Quantum electromagnetic field in a three-dimensional oscillating cavity

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We compute the photon creation inside a perfectly conducting, three-dimensional oscillating cavity, taking the polarization of the electromagnetic field into account. As the boundary conditions for this field are both of Dirichlet and (generalized) Neumann type, we analyze as a preliminary step the dynamical Casimir effect for a scalar field satisfying generalized Neumann boundary conditions. We show that particle production is enhanced with respect to the case of Dirichlet boundary conditions. Then we consider the transverse electric and transverse magnetic polarizations of the electromagnetic field. For resonant frequencies, the total number of photons grows exponentially in time for both polarizations, the rate being greater for transverse magnetic modes.

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I. INTRODUCTION

The existence of an attractive force between two uncharged, perfectly conducting parallel plates was predicted by Casimir in 1948 [1] and has recently been measured at the 15% precision level using state-of-the-art cantilevers [2]. A similar force between a conducting plane and a sphere has also been measured with progressively higher precision in the last years using torsion balances [3], atomic force microscopes [4], and capacitance bridges [5,6], with the latter reference showing the relevance of Casimir forces in nanotechnology. For a recent review of experimental and theoretical developments, see Ref. [7].

The dynamical effect consists in the generation of photons due to the instability of the vacuum state of the electromagnetic field in the presence of time-dependent boundaries. In the literature it is referred to as dynamical Casimir effect [8] or motion-induced radiation [9]. The dynamical effect has been recently reviewed in Ref. [10]. Up till now no concrete experiment has been performed to confirm this photon generation, but an experimental verification is not out of reach. From the theoretical point of view it is widely accepted that the most favorable configuration in order to observe the phenomenon is a vibrating cavity in which it is possible to produce resonant effects between the mechanical and field oscillations.

Many previous papers have focused their attention in the scalar field quantization within a one-dimensional cavity [11,12]. Recently, we analyzed in detail the case of a three-dimensional cavity [13], but still considering a scalar field (in this and other [14] previous works Dirichlet boundary conditions are assumed). The main difference between one and three-dimensional cavities is that, while in one dimension the cavity's frequency spectrum is equidistant and leads to strong intermode interactions, in three dimensions the spectrum is in general nonequidistant, and only a few modes may be coupled. The relevance of this coupling has been pointed out only recently (see Refs. [13,15]) . The aim of this paper is to extend the results of Ref. [13] to the realistic case of the electromagnetic field, properly taking into account the polarization of the different modes [transverse electric (TE) and transverse magnetic (TM) polarizations].

As we will see in Sec. II, the electromagnetic field involves both Dirichlet and (generalized) Neumann boundary conditions. For this reason, it is of interest to analyze the case of a massless scalar field satisfying this latter type of boundary conditions, which we do in Sec. III. Assuming a resonant vibration of the cavity and using multiple scale analysis [16] we will show that the number of particles produced is much larger than for Dirichlet boundary conditions. We study in detail the resonant case in which the cavity oscillates at twice the frequency of some field mode. In Sec. IV we show that TE modes of the electromagnetic field behave as a scalar field with Dirichlet boundary conditions, while TM modes are analogous to the scalar case with Neumann boundary conditions. Sec. V contains our main conclusions.

II. THE BOUNDARY CONDITIONS

We consider a rectangular cavity formed by perfectly conducting walls with dimensions L_x, L_y, and L_z. The wall placed at $x=L_x$ is at rest for $t<0$ and begins to move following a given trajectory, $L_x(t)$, at $t=0$. We assume this trajectory as prescribed for the problem (not a dynamical variable) and that it works as a time-dependent boundary condition for the field. Moreover, we will assume a nonrelativistic motion of the wall with $L_x(t)=L_0[1+\epsilon f(t)]$ with $\epsilon \ll 1$ and $f(t)$ a bounded function. We use units $\hbar = c = 1$.

Let us consider the problem of finding the electromagnetic field inside the cavity in terms of the four-vector potential $A_\mu = (\varphi, \mathbf{A})$. In the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, the scalar potential φ vanishes and the vector potential satisfies the

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wave equation $\Box \mathbf{A}=0$. For the static walls, the boundary conditions are the usual ones

$$\mathbf{E}_{\parallel}=\mathbf{0};\quad \mathbf{B}_{\perp}=\mathbf{0}, \tag{1}$$

where $\parallel$ and $\perp$, respectively, denote the components of the field, parallel and perpendicular to the wall. Note that these conditions follow from Faraday's law and from the fact that the divergence of the magnetic field vanishes (i.e., the source-free Maxwell equations).

On the moving wall, these boundary conditions must be imposed in a Lorentz frame in which the mirror is instantaneously at rest. As the mirror moves in the x direction, it will be convenient to decompose the electromagnetic fields into TE and TM modes with respect to the x axis. The TE fields are defined as the solutions to Maxwell equations with $\mathbf{E}^{(\mathrm{TE})}\cdot\hat{\mathbf{x}}=0$. Analogously, the TM fields satisfy $\mathbf{B}^{(\mathrm{TM})}\cdot\hat{\mathbf{x}}=0$.

It is useful to introduce a *different* vector potential for each polarization through the equations [17,18]

$$\mathbf{E}^{(\mathrm{TE})}=-\partial_t\mathbf{A}^{(\mathrm{TE})},\quad \mathbf{B}^{(\mathrm{TE})}=\boldsymbol{\nabla}\times\mathbf{A}^{(\mathrm{TE})}, \tag{2}$$

$$\mathbf{B}^{(\mathrm{TM})}=\partial_t\mathbf{A}^{(\mathrm{TM})},\quad \mathbf{E}^{(\mathrm{TM})}=\boldsymbol{\nabla}\times\mathbf{A}^{(\mathrm{TM})}. \tag{3}$$

Both potentials satisfy the Coulomb gauge and have vanishing x component. As $\mathbf{A}\cdot\hat{\mathbf{x}}=0$ and $\varphi=0$, the vector potentials are invariant under a boost in the x direction. The same is true for the Coulomb gauge. In terms of these potentials, the boundary conditions are relatively simple [18]. Let us denote by S the laboratory frame and by S' the instantaneous comoving frame. In S' the TE vector potential satisfies Dirichlet boundary conditions $\mathbf{A}'^{(\mathrm{TE})}(x'=0,y',z',t')=\mathbf{0}$. Therefore, on the moving mirror,

$$\mathbf{A}^{(\mathrm{TE})}[x=L_x(t),y,z,t]=\mathbf{0}. \tag{4}$$

On the other hand, the TM vector potential satisfies

$$n^{\mu'}\partial_{\mu'}\mathbf{A}'^{(\mathrm{TM})}(x'=0,y',z',t')=\mathbf{0}, \tag{5}$$

where $n^{\mu'}=(0,1,0,0)$. As a consequence, for a nonrelativistic motion of the mirror,

$$\begin{aligned} n^{\mu}\partial_{\mu}\mathbf{A}^{(\mathrm{TM})}[x=L_x(t),y,z,t] &= [\partial_x+\dot{L}_x(t)\partial_t]\mathbf{A}^{(\mathrm{TM})}[x=L_x(t),y,z,t] \\ &= \mathbf{0}, \end{aligned} \tag{6}$$

i.e., a "generalized" Neumann boundary condition with $n^{\mu}=(\dot{L}_x,1,0,0)$. On the static mirrors the boundary conditions for the TE vector potential are given by

$$\begin{gathered} \mathbf{A}^{(\mathrm{TE})}(x=0,y,z,t)=\mathbf{0}, \\ A_y^{(\mathrm{TE})}(x,y,\{z=0,L_z\},t)=A_z^{(\mathrm{TE})}(x,\{y=0,L_y\},z,t)=0. \end{gathered} \tag{7}$$

