Unconditional continuous-variable dense coding

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We investigate the conditions under which unconditional dense coding can be achieved using continuous-variable entanglement. We consider the effect of entanglement impurity and detector efficiency and discuss experimental verification. We conclude that the requirements for a strong demonstration are not as stringent as previously thought, and are within the reach of present technology.

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I. INTRODUCTION

The classical channel capacity of a quantum channel can be enhanced if the sender and recipient of the information, Alice and Bob, repectively, share an entangled state. This effect is known as quantum dense coding [1], and can be thought of as the converse problem to quantum teleportation [2] where, effectively, the quantum capacity of a classical channel is enhanced by the use of entanglement.

Dense coding was originally introduced for discrete variables and an experimental demonstration of the effect has been made using photonic polarization entanglement [3]. One drawback of this demonstration was, due to the low efficiency of entanglement production and detection, the demonstration was conditional on Bob detecting a pair of photons, a rare event. In contrast a dense coding scheme based on continuous variables, such as the quadrature amplitudes of a light field, which has recently been proposed, would in principle demonstrate an unconditional improvement in classical channel capacity [4]. Ultimately this scheme can beat, under certain conditions, the maximum channel capacity given by Fock-state encoding. However, the conditions for this strong violation found in Ref. [4] required unrealistic levels of squeezing.

A number of groups have taken steps towards the experimental implementation of this scheme [5,6]. In these experiments increased signal to noise was demonstrated with the addition of entanglement and conclusions were drawn about the violation of coherent-state classical capacity, based on the results of Ref. [4]. However, unit entanglement state purity and detection efficiency were assumed in Ref. [4], which is unlikely to have been the case experimentally. Also no attempt was made to experimentally quantify the number of quanta used in the communication channel. There is thus a need for a more detailed analysis.

In this paper we make such an investigation. We come to the rather surprizing conclusion that in fact the conditions required for a strong demonstration of the effect, i.e., beating the ultimate channel capacity given by Fock-state encoding, are not as stringent as previously thought, even taking into account lack of state purity and nonunit detection efficiency.

II. IDEAL CHANNEL CAPACITIES

We begin by rederiving the channel capacities of Gaussian quantum channels and continuous variable dense coding using quadrature spectral variances. Such variances are directly measureable in an experiment. The Shannon capacity [7] of a communication channel with Gaussian noise of power (variance) N and Gaussian distributed signal power S operating at the bandwidth limit is

$$C=\frac{1}{2}\log_2\left[1+\frac{S}{N}\right]. \tag{1}$$

Equation (1) can be used to calculate the channel capacities of quantum states with Gaussian probability distributions such as coherent states and squeezed states [8,9]. Consider first a signal composed of a Gaussian distribution of coherent-state amplitudes all with the same quadrature angle [see Fig. 1 (a)]. The signal power V_s is given by the variance of the distribution. The noise is given by the intrinsic quantum noise of the coherent states and is defined to be $V_n=1$. Because the quadrature angle of the signal is known, homodyne detection can, in principle, detect the the signal without further penalty. Thus the measured signal-to-noise ratio is $S/N=V_s/V_n=V_s$.

In general the average photon number per bandwidth per second of a light beam is given by

$$\bar{n}=\frac{1}{4}(V^+ + V^-)-\frac{1}{2}, \tag{2}$$

where V^+ (V^-) are the variances of the maximum (minimum) quadrature projections of the noise ellipse of the state. These projections are orthogonal quadratures, such as amplitude and phase, and obey the uncertainty principle $V^+V^-\geqslant 1$. In the above example one quadrature is made up of signal plus quantum noise such that $V^+=V_s+1$ whilst the orthogonal quadrature is just quantum noise so $V^-=1$.

