of S'_f of this kind of wave functions is also quite easy. Our result for $N=3$ case is shown in Fig. 2. It can be seen from Fig. 2 that S'_f also increases when m is increasing. However, we can see that at this time $S'_f(|\psi'_3\rangle)$ is no longer equal to $S'_f(|\phi'_1\rangle)$. Comparing the two curves in Fig. 2 we can find that for each m , the relation $S'_f(|\psi'_m\rangle) > S'_f(|\phi'_m\rangle)$ still holds.

The comparison of $N=2$ and $N=3$ cases for the Laughlin wave function is shown in Fig. 3. It can be seen that the amount of entanglement between one electron and the other electrons increases with the electron number N . The K matrix case is shown in Fig. 4. However, this case is quite sophisticated since the amount of entanglement between one elec-

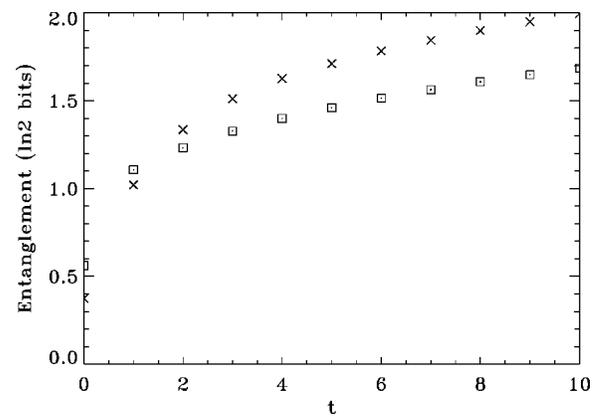


FIG. 4. The variation of $S_f(|\phi_m\rangle)$ and $S'_f(|\phi'_m\rangle)$ of the wave functions generated by the $(m=3)$ K matrix with $t=(m-1)/2$ in the units of $\ln 2$ bits. The boxes represent the value of $S_f(|\phi_m\rangle)$ of the $N=2$ case and the crosses represent the value of $S'_f(|\phi'_m\rangle)$ of the $N=3$ case.

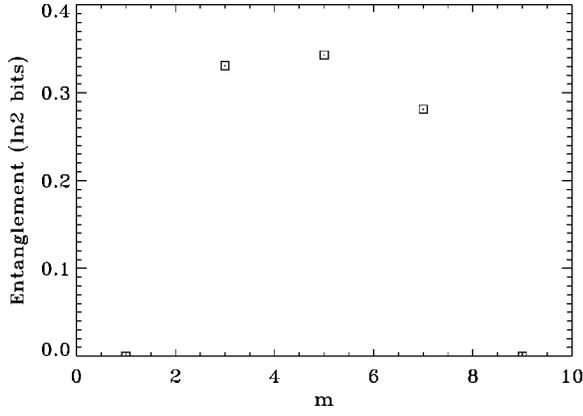


FIG. 5. The $N=4$ case: The variation of $S'_f(|\chi_m\rangle)$ of the wave function generated by the K matrix in the units of $\ln 2$ bits.

tron and the other electrons only increases with the electron number N besides the case $m=3$.

Now we turn to another kind of dancing patterns of these strongly correlated electrons. We have already shown that the $\nu=1$ state is a separable state, while if there exists some quasihole excitation above it, and then these quasiholes form a Laughlin-type state with filling fraction $\nu=1/(m-1)$, where m is an odd integer. It will result in entanglement between electrons. The K matrix is written as $\begin{pmatrix} 1 & \\ & -(m-1) \end{pmatrix}$, and the corresponding filling fractions are $\nu=1/[1-(1/m-1)]$. Thus the wave functions associated with this K matrix take the form (take the case $N=4$, for example)

$$\begin{aligned} \chi_m(z_1, z_2, z_3, z_4) = & (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)(z_1 - z_4)(z_2 \\ & - z_4)(z_3 - z_4) \exp\left[-\frac{1}{4}(|z_1|^2 + |z_2|^2 \right. \\ & \left. + |z_3|^2 + |z_4|^2)\right] \int \int d\xi_1 d\xi_2 (\xi_1 - z_1)(\xi_1 \\ & - z_2)(\xi_1 - z_3)(\xi_1 - z_4)(\xi_2 - z_1)(\xi_2 - z_2) \\ & \times (\xi_2 - z_3)(\xi_2 - z_4)(\xi_1^* - \xi_2^*)^{(m-1)} \\ & \times \exp\left[-\frac{1}{3}(|\xi_1|^2 + |\xi_2|^2)\right]. \end{aligned} \quad (30)$$

Using the method above we can also get some information of the entanglement properties these wave functions. For $S'_f(|\chi_m(z_1, z_2, z_3, z_4)\rangle)$, our result is shown in Fig. 5. It is similar to that of Fig. 1; this is consistent with the particle-

hole duality picture in quantum Hall liquid. The eventually vanishing of S'_f is due to the effect of finite number of electrons. It is easy to find that when $m > 2N+1$

$$\begin{aligned} & \int \int d\xi_1 d\xi_2 \prod_{i=1}^N (\xi_1 - z_i)(\xi_2 - z_i)(\xi_1^* - \xi_2^*)^{(m-1)} \\ & \times \exp\left(-\frac{1}{3}(|\xi_1|^2 + |\xi_2|^2)\right) = 0. \end{aligned} \quad (31)$$

This fact means that the $\nu=1$ N -electron system cannot support a Laughlin-type wave function of quasiholes when m is larger than $2N+1$.

V. DISCUSSIONS

We calculated the entanglement of the Laughlin wave function and the wave functions that are generated by the K matrix using the modified entanglement measure of indistinguishable fermions that is first proposed by Pařkauskas and You [3] through the second quantized approach in this paper. It is noticed that the fractional quantum Hall effect occurs when the external magnetic field is sufficiently high, the degeneracy of the lowest Landau level is large enough that the problem can be treated in the subspace of lowest Landau level. This leads to the fact that the wave function can be expressed in an elegant form which is an analytical homogeneous polynomial of its arguments apart from a Gaussian factor, and results in a clean second quantized form without any off-diagonal elements. This property enables us to write down the entanglement measure in an analytical way.

Our result shows that for both kinds of wave functions, the amount of entanglement contained in each kind of wave function increases with the increase of the parameter m . However, since it is well known that the filling fraction changes dramatically with the increase of external magnetic field B , the entanglement properties is indeed very sophisticated in the system of quantum Hall liquid. It is expected that our results and methods can shed light on further studies of quantum orders in quantum Hall liquid and other physical systems.

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