For the TM potential we have

$$\begin{gathered} \partial_x A_z^{(\mathrm{TM})}(x=0,y,z,t)=\partial_x A_y^{(\mathrm{TM})}(x=0,y,z,t)=0, \\ A_y^{(\mathrm{TM})}(x,\{y=0,L_y\},z,t)=\partial_y A_z^{(\mathrm{TM})}(x,\{y=0,L_y\},z,t)=0, \\ A_z^{(\mathrm{TM})}(x,y,\{z=0,L_z\},t)=\partial_z A_y^{(\mathrm{TM})}(x,y,\{z=0,L_z\},t)=0. \end{gathered} \tag{8}$$

From these boundary conditions it is clear that the behavior of each component of the TE vector field is related to the problem of a scalar field subjected to Dirichlet boundary conditions. For the TM vector field it is necessary to deal with the generalized Neumann boundary conditions given in Eq. (6). The former problem was extensively studied in our previous paper [13], and the latter will be treated in the following section.

In the derivation of the boundary conditions above we have assumed that the perfect conductor boundary conditions must be imposed in the Lorentz frame in which the moving mirror is instantaneously at rest. This is the usual procedure (see Refs. [17–19]). One might argue that the acceleration of the mirror could induce modifications to the boundary conditions. However, this is not the case. The boundary conditions can be imposed in a noninertial frame in which the mirror is at rest all the time. In this frame, the electromagnetic tensor $F_{\mu\nu}$ is written as $F_{\mu\nu}=A_{\mu;\nu}-A_{\nu;\mu}$, where ; denotes the covariant derivative. It is easy to show that the connection coefficients contained in the covariant derivative cancel out, and then $F_{\mu\nu}$ can be written using ordinary derivatives, $F_{\mu\nu}=\partial_{\nu}A_{\mu}-\partial_{\mu}A_{\nu}$. The source-free Maxwell equations follow from this identity, and therefore have the same form both in the noninertial frame and in the instantaneous Lorentz frame. Consequently there are no corrections to the boundary conditions due to the acceleration of the mirror.

III. SCALAR FIELD WITH NEUMANN BOUNDARY CONDITIONS

Let us consider the problem of a massless scalar field $\phi(\mathbf{x},t)$ satisfying the wave equation $\Box\phi=0$ and (generalized) Neumann boundary conditions on all surfaces of the cavity. In the comoving frame the Neumann boundary condition is $n^{\mu'}\partial_{\mu'}\phi'=0$. In the laboratory frame, this condition becomes $n^{\mu}\partial_{\mu}\phi=0$, where $n^{\mu}=(\dot{L}_x,1,0,0)$. Therefore we have

$$\begin{gathered} \partial_x\phi(x=0,y,z,t)=0;\ (\partial_x+\dot{L}_x\partial_t)\phi[x=L_x(t),y,z,t]=0; \\ \partial_y\phi(x,\{y=0,L_y\},z,t)=0;\quad \partial_z\phi(x,y,\{z=0,L_z\},t)=0. \end{gathered} \tag{9}$$

A. Instantaneous basis

The Fourier expansion of the field for an arbitrary moment of time can be written in terms of creation and annihilation operators as

$$\phi(\mathbf{x},t)=\sum_{\mathbf{n}} a_{\mathbf{n}}^{\mathrm{in}}u_{\mathbf{n}}(\mathbf{x},t)+\mathrm{H.c.}, \tag{10}$$

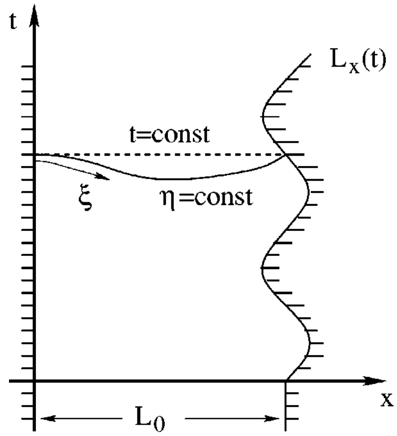

FIG. 1. Worldlines of the mirrors along the x direction. The line η=const is orthogonal to the worldlines at $x=0$ and $x=L_x(t)$, and the coordinate ξ measures the distance from the static mirror along the η=const line.

where the mode functions $u_{\mathbf{n}}(\mathbf{x},t)$ form a complete orthonormal set of solutions of the wave equation with Neumann boundary conditions. When $t\leqslant 0$ (static cavity) each field mode is determined by three nonnegative integers n_x, n_y, and n_z. Namely,

$$u_{\mathbf{n}}(\mathbf{x},t<0)=\frac{1}{\sqrt{2\omega_{\mathbf{n}}}}\sqrt{\frac{2}{L_x}}\cos\left(\frac{n_x\pi}{L_x}x\right)\sqrt{\frac{2}{L_y}}\times\cos\left(\frac{n_y\pi}{L_y}y\right)\sqrt{\frac{2}{L_z}}\cos\left(\frac{n_z\pi}{L_z}z\right)e^{-i\omega_{\mathbf{n}}t}, \tag{11}$$

$$\omega_{\mathbf{n}}=\pi\sqrt{\left(\frac{n_x}{L_x}\right)^2+\left(\frac{n_y}{L_y}\right)^2+\left(\frac{n_z}{L_z}\right)^2}, \tag{12}$$

with the shorthand $\mathbf{n}=(n_x,n_y,n_z)$. In order to satisfy the boundary conditions for $t>0$ it is useful to expand the mode functions in Eq. (10) with respect to an *instantaneous basis*. If the boundary condition on the moving mirror were the instantaneous Neumann condition $\partial_x\phi[x=L_x(t),y,z,t]=0$, the trivial choice for the instantaneous basis would be

$$\sqrt{\frac{2}{L_x(t)}}\cos\left(\frac{k_x\pi}{L_x(t)}x\right)\sqrt{\frac{2}{L_y}}\cos\left(\frac{k_y\pi}{L_y}y\right)\times\sqrt{\frac{2}{L_z}}\cos\left(\frac{k_z\pi}{L_z}z\right). \tag{13}$$

However, as the generalized Neumann condition in Eq. (9) involves the time derivative of the field, the situation is more complex.