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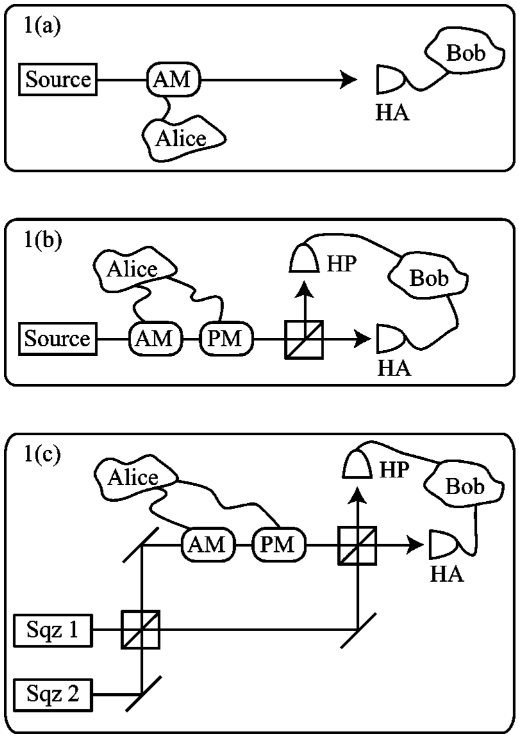

FIG. 1. Schematic diagrams of (a) coherent homodyne, (b) coherent heterodyne, (c) dense coding schemes for a communication channel. The abbreviations are AM, amplitude modulation; PM, phase modulation; HA, coherent homodyne detection of the amplitude quadrature; HP, coherent homodyne detection of the phase quadrature. The beam splitters are taken to be 50% transmitting and the two squeezed sources are squeezed in orthogonal quadratures.

Hence $\bar{n}=1/4V_s$ and so the channel capacity of a coherent state with single quadrature encoding and homodyne detection is

$$C_c=\log_2[\sqrt{1+4\bar{n}}]. \tag{3}$$

Establishing in an experiment that a particular optical mode has this capacity would involve: (i) measuring the quadrature amplitude variances of the beam, V^+ and V^-, (ii) calibrating Alice's signal variance, and (iii) measuring Bob's signal-to-noise ratio. If these measurments agreed with the theoretical conditions above then Shannons theorem tells us that an encoding scheme exists which could realize the channel capacity of Eq. (3). An example of such an encoding is given in Ref. [10].

For photon numbers $\bar{n}>2$ improved channel capacity can be obtained by encoding symmetrically on both quadratures and detecting both quadratures simultaneously using heterodyne detection or dual homodyne detection [see Fig. 1(b)]. Because of the noncommutation of orthogonal quaratures there is a penalty for their simultaneous detection which reduces the signal-to-noise ratio of each quadrature to $S/N=1/2V_s$. Also because there is signal on both quadratures the average photon number of the beam is now $\bar{n}=1/2V_s$. On the other hand the total channel capacity will now be the sum of the two independent channels carried by the two quadratures. Thus the channel capacity for a coherent state with dual quadrature encoding and heterodyne detection is

$$C_{ch}=\frac{1}{2}\log_2\left[1+\frac{S}{N}^+\right]+\frac{1}{2}\log_2\left[1+\frac{S}{N}^-\right]=\log_2[1+\bar{n}], \tag{4}$$

which exceeds that of the homodyne technique [Eq. (3)] for $\bar{n}>2$.

The above channel capacities are the best achievable if we restrict ourselves to a semiclassical treatment of light. However, the channel capacity of the homodyne technique [Fig. 1(a)] can be improved by the use of nonclassical, squeezed light. With squeezed light the noise variance of the encoded quadrature can be reduced such that $V_{ne}<1$, while the noise of the unencoded quadrature is increased such that $V_{nu}\geqslant 1/V_{ne}$. As a result the signal-to-noise ratio is improved to $S/N=V_s/V_{ne}$ while the photon number is now given by Eq. (2) but with $V^+=V_s+V_{ne}$ and $V^-=1/V_{ne}$ where a pure (i.e., minimum uncertainty) squeezed state has been assumed. Maximizing the signal-to-noise ratio for fixed $\bar{n}$ leads to $S/N=4(\bar{n}+\bar{n}^2)$ for a squeezed quadrature variance of $V_{ne,opt}=1/(1+2\bar{n})$. Hence the channel capacity for a squeezed beam with homodyne detection is

$$C_{sh}=\log_2[1+2\bar{n}], \tag{5}$$

which exceeds both coherent homodyne and heterodyne for all values of $\bar{n}$.