We consider new variables (η,ξ) in the (t,x) plane in order to reduce the problem of generalized Neumann boundary conditions to the case of "standard" (i.e., no time derivative of the field) Neumann boudary conditions, for which we know how to choose the instantaneous basis. We define the line η=const as a slight modification of the corresponding t=const line, in such a way that it is orthogonal to the worldlines of the mirrors at $x=0$ and $x=L_x(t)$ (see Fig. 1). The variable ξ is defined as the distance, on the line η=const, from $x=0$ to x. In these coordinates, the generalized boundary condition on the two mirrors becomes the standard one, namely $\partial_\xi\phi(\xi,y,z,\eta)=0$ both at $\xi=0$ and at $\xi=l(\eta)$, where $l(\eta)$ is the value of the coordinate ξ on the moving mirror. Therefore, an instantaneous basis to describe the field is

$$\sqrt{\frac{2}{l(\eta)}}\cos\left(\frac{k_x\pi}{l(\eta)}\xi\right)\sqrt{\frac{2}{L_y}}\cos\left(\frac{k_y\pi}{L_y}y\right)\times\sqrt{\frac{2}{L_z}}\cos\left(\frac{k_z\pi}{L_z}z\right). \tag{14}$$

To find a concrete form for the new coordinates we write $\eta=t+g(x,t)$. Therefore ξ is given by

$$\xi=\int_0^x dx'\sqrt{1+\frac{g'^2(x',t)}{[1+\dot{g}(x',t)]^2}}. \tag{15}$$

At this point it is important to note that, since we are considering motions of the wall that are small $[O(\epsilon)]$ deviations from the initial static position, terms of order $O(\epsilon^2)$ or higher will be neglected in what follows. Moreover, $g(x,t)=O(\epsilon)$, $\xi=x+O(\epsilon^2)$, and $l(\eta)=L_x(t)+O(\epsilon^2)$. With this in mind, it is easy to show that, in order to fulfill the assumed orthogonality between the line η=const and the mirrors' worldlines, the function $g(x,t)$ must satisfy

$$g(x=0,t)=0;\quad g[x=L_x(t),t]=0;$$

$$\partial_x g(x=0,t)=0;\quad \partial_x g[x=L_x(t),t]=-\dot{L}. \tag{16}$$

There are many solutions to the above conditions, that can be written in the form

$$g(x,t)=\dot{L}_x(t)L_x(t)v[x/L_x(t)], \tag{17}$$

where $v(0)=v(1)=0$, $v'(0)=0$, and $v'(1)=-1$ (the prime denotes derivation with respect to x). For example, a possible solution is $v(z)=\frac{1}{2}(z^2-z^4)$. The freedom of selecting the function $g(x,t)$ means that one can choose different instantaneous basis to describe the same field, and for each of them one has in principle a different set of modes. However, physical quantities like the number of created particles or the energy density inside the cavity should not depend on the choice of $g(x,t)$. We will keep a general function as a benchmark for our calculations.

Finally, the mode functions in Eq. (10) can be expanded in terms of the instantaneous basis Eq. (14) as

$$u_{\mathbf{n}}(\mathbf{x},t>0)=\sum_{\mathbf{k}} Q_{\mathbf{k}}^{(\mathbf{n})}(\eta)\sqrt{\frac{2}{l(\eta)}}\cos\left(\frac{k_x\pi}{l(\eta)}\xi\right)\sqrt{\frac{2}{L_y}}\cos\left(\frac{k_y\pi}{L_y}y\right)\sqrt{\frac{2}{L_z}}\cos\left(\frac{k_z\pi}{L_z}z\right)$$

$$\simeq\sum_{\mathbf{k}}[Q_{\mathbf{k}}^{(\mathbf{n})}(t)+\dot{Q}_{\mathbf{k}}^{(\mathbf{n})}(t)g(x,t)]\sqrt{\frac{2}{L_x(t)}}\cos\left(\frac{k_x\pi}{L_x(t)}x\right)\sqrt{\frac{2}{L_y}}\cos\left(\frac{k_y\pi}{L_y}y\right)\sqrt{\frac{2}{L_z}}\cos\left(\frac{k_z\pi}{L_z}z\right)$$

$$\equiv\sum_{\mathbf{k}}[Q_{\mathbf{k}}^{(\mathbf{n})}(t)+\dot{Q}_{\mathbf{k}}^{(\mathbf{n})}(t)g(x,t)]\phi_{\mathbf{k}}[\mathbf{x},L_x(t)], \tag{18}$$

where the functions $Q_{\mathbf{k}}^{(\mathbf{n})}(t)$ depend on the choice for $g(x,t)$. The initial conditions are given by

$$Q_{\mathbf{k}}^{(\mathbf{n})}(0)=\frac{1}{\sqrt{2\omega_{\mathbf{n}}}}\delta_{\mathbf{k},\mathbf{n}},\quad \dot{Q}_{\mathbf{k}}^{(\mathbf{n})}(0)=-i\sqrt{\frac{\omega_{\mathbf{n}}}{2}}\delta_{\mathbf{k},\mathbf{n}}, \tag{19}$$

provided that $L_x(t)$ and $\dot{L}_x(t)$ are continuous at $t=0$, and that the initial acceleration $\ddot{L}_x(0)=O(\epsilon^2)$. In this way we ensure that each field mode and its time derivative are also continuous functions at $t=0$.

B. Dynamical equations

We now study the trajectory $L_x(t)=L_0[1+\epsilon\sin(\Omega t)]$.[1] The equations for the modes $Q_{\mathbf{k}}^{\mathbf{n}}(t)$ can be obtained from Eq. (18), since $\Box u_{\mathbf{n}}(\mathbf{x},t>0)=0$. We first apply the Dalambertian, and then multiply by $\phi_{\mathbf{k}}$ and integrate over the cavity. To order $O(\epsilon^2)$, the equations read

$$\begin{aligned}\ddot{Q}_{\mathbf{k}}^{(\mathbf{n})}+\omega_{\mathbf{n}}^2(t)Q_{\mathbf{k}}^{(\mathbf{n})}=&-2\lambda(t)\sum_{\mathbf{j}} g_{\mathbf{jk}}\dot{Q}_{\mathbf{j}}^{(\mathbf{n})}-\dot{\lambda}(t)\sum_{\mathbf{j}} g_{\mathbf{jk}}Q_{\mathbf{j}}^{(\mathbf{n})}\\ &-2\dot{\lambda}(t)L_x^2(t)\sum_{\mathbf{j}} r_{\mathbf{jk}}\ddot{Q}_{\mathbf{j}}^{(\mathbf{n})}\\ &-\sum_{\mathbf{j}}\dot{Q}_{\mathbf{j}}^{(\mathbf{n})}[r_{\mathbf{jk}}\ddot{\lambda}(t)L_x^2(t)-\lambda(t)\eta_{\mathbf{jk}}]\\ &-\lambda(t)L_x^2(t)\sum_{\mathbf{j}} r_{\mathbf{jk}}\partial_t\ddot{Q}_{\mathbf{j}}^{(\mathbf{n})},\end{aligned} \tag{20}$$

where

$$\omega_{\mathbf{k}}(t)=\pi\sqrt{\left(\frac{k_x}{L_x(t)}\right)^2+\left(\frac{k_y}{L_y}\right)^2+\left(\frac{k_z}{L_z}\right)^2}, \tag{21}$$

$$\lambda(t)=\frac{\dot{L}_x(t)}{L_x(t)}, \tag{22}$$

$$r_{\mathbf{jk}}=\int_0^{L_x(t)}dx\int_0^{L_y}dy\int_0^{L_z}dz\, v\,\phi_{\mathbf{j}}\phi_{\mathbf{k}}, \tag{23}$$

$$\eta_{\mathbf{jk}}=L_x^2(t)\int_0^{L_x(t)}dx\int_0^{L_y}dy\int_0^{L_z}dz\times[(v''-\omega_{\mathbf{j}}^2 v)\phi_{\mathbf{j}}\phi_{\mathbf{k}}+2v'\phi_{\mathbf{j}}'\phi_{\mathbf{k}}]. \tag{24}$$

Here, $\omega_{\mathbf{j}}$ is the frequency of the mode evaluated at $\epsilon=0$. As before, the prime denotes derivation with respect to x. The coefficients $g_{\mathbf{jk}}$ are defined by

$$g_{\mathbf{jk}}=L_x(t)\int_0^{L_x(t)}dx\int_0^{L_y}dy\int_0^{L_z}dz\frac{\partial\phi_{\mathbf{j}}}{\partial L_x}\phi_{\mathbf{k}} =\begin{cases}(-1)^{k_x+j_x}\dfrac{2j_x^2}{k_x^2-j_x^2}\delta_{k_yj_y}\delta_{k_zj_z} & \text{if } k_x\neq j_x,\\ -\delta_{k_yj_y}\delta_{k_zj_z} & \text{if } k_x=j_x.\end{cases} \tag{25}$$

Had we considered Dirichlet boundary conditions on the static walls $(y=0,L_y;z=0,L_z)$ we would have obtained the same dynamical equations for the modes $Q_{\mathbf{k}}^{\mathbf{n}}(t)$; i.e., the form of the equations only depends on the boundary conditions imposed along the x direction. This is because the coefficients $r_{\mathbf{jk}},\eta_{\mathbf{jk}}$, and $g_{\mathbf{jk}}$ do not depend on the particular form of the $\phi_{\mathbf{k}}$ in the plane y-z, as long as they are properly normalized in this plane. However, when Dirichlet boundary conditions on $x=0$ and $x=L_x(t)$ are considered, the equations for the modes are different [see Ref. [13] and Eqs. (48) and (49) below]. Note that the coefficients $g_{\mathbf{jk}}$ for Neumann boundary conditions are different from those for Dirichlet boundary conditions.

When the mirror returns to its initial position for $t>t_{\rm final}$ the rhs of Eq. (20) vanishes and the solution reads

$$Q_{\mathbf{k}}^{(\mathbf{n})}(t>t_{\rm final})=A_{\mathbf{k}}^{(\mathbf{n})}e^{i\omega_{\mathbf{k}}t}+B_{\mathbf{k}}^{(\mathbf{n})}e^{-i\omega_{\mathbf{k}}t}, \tag{26}$$

with $A_{\mathbf{k}}^{(\mathbf{n})}$ and $B_{\mathbf{k}}^{(\mathbf{n})}$ being some constant coefficients to be determined by the continuity conditions at $t=t_{\rm final}$. The number of particles in the mode $\mathbf{k}$ is given by

$$\langle\mathcal{N}_{\mathbf{k}}\rangle=\sum_{\mathbf{n}} 2\omega_{\mathbf{k}}|A_{\mathbf{k}}^{(\mathbf{n})}|^2. \tag{27}$$

[1]Strictly speaking, we should add to $L_x(t)$ some decaying function in order to meet the continuity conditions at $t=0$. Since we will be interested in a resonant behavior of the field, this additional function will not contribute, being irrelevant for what follows. For a more detailed discussion of this point see Ref. [13].

C. Multiple scale analysis

In order to find a solution to Eq. (20) we use the multiple scale analysis technique [16], which we have already applied in our previous paper [13]. We first introduce a second time scale $\tau=\epsilon t$ and expand $Q_{\mathbf{k}}^{(\mathbf{n})}=Q_{\mathbf{k}}^{(\mathbf{n})(0)}+\epsilon Q_{\mathbf{k}}^{(\mathbf{n})(1)}$. Replacing this into Eq. (20) we obtain, as zeroth order solution, $Q_{\mathbf{k}}^{(\mathbf{n})(0)}=A_{\mathbf{k}}^{(\mathbf{n})}(\tau)e^{i\omega_{\mathbf{k}}t}+B_{\mathbf{k}}^{(\mathbf{n})}(\tau)e^{-i\omega_{\mathbf{k}}t}$. The functions $A_{\mathbf{k}}^{(\mathbf{n})}(\tau)$ and $B_{\mathbf{k}}^{(\mathbf{n})}(\tau)$ will be obtained from imposing that no secularities appear in the equation for $Q_{\mathbf{k}}^{(\mathbf{n})(1)}$. This reads

$$\ddot{Q}_{\mathbf{k}}^{(\mathbf{n})(1)}+\omega_{\mathbf{k}}^2Q_{\mathbf{k}}^{(\mathbf{n})(1)}=-2\partial_\tau\dot{Q}_{\mathbf{k}}^{(\mathbf{n})(0)}+2\left(\frac{\pi k_x}{L_0}\right)^2\sin(\Omega t)Q_{\mathbf{k}}^{(\mathbf{n})(0)}+2\Omega\cos(\Omega t)\dot{Q}_{\mathbf{k}}^{(\mathbf{n})(0)}-\Omega^2\sin(\Omega t)Q_{\mathbf{k}}^{(\mathbf{n})(0)}$$
$$-2L_0^2\Omega^2\sin(\Omega t)r_{\mathbf{kk}}\omega_{\mathbf{k}}^2Q_{\mathbf{k}}^{(\mathbf{n})(0)}+L_0^2\Omega\cos(\Omega t)r_{\mathbf{kk}}\omega_{\mathbf{k}}^2\dot{Q}_{\mathbf{k}}^{(\mathbf{n})(0)}+L_0^2\Omega\cos(\Omega t)\dot{Q}_{\mathbf{k}}^{(\mathbf{n})(0)}\left[\Omega^2r_{\mathbf{kk}}+\frac{1}{L_0^2}\eta_{\mathbf{kk}}\right]$$
$$+\Omega^2\sin(\Omega t)\sum_{\mathbf{j}\neq\mathbf{k}}g_{\mathbf{jk}}Q_{\mathbf{j}}^{(\mathbf{n})(0)}-2\Omega\cos(\Omega t)\sum_{\mathbf{j}\neq\mathbf{k}}g_{\mathbf{jk}}\dot{Q}_{\mathbf{j}}^{(\mathbf{n})(0)}-2L_0^2\Omega^2\sin(\Omega t)\sum_{\mathbf{j}\neq\mathbf{k}}r_{\mathbf{jk}}\omega_{\mathbf{j}}^2Q_{\mathbf{j}}^{(\mathbf{n})(0)}$$
$$+L_0^2\Omega\cos(\Omega t)\sum_{\mathbf{j}\neq\mathbf{k}}r_{\mathbf{jk}}\omega_{\mathbf{j}}^2\dot{Q}_{\mathbf{j}}^{(\mathbf{n})(0)}+L_0^2\Omega\cos(\Omega t)\sum_{\mathbf{j}\neq\mathbf{k}}\dot{Q}_{\mathbf{j}}^{(\mathbf{n})(0)}\left[\Omega^2r_{\mathbf{jk}}+\frac{1}{L_0^2}\eta_{\mathbf{jk}}\right], \tag{28}$$