A final improvement in channel capacity can be obtained by allowing non-Gaussian states. The absolute maximum channel capacity for a single mode is given by the Holevo bound and can be realized by encoding in a maximum entropy ensemble of Fock states and using photon number detection [8,9,11]. This ultimate channel capacity is

$$C_{Fock}=(1+\bar{n})\log_2[(1+\bar{n})]-\bar{n}\log_2[\bar{n}], \tag{6}$$

which is the maximal channel capacity at all values of $\bar{n}$.

We now turn to dense coding. At low-average photon numbers the single-channel capacities are always best. However, we will find that for sufficiently high-average photon numbers dense coding can give superior capacities. The setup is depicted in Fig. 1(c). Entanglement is generated in the standard way by mixing two squeezed states, with their squeezing ellipses orthogonal, on a 50:50 beam splitter [12]. One half of the entangled pair is sent to Alice who encodes on both quadratures in the manner of coherent heterodyne. She sends the beam on to Bob who has also received the other half of the entangled pair. He uses a dual homodyne technique to measure both quadratures of the beam from Alice, but injects his entangled beam into the empty port of the dual homodyne beam splitter. The resulting signal-to-noise ratio for the two quadrature channels is $S/N=\frac{1}{2}(V_s/V_{ne})$, where now V_{ne} is the variance of the squeezed quadrature of the beams used to create the entanglement. The photon number is just that of the beam carrying the signal (the cost of distributing the entanglement is not taken into account) and so is given by Eq. (2) with $V^{+}=\frac{1}{2}V_s+\frac{1}{4}V_{ne}$ and $V^{-}=1/V_{ne}$. Once again pure squeezed states are assumed. Maximizing the signal-to-noise ratio for fixed $\bar{n}$ gives $S/N=\bar{n}+\bar{n}^2$ for a squeezed quadrature variance of $V_{ne,opt}=1/(1+2\bar{n})$. So the optimum channel capacity for dense coding is

$$C_{dc}^{opt}=\log_2[1+\bar{n}+\bar{n}^2], \tag{7}$$

which exceeds the coherent-state homodyne for $\bar{n}>0.478$, which can be achieved with $V_{ne}\approx 0.5$ (about 50% squeezing), and always exceeds the coherent-state heterodyne channel capacity. Dense coding beats the squeezed state channel capacity with $\bar{n}>1$ (achieved with $V_{ne}\approx 0.33$ or about 67% squeezing) and beats Fock-state encoding when $\bar{n}>1.88$ (achieved with $V_{ne}\approx 0.2$ or squeezing of about 80%).

Some comments are in order concerning the analysis to date. First, notice the boundaries of the previous analysis were for pure squeezed states which saturate the uncertainty inequality. In contrast the states produced in experiments are rarely pure, sometimes because of technical noise [13], sometimes due to the type of squeezing mechanism [14], and sometimes simply due to loss in the nonlinear crystal [15]. Loss in the optical elements used to produce the entanglement from the squeezing will also reduce the purity (as well as the effective entanglement). Therefore, in an experiment we will have that $V_{nu}=1/V_{ne}+b$, where b represents excess noise. This means that a particular level of entanglement is accompanied by more photons than in the pure case. Hence channel capacities will be lowered[1]. Further, unit detection efficiency was assumed. Again, this is unlikely in an experiment. As a result Bob's detected variances will be given by $V_{det}=\eta V+1-\eta$, where η is the detection efficiency. Nonunit detection efficiency will lower signal-to-noise ratio and once again decrease the effective channel capacity. Propagation loss (assumed equal in the two channels) has the same effect as detection efficiency and so can be rolled into the value of η.