where we have used that, to zeroth order, $\ddot{Q}_{\mathbf{k}}^{(\mathbf{n})(0)}=-\omega_{\mathbf{k}}^2Q_{\mathbf{k}}^{(\mathbf{n})(0)}$.

The equations for $A_{\mathbf{k}}^{(\mathbf{n})}(\tau)$ and $B_{\mathbf{k}}^{(\mathbf{n})}(\tau)$ are obtained imposing the condition that any term in the right-hand side of Eq. (28) with a time dependency of the form $e^{\pm i\omega_{\mathbf{k}}\mathbf{t}}$ must vanish. We get

$$\frac{dA_{\mathbf{k}}^{(\mathbf{n})}}{d\tau}=-\frac{1}{2\omega_{\mathbf{k}}}\left[\frac{k_x^2\pi^2}{L_0^2}-2\omega_{\mathbf{k}}^2\right]B_{\mathbf{k}}^{(\mathbf{n})}\delta(2\omega_{\mathbf{k}}-\Omega)+\sum_{\mathbf{j}\neq\mathbf{k}}\left[-\left(-\omega_{\mathbf{j}}+\frac{\Omega}{2}\right)g_{\mathbf{jk}}+\delta_{k_yj_y}\delta_{k_zj_z}\omega_{\mathbf{j}}\right]\delta(-\omega_{\mathbf{k}}-\omega_{\mathbf{j}}+\Omega)\frac{\Omega}{2\omega_{\mathbf{k}}}B_{\mathbf{j}}^{(\mathbf{n})}$$
$$+\sum_{\mathbf{j}\neq\mathbf{k}}\left[-\left(\omega_{\mathbf{j}}+\frac{\Omega}{2}\right)g_{\mathbf{jk}}-\delta_{k_yj_y}\delta_{k_zj_z}\omega_{\mathbf{j}}\right]\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{j}}-\Omega)\frac{\Omega}{2\omega_{\mathbf{k}}}A_{\mathbf{j}}^{(\mathbf{n})}+\sum_{\mathbf{j}\neq\mathbf{k}}\left[-\left(\omega_{\mathbf{j}}-\frac{\Omega}{2}\right)g_{\mathbf{jk}}-\delta_{k_yj_y}\delta_{k_zj_z}\omega_{\mathbf{j}}\right]$$
$$\times\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{j}}+\Omega)\frac{\Omega}{2\omega_{\mathbf{k}}}A_{\mathbf{j}}^{(\mathbf{n})}, \tag{29}$$

and an analogous equation for $B_{\mathbf{k}}^{(\mathbf{n})}$, obtained by the interchange $A_{\mathbf{k}}^{(\mathbf{n})}\leftrightarrow B_{\mathbf{k}}^{(\mathbf{n})}$. Note that Eq. (29) is independent of $g(x,t)$. This nontrivial check of our calculations follows from two identities we used to derive Eq. (29), namely,

$$-\frac{1}{2\omega_{\mathbf{k}}}L_0^2\omega_{\mathbf{k}}^2B_{\mathbf{k}}^{(\mathbf{n})}\delta(2\omega_{\mathbf{k}}-\Omega)\int_0^{L_x(t)}dx\int_0^{L_y}dy\int_0^{L_z}dz(\phi_{\mathbf{k}}^2v')'=\omega_{\mathbf{k}}B_{\mathbf{k}}^{(\mathbf{n})}\delta(2\omega_{\mathbf{k}}-\Omega), \tag{30}$$

$$\omega_{\mathbf{k}}^2r_{\mathbf{jk}}+\frac{1}{L_0^2}\eta_{\mathbf{jk}}=\int_0^{L_x(t)}dx\int_0^{L_y}dy\int_0^{L_z}dz[v'\phi_{\mathbf{j}}\phi_{\mathbf{k}}+v(\phi_{\mathbf{k}}\phi_{\mathbf{j}}'-\phi_{\mathbf{k}}'\phi_{\mathbf{j}})]'=-\frac{2}{L_0^2}\delta_{k_yj_y}\delta_{k_zj_z}, \tag{31}$$

where we have used the boundary conditions $v'(0)=0$ and $v'(1)=-1$.

D. Examples

Let us consider the "parametric resonant case," in which the external frequency Ω is twice the frequency of an unperturbed mode $\mathbf{k}$, $\Omega=2\omega_{\mathbf{k}}$. A second mode $\mathbf{j}$ will be coupled to the mode $\mathbf{k}$ iff $|\omega_{\mathbf{k}}\pm\omega_{\mathbf{j}}|=\Omega$. We first assume this is not the case. Therefore the evolution equations become

$$\frac{dA_{\mathbf{k}}^{(\mathbf{n})}}{d\tau}=-\frac{1}{2\omega_{\mathbf{k}}}\left[\frac{k_x^2\pi^2}{L_0^2}-2\omega_{\mathbf{k}}^2\right]B_{\mathbf{k}}^{(\mathbf{n})},$$

$$\frac{dB_{\mathbf{k}}^{(\mathbf{n})}}{d\tau}=-\frac{1}{2\omega_{\mathbf{k}}}\left[\frac{k_x^2\pi^2}{L_0^2}-2\omega_{\mathbf{k}}^2\right]A_{\mathbf{k}}^{(\mathbf{n})}. \tag{32}$$

It is easy to check from these equations that $A_{\mathbf{k}}^{(\mathbf{n})}$ and $B_{\mathbf{k}}^{(\mathbf{n})}$ grow exponentially as $e^{\lambda_N\tau}$, with a rate $\lambda_N=(1/2\omega_{\mathbf{k}})(\omega_{\mathbf{k}}^2+\omega_p^2)$, where $\omega_p^2=\omega_{\mathbf{k}}^2-k_x^2\pi^2/L_0^2$. It is interesting to compare this rate with that for Dirichlet conditions, which is given by $\lambda_D=(1/2\omega_{\mathbf{k}})(\omega_{\mathbf{k}}^2-\omega_p^2)$ [13]. We have

$$\frac{\lambda_N}{\lambda_D}=\frac{\omega_{\mathbf{k}}^2+\omega_p^2}{\omega_{\mathbf{k}}^2-\omega_p^2}>1. \tag{33}$$

For a given mode, the rate for Neumann boundary conditions is always bigger than the rate for Dirichlet conditions.

Let us now assume the existence of one mode, say **j**, that satisfies $\omega_{\mathbf{j}}=3\omega_{\mathbf{k}}$ and $j_y=k_y$, $j_z=k_z$. We obtain for $A_{\mathbf{k}}^{(\mathbf{n})}$ and $B_{\mathbf{k}}^{(\mathbf{n})}$

$$\frac{dA_{\mathbf{k}}^{(\mathbf{n})}}{d\tau}=-\frac{1}{2\omega_{\mathbf{k}}}\left[\frac{k_x^2\pi^2}{L_0^2}-2\omega_{\mathbf{k}}^2\right]B_{\mathbf{k}}^{(\mathbf{n})}+\frac{1}{2\omega_{\mathbf{k}}}\left[(-1)^{k_x+j_x}\frac{j_x^2\pi^2}{L_0^2}-6\omega_{\mathbf{k}}^2\right]A_{\mathbf{j}}^{(\mathbf{n})},$$

$$\frac{dB_{\mathbf{k}}^{(\mathbf{n})}}{d\tau}=-\frac{1}{2\omega_{\mathbf{k}}}\left[\frac{k_x^2\pi^2}{L_0^2}-2\omega_{\mathbf{k}}^2\right]A_{\mathbf{k}}^{(\mathbf{n})}+\frac{1}{2\omega_{\mathbf{k}}}\left[(-1)^{k_x+j_x}\frac{j_x^2\pi^2}{L_0^2}-6\omega_{\mathbf{k}}^2\right]B_{\mathbf{j}}^{(\mathbf{n})}. \quad (34)$$

We also assume that the spectrum is such that the mode **j** is only coupled to the mode **k**. The equations for $A_{\mathbf{j}}^{(\mathbf{n})}$ and $B_{\mathbf{j}}^{(\mathbf{n})}$ are therefore

$$\frac{dA_{\mathbf{j}}^{(\mathbf{n})}}{d\tau}=-\frac{1}{6\omega_{\mathbf{k}}}\left[(-1)^{k_x+j_x}\frac{k_x^2\pi^2}{L_0^2}+2\omega_{\mathbf{k}}^2\right]A_{\mathbf{k}}^{(\mathbf{n})},$$

$$\frac{dB_{\mathbf{j}}^{(\mathbf{n})}}{d\tau}=-\frac{1}{6\omega_{\mathbf{k}}}\left[(-1)^{k_x+j_x}\frac{k_x^2\pi^2}{L_0^2}+2\omega_{\mathbf{k}}^2\right]B_{\mathbf{k}}^{(\mathbf{n})}. \quad (35)$$

We write the system of equations in matrix form

$$\frac{d\mathbf{v}(\tau)}{d\tau}=\mathcal{M}\,\mathbf{v}(\tau), \quad (36)$$

where

$$\mathbf{v}(\tau)=\begin{pmatrix}A_{\mathbf{k}}^{(\mathbf{n})}(\tau)\\ B_{\mathbf{k}}^{(\mathbf{n})}(\tau)\\ A_{\mathbf{j}}^{(\mathbf{n})}(\tau)\\ B_{\mathbf{j}}^{(\mathbf{n})}(\tau)\end{pmatrix},\quad \mathcal{M}=\frac{1}{2\omega_{\mathbf{k}}}\begin{pmatrix}0&a&b&0\\ a&0&0&b\\ c&0&0&0\\ 0&c&0&0\end{pmatrix}, \quad (37)$$

where $a=[-k_x^2\pi^2/L_0^2+2\omega_{\mathbf{k}}^2]$, $b=[(-1)^{k_x+j_x}(j_x^2\pi^2/L_0^2)-6\omega_{\mathbf{k}}^2]$, and $c=-\frac{1}{3}[(-1)^{k_x+j_x}(k_x^2\pi^2/L_0^2)+2\omega_{\mathbf{k}}^2]$. The solution to this system can be easily obtained after diagonalizing the matrix $\mathcal{M}$. The eigenvalues are given by

$$\lambda=\frac{1}{4\omega_{\mathbf{k}}}(\pm a\pm\sqrt{a^2+4bc}). \quad (38)$$

We note that the exponential growth rate in the uncoupled case is given by $\lambda_N=a/2\omega_{\mathbf{k}}$. When two modes are coupled, the rate is given by the real part of the biggest eigenvalue in Eq. (38). When $a^2+4bc<0$, the rate is half the one expected for the resonant mode when the coupling is neglected, as for Dirichlet boundary conditions [13]. It is easy to show that this is the case if $(-1)^{k_x+j_x}=+1$. However, in the opposite case, $bc>0$, and the growth rate for coupled modes is bigger than $\lambda_N=a/2\omega_{\mathbf{k}}$.

A relevant case where two modes are coupled is the cubic cavity $L_x=L_y=L_z=L$. We fix Ω as twice the lowest cavity frequency,

$$\Omega=2\omega_{(1,1,1)}=\frac{2\pi\sqrt{3}}{L}. \quad (39)$$

The fundamental mode $\mathbf{k}=(1,1,1)$ is coupled to $\mathbf{j}=(5,1,1)$ because $\omega_{(5,1,1)}=3\omega_{(1,1,1)}$. Only these two modes are coupled, since there does not exist in the spectrum any mode **s** satisfying $\omega_{\mathbf{s}}=5\omega_{(1,1,1)}$. For this particular case, the four eigenvalues are

$$\lambda=\frac{\pi}{4\sqrt{3}L}(\pm5\pm6.35i). \quad (40)$$

Had we neglected the intermode coupling, we would have concluded that the growth rate in the fundamental mode would be $\lambda=2.5\pi/\sqrt{3}L$. The growth rate in the coupled case is one half of this.

One striking new feature is the possibility to enhance the exponential growth rate by means of mode coupling, provided that the two coupled modes satisfy $(-1)^{k_x+j_y}=-1$. As an example let us consider a cavity with dimensions $L_y=L_z=4L_x$. We set the external frequency to be

$$\Omega=2\omega_{(0,1,1)}=\frac{2\pi}{\sqrt{8}L_x}. \quad (41)$$

If this is the case, then the mode $\mathbf{k}=(0,1,1)$ is coupled to $\mathbf{j}=(1,1,1)$. The four eigenvalues are

$$\lambda=\frac{\sqrt{8}\pi}{16L_x}(\pm1\pm\sqrt{31/3}). \quad (42)$$

This means that the exponential growth is at the rate $0.74\pi/L_x$, which is more than twice the value we would had predicted had we neglected the coupling $(\pi/\sqrt{8}L_x)$.