To achieve *unconditional dense coding* we require that even in the presence of these kinds of imperfections, the dense coding channel capacity exceeds that of the ideal single-channel capacities. The levels of squeezing apparently required in the ideal case are already at the boundary of what is currently achievable experimentally—experiments regularly achieve squeezing greater than 3 dB (50%) [16], but stable measured squeezing of approximately 5 dB (68%) has only been reported recently [17]. Imperfections appear to only further increase the stringent experimental requirements

[1]For example, the optimum signal-to-noise ratio of the dense coding scheme when considering entanglement impurity is $S/N=\bar{n}+\bar{n}^2-b(0.25+0.5\bar{n})+0.0625b^2$. At the optimum squeezed quadrature variance of $V_{ne,opt}=2/(4\bar{n}+2-b)$ (with $b<4\bar{n}+2$) the new, more general expression for the optimum dense coding capacity is $C_{dci}^{opt}=\log_2[1+\bar{n}+\bar{n}^2-b(0.25+0.5\bar{n})+0.0625b^2]$ which will be less than C_{dc}^{opt} for any amount of excess noise.

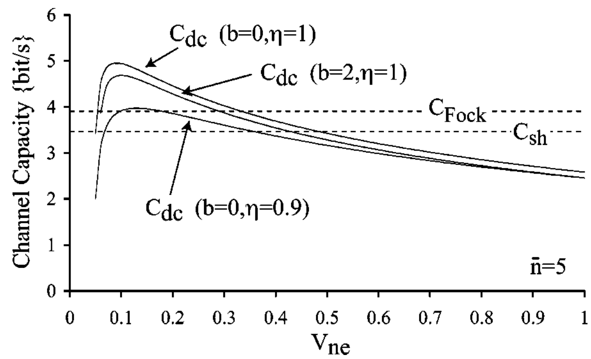

FIG. 2. Plots of the dense coding channel capacity C_{dc} as a function of input squeezing, V_{ne}, for an average photon number of $\bar{n}=5$. Regions in which the dense coding channel capacity exceed the Fock-state channel capacity display unconditional dense coding with respect to the Fock-state scheme. Regions in which $C_{dc}>C_{sh}$ display unconditional dense coding with respect to the squeezed state scheme.

and it would seem that an experimental demonstration of unconditional dense coding is beyond current technology. However, in the following section we will show that this is not the case, and that a demonstration is within the reach of current continuous-variable technology.

III. DEMONSTRATING UNCONDITIONAL DENSE CODING

Notice that the preceeding analysis and that of Ref. [4] asked the question: "what is the minimum *photon number* for which we can demonstrate dense coding?" We will now show that a different answer is obtained if we ask the question: "what is the minimum *squeezing* required to demonstrate dense coding?" Rather than maximizing the signal-to-noise ratio for a fixed $\bar{n}$ we now allow an arbitrary relationship between the squeezed quadrature variance, average photon number and excess noise. The detected dense coding signal-to-noise ratio may then be explicitly written as $[\eta(4\bar{n}-V_{ne}-1/V_{ne}-b+2)]/(4\eta V_{ne}+4-4\eta)$. Hence a more general expression for the dense coding capacity is

$$C_{dc}=\log_2\left[1+\frac{\eta(4\bar{n}-V_{ne}-1/V_{ne}-b+2)}{4(\eta V_{ne}+1-\eta)}\right]. \tag{8}$$

Figure 2 shows the channel capacity of the dense coding scheme, C_{dc} as a function of the squeezed quadrature variance at an average photon number of $\bar{n}=5$. For the moment, focus on the topmost curve (labeled "C_{dc} $(b=0,\eta=1)$") which represents the channel capacity in the absence of excess noise and for perfect detection efficiency. This curve represents the best possible scenario for that photon number. Also shown in this figure are the ideal maximum channel capacities for the squeezed state scheme, C_{sh} and the Fock state scheme C_{Fock} for this photon number. As illustrated in Fig. 2 and indicated by Eq. (7), at $\bar{n}=5$ the optimum dense coding capacity exceeds the capacity of the squeezed or Fock-state schemes. However, Fig. 2 illustrates a point that is