IV. THE ELECTROMAGNETIC FIELD

A. Transverse electric modes

For the TE case, the expansion of the vector potential for an arbitrary moment of time, in terms of creation and annihilation operators, can be written as

$$\mathbf{A}^{(\mathrm{TE})}(\mathbf{x},t)=\sum_{\mathbf{n}}a_{\mathbf{n}}^{\mathrm{in}}\mathbf{u}_{\mathbf{n}}^{(\mathrm{TE})}(\mathbf{x},t)+\mathrm{H.c.} \quad (43)$$

For $t\leqslant0$ the cavity is static, and each field mode is given by

$$\mathbf{u}_{\mathbf{n}}^{(\mathrm{TE})}(\mathbf{x},t\leqslant 0)=\frac{1}{\sqrt{2\omega_{\mathbf{n}}}}\sqrt{\frac{8}{L_xL_yL_z}}\left(0,\alpha_{\mathbf{n}}\sin\left(\frac{\pi n_x}{L_x}x\right)\times\cos\left(\frac{\pi n_y}{L_y}y\right)\sin\left(\frac{\pi n_z}{L_z}z\right),\beta_{\mathbf{n}}\sin\left(\frac{\pi n_x}{L_x}x\right)\times\sin\left(\frac{\pi n_y}{L_y}y\right)\cos\left(\frac{\pi n_z}{L_z}z\right)\right)e^{-i\omega_{\mathbf{n}}t}, \quad (44)$$

where n_x, n_y, and n_z are integers such that $n_x\geqslant 1$, $n_y,n_z\geqslant 0$, and n_y,n_z cannot be simultaneously zero. The constants $\alpha_{\mathbf{n}}$ and $\beta_{\mathbf{n}}$ are components of the polarization vector for the electromagnetic field, satisfying the normalization condition $\alpha_{\mathbf{n}}^2+\beta_{\mathbf{n}}^2=1$, and the Coulomb gauge condition, $\alpha_{\mathbf{n}}n_y/L_y+\beta_{\mathbf{n}}n_z/L_z=0$.

When $t>0$, we expand the mode functions in Eq. (43) with respect to an *instantaneous basis*

$$\mathbf{u}_{\mathbf{n}}^{(\mathrm{TE})}(\mathbf{x},t>0)=\sum_{\mathbf{k}} Q_{\mathbf{k},\mathrm{TE}}^{(\mathbf{n})}(t)\sqrt{\frac{2}{L_x(t)}}\sin\left(\frac{\pi n_x}{L_x(t)}x\right)\mathbf{\Phi}_{k_yk_z}^{(\mathrm{TE})}(y,z), \quad (45)$$

where $\mathbf{\Phi}_{k_yk_z}^{(\mathrm{TE})}$ is

$$\mathbf{\Phi}_{k_yk_z}^{(\mathrm{TE})}(y,z)=\sqrt{\frac{4}{L_yL_z}}\left(0,\alpha_{\mathbf{k}}\cos\left(\frac{\pi k_y}{L_y}y\right)\sin\left(\frac{\pi k_z}{L_z}z\right),\beta_{\mathbf{k}}\sin\left(\frac{\pi k_y}{L_y}y\right)\cos\left(\frac{\pi k_z}{L_z}z\right)\right). \quad (46)$$

The functions $\mathbf{\Phi}_{k_yk_z}^{(\mathrm{TE})}$ form a complete set satisfying

$$\int_0^{L_y}dy\int_0^{L_z}dz\,\mathbf{\Phi}_{k_yk_z}^{(\mathrm{TE})}\cdot\mathbf{\Phi}_{j_yj_z}^{(\mathrm{TE})\star}=\delta_{k_yj_y}\delta_{k_zj_z}. \quad (47)$$

From the above Eq. (45) it is easy to obtain the dynamical equations for the modes $Q_{\mathbf{k},\mathrm{TE}}^{(\mathbf{n})}$. We get

$$\ddot{Q}_{\mathbf{k},\mathrm{TE}}^{(\mathbf{n})}+\omega_{\mathbf{n}}^2(t)\,Q_{\mathbf{k},\mathrm{TE}}^{(\mathbf{n})}=2\lambda(t)\sum_{\mathbf{j}} g_{\mathbf{kj}}\dot{Q}_{\mathbf{j},\mathrm{TE}}^{(\mathbf{n})}+\dot{\lambda}(t)\sum_{\mathbf{j}} g_{\mathbf{kj}}Q_{\mathbf{j},\mathrm{TE}}^{(\mathbf{n})}(t), \quad (48)$$

where

$$g_{\mathbf{kj}}=-g_{\mathbf{jk}}=\begin{cases}(-1)^{k_x+j_x}\dfrac{2k_xj_x}{j_x^2-k_x^2}\,\delta_{k_yj_y}\,\delta_{k_zj_z} & \text{if } k_x\neq j_x,\\ 0 & \text{if } k_x=j_x.\end{cases} \quad (49)$$

As expected, these equations are exactly those corresponding to a scalar field satisfying Dirichlet boundary conditions on the surfaces $x=0,L_x(t)$ [13]. Therefore, the number of created photons in the TE mode equals the number of created Dirichlet scalar particles.

As an example, let us consider the parametric resonant case $\Omega=2\omega_{\mathbf{k}}$ for a cubic cavity. For uncoupled $\mathbf{k}$ modes (such as either of the two fundamental TE modes, $\mathbf{k}=(1,1,0)$ and $\mathbf{k}=(1,0,1)$), the number of TE photons grows exponentially as

$$\langle\mathcal{N}_{\mathbf{k},\mathrm{TE}}\rangle=\sinh^2(\lambda_D\epsilon t), \quad (50)$$

where λ_{D} is the growth rate for Dirichlet scalar particles, introduced in Sec. III D. For the above mentioned fundamental modes, $\lambda_D=\pi/2\sqrt{2}L$. The first coupled TE mode is $\mathbf{k}=(1,1,1)$, which only couples to the TE mode $\mathbf{j}=(5,1,1)$. At large times $\epsilon t/L\gg 1$ the number of TE photons in those modes grows as $\langle\mathcal{N}_{\mathbf{k},\mathrm{TE}}\rangle\approx\langle\mathcal{N}_{\mathbf{j},\mathrm{TE}}\rangle\approx e^{0.9\epsilon t/L}$ [13].

B. Transverse magnetic modes

The expansion in terms of creation and annihilation operators is again of the form Eq. (43), but now for $t\leqslant 0$ each field mode is given by

$$\mathbf{u}_{\mathbf{n}}^{(\mathrm{TM})}(\mathbf{x},t\leqslant 0)=\frac{1}{\sqrt{2\omega_{\mathbf{n}}}}\sqrt{\frac{8}{L_xL_yL_z}}\left(0,\alpha_{\mathbf{n}}\cos\left(\frac{\pi n_x}{L_x}x\right)\sin\left(\frac{\pi n_y}{L_y}y\right)\cos\left(\frac{\pi n_z}{L_z}z\right),\ \beta_{\mathbf{n}}\cos\left(\frac{\pi n_x}{L_x}x\right)\cos\left(\frac{\pi n_y}{L_y}y\right)\sin\left(\frac{\pi n_z}{L_z}z\right)\right)e^{-i\omega_{\mathbf{n}}t}. \quad (51)$$

Here n_x, n_y, and n_z are nonnegative integers, and n_y and n_z cannot be simultaneously zero.