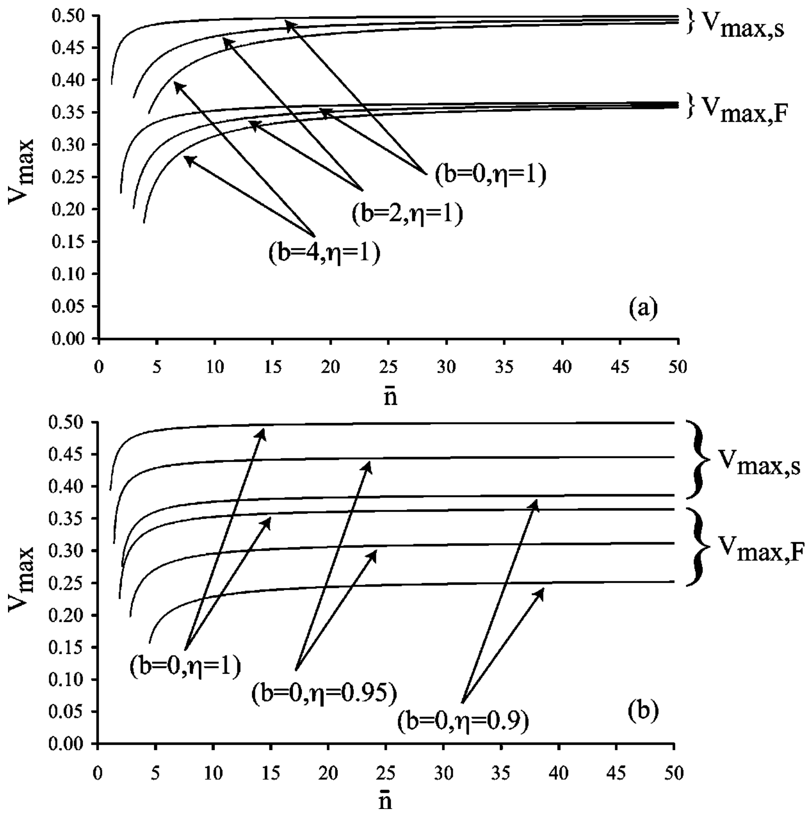

FIG. 3. Plots of the maximum squeezed quadrature variances at which unconditional dense coding may be demonstrated with respect to C_{sh} and C_{Fock}. These are labeled $V_{max,s}$ and $V_{max,F}$, respectively.

not clear from Eq. (7)—the dense coding channel capacity exceeds that of C_{sh} or C_{Fock} for a number of values of V_{ne}, not just the optimum.

Most significantly, Fig. 2 shows that it is possible to demonstrate unconditional dense coding with a relatively modest amount of squeezing. For example, it is seen from Fig. 2 that unconditional dense coding may be demonstrated with respect to the squeezed state system for $V_{ne}\approx 0.48$, and it may be demonstrated with respect to the Fock-state scheme for $V_{ne}\approx 0.33$. These levels of squeezing are far more experimentally feasible than those found by simply considering the optimum. Even more heartening from an experimental perspective is that the levels of squeezing required may be reduced by increasing $\bar{n}$. Experimentally, given a minimum amount of squeezing, this amounts to simply increasing the signal strength.

Denoting the maximum squeezed quadrature variances at which unconditional dense coding may be demonstrated with respect to C_{sh} and C_{Fock} as $V_{max,s}$ and $V_{max,F}$, respectively, Fig. 3 shows $V_{max,s}$ and $V_{max,F}$ as a function of the photon number. Again, focusing for the moment on the curves for pure entanglement and perfect detection efficiency [labeled "$(b=0,\eta=1)$]," it is seen that $V_{max,s}$ and $V_{max,F}$ asymptote to values of $\frac{1}{2}$ and $1/e$, respectively. This is quite a surprizing result. This figure shows that, in a perfect experiment, it is possible to demonstrate unconditional dense coding with 50% squeezing with respect to the the squeezed state channel capacity and 63% squeezing with respect to the Fock-state scheme.

Turning now to experimental issues such as excess noise or imperfect detection efficiency, Fig. 2 also shows curves representing the dense coding channel capacity, C_{dc} when excess noise, labeled "$(b=2,\eta=1)$" and imperfect detection efficiency, labeled "$(b=0,\eta=0.9)$," are considered. First note that Fig. 2 shows that both excess noise and imperfect detection efficiency decrease the effective channel capacity of the dense coding scheme. Indeed, sufficient amounts of either of these imperfections will render a demonstration of unconditional dense coding impossible for low photon numbers. However, given a minimum level of squeezing, this may be solved by increasing the photon number, i.e., by increasing the signal strength. The minimum level of squeezing required for each photon number depends quite strongly on the entanglement impurity and detection efficiency. This effect is shown in Fig. 3, where $V_{max,F}$ and $V_{max,s}$ are plotted as a function of the photon number for a number of values of excess noise in part (a) and a number of values of detection efficiency in part (b).

Figure 3 shows that, for a given photon number, either of these imperfections will mean that more squeezing is required than for the best possible scenario. Alternatively, for a given level of squeezing, entanglement impurity or imperfect detection efficiency will require that more photons must be used to demonstrate unconditional dense coding.

Perhaps more importantly, Fig. 3 shows that the experimental issue of greatest concern is that of imperfect detection efficiency. Excess noise will certainly have an effect on the performance of the dense coding scheme for small photon numbers. However, the asymptotes of $V_{max,F}$ and $V_{max,s}$ do not depend on the entanglement purity. By contrast, the asymptotes of $V_{max,F}$ and $V_{max,s}$ depend very strongly on the detection efficiency. In practical terms, this means that much greater levels of squeezing will be required to demonstrate unconditional dense coding when the detection efficiency is poor. Indeed the amount of squeezing required increases exponentially around a characteristic value of detection efficiency. This suggests that practical systems must exceed a minimum detection efficiency in order to demonstrate unconditional dense coding. When comparing the dense coding channel capacity to the squeezed state system, the minimum detection efficiency required is $\eta_{min,s}=\frac{2}{3}$. The minimum detection efficiency increases to $\eta_{min,F}=e/(1+e)$ in order to demonstrate unconditional dense coding with respect to the Fock-state system.

Taking these effects into account, it appears that an experimental demonstration of unconditional dense coding with respect to the squeezed state system is currently feasible. For example, with experimentally realistic detection efficiencies of 85–95 % the maximum squeezing required would be approximately 68–55 %, respectively. These levels of squeezing are now quite commonly achieved [16,17]. On the other hand, an experimental demonstration of unconditional dense coding with respect to the Fock scheme is a rather more ambitious, but not unattainable, goal. With experimentally realistic detection efficiencies of 85–95 % the maximum squeezing required would be approximately 81–68 %, respectively. Ref. [17] reported measured squeezing of 5 dB with a detection efficiency of approximately 87%. Assuming a dense coding scheme with approximately the same detection efficiency as quoted in Ref. [17], we conclude that this would have been just sufficient to demonstrate unconditional dense coding with respect to the Fock-state scheme. Thus the levels of squeezing required for a strong demonstration of unconditional dense coding are within the boundaries of current technology.

IV. CONCLUSION

We have shown that by working in the large signal regime a demonstration of unconditional dense coding appears possible with present technology. We believe that such a demonstration would represent a bench-mark experiment in continuous-variable quantum information technology. It is interesting to note relationships between the entanglement requirements of dense coding and teleportation. Beating the coherent-state channel capacity with dense coding can be achieved with any level of squeezing in the entanglement. Similarly an improvement over the classical fidelity limit for teleportation of coherent states is achieved with any finite level of squeezing. However, the preservation of nonclassical properties of the state like squeezing requires greater than 50% squeezing in teleportation, corresponding to the requirement of 50% squeezing to beat the squeezed state channel capacity in dense coding. It is interesting to muse as to whether the $1/e$ entanglement requirement for unconditional dense coding corresponds to the passing of some other tangible limit in teleportation.

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