On the other hand, when $t>0$ we introduce an instantaneous basis similar to that of the scalar field in Sec. III. We write

$$\mathbf{u}_{\mathbf{n}}^{(\mathrm{TM})}(\mathbf{x},t>0)=\sum_{\mathbf{k}}\left(Q_{\mathbf{k},\mathrm{TM}}^{(\mathbf{n})}(t)+\dot{Q}_{\mathbf{k},\mathrm{TM}}^{(\mathbf{n})}(t)g(x,t)\right)\times\sqrt{\frac{2}{L_x(t)}}\cos\left(\frac{\pi n_x}{L_x(t)}x\right)\mathbf{\Phi}_{k_yk_z}^{(\mathrm{TM})}(y,z), \quad (52)$$

where the functions $\Phi^{(\mathrm{TM})}_{k_y k_z}$ are similar to their TE counterparts (they can be obtained by interchanging cos and sin in the rhs of Eq. (46). Since all TE modes have $n_x=0$, the first mode of the cavity that can be excited by the external frequency is a TM mode. In particular, for a cavity such that $L_x \ll L_y, L_z$ only TM modes can be excited.

From the above equation, it is now clear that the dynamical evolution of the TM modes is that of a scalar field satisfying generalized Neumann boundary conditions. As a consequence, the number of created photons in the TM mode equals the number of created Neumann scalar particles. Again, let us consider the parametric resonant case $\Omega = 2\omega_{\mathbf{k}}$ for a cubic cavity. For uncoupled $\mathbf{k}$ modes [such as either of the two fundamental TM modes, $\mathbf{k}=(0,1,0)$ and $\mathbf{k}=(0,0,1)$], the number of TM photons grows exponentially as

$$\langle \mathcal{N}_{\mathbf{k},\mathrm{TM}} \rangle = \sinh^2(\lambda_N \epsilon t), \tag{53}$$

where λ_N is the growth rate for Neumann scalar particles, also introduced in Sec. III D. For the fundamental modes $\lambda_N = \pi/L$. The first coupled TM mode is $\mathbf{k}=(0,1,1)$, which only couples to the TM mode $\mathbf{j}=(4,1,1)$. For large times $(\epsilon t/L \gg 1)$ the number of particles in these modes grows as $\langle \mathcal{N}_{\mathbf{k},\mathrm{TM}} \rangle \approx \langle \mathcal{N}_{\mathbf{j},\mathrm{TM}} \rangle \approx e^{4.4\epsilon t/L}$. The next coupled TM mode is the same as the TE mode, namely $\mathbf{k}=(1,1,1)$, coupled to $\mathbf{j}=(5,1,1)$. The exponential growth is $\langle \mathcal{N}_{\mathbf{k},\mathrm{TM}} \rangle \approx \langle \mathcal{N}_{\mathbf{j},\mathrm{TM}} \rangle \approx e^{4.5\epsilon t/L}$, the growth rate for these modes being greater than that for the TE case.

V. DISCUSSION

In this paper we have computed the resonant photon creation inside a three-dimensional oscillating cavity taking the vector nature of the electromagnetic field into account. Previous works studied the case of a scalar field with Dirichlet boundary conditions. As the electromagnetic field involves both Neumann and Dirichlet boundary conditions, we first analyzed a massless scalar field satisfying (generalized) Neumann boundary conditions. We have shown that in this case it is also possible to expand the field modes in terms of an instantaneous basis, the difference with the Dirichlet case being that the expansion is not unique—it depends on an arbitrary function $g(x,t)$ satisfying the boundary conditions Eq. (16). However, physical quantities like the number of created particles or the energy density inside the cavity are independent of the choice of such a function. After treating the Neumann scalar case we considered the full electromagnetic problem and showed that the TE modes of the electromagnetic field are essentially described by a Dirichlet scalar field, while the TM modes correspond to a Neumann scalar field.

We have studied in detail the resonant situation $\Omega = 2\omega_{\mathbf{k}}$ for two cases: an uncoupled resonant mode and two coupled resonant modes. In both cases, the exponential growth of created photons is greater for TM modes. For the uncoupled case, we have found that

$$\frac{\lambda_{\mathrm{TM}}}{\lambda_{\mathrm{TE}}} = \frac{\omega_{\mathbf{k}}^2 + \omega_p^2}{\omega_{\mathbf{k}}^2 - \omega_p^2}. \tag{54}$$

For a cavity with $L_x \simeq L_y \simeq L_z$, $\omega_p^2 \simeq \frac{2}{3}\omega_{\mathbf{k}}^2$ so $\lambda_{\mathrm{TM}} \simeq 5\lambda_{\mathrm{TE}}$. We can estimate the number of created TE and TM photons given by Eqs. (50) and (53) using typical values for the maximal dimensionless displacement ϵ that may be obtained in conceivable future experiments. For 3D cubic cavities of linear dimensions of the order of 1–10 cm, the lowest resonant frequency is of the order of GHz. It may turn out to be very difficult, if not impossible, to make the cavity oscillate as a whole at such a high frequency. To overcome this difficulty a different experimental scenario was proposed in Ref. [12], consisting of strong acoustic waves excited on the surface of the cavity wall. Typical materials cannot bear relative amplitude deformations in excess of $\delta_{\max}=10^{-2}$. This sets a limit to the maximum velocity of the boundary, $v_{\max} = \delta_{\max} v_s \approx 50$ m/s, (v_s is the speed of sound in the material), and consequently to the maximal dimensionless displacement $\epsilon_{\max} = v_{\max}/\Omega L$. For example, for a cavity with $L = 10$ cm whose lowest mode [i.e., either of the two TM modes $\mathbf{k}=(0,1,0)$ or $\mathbf{k}=(0,0,1)$] is being excited ($\Omega = 2\pi c/L = 18$ GHz), we get $\epsilon_{\max} \approx 10^{-8}$. Even for a value of ϵ, ten times smaller than this, one gets an exponentially large number of created photons $\langle \mathcal{N}_{\mathbf{k},\mathrm{TM}} \rangle = \sinh^2(10t/\mathrm{s})$ which, after 1 sec, gives a total of approximately 10^8 photons created in that mode. We can also compare the number of photons produced for an uncoupled mode $\mathbf{k}$, common to both kind of polarizations TE and TM. For example, for the mode $\mathbf{k}=(1,1,0)$ one gets $\langle \mathcal{N}_{\mathbf{k},\mathrm{TE}} \rangle \approx \sinh^2(3t/\mathrm{s})$ and $\langle \mathcal{N}_{\mathbf{k},\mathrm{TM}} \rangle \approx \sinh^2(10t/\mathrm{s})$, which after one second produces a total of 10^2 TE photons and 10^8 TM photons. For the case of two coupled modes we have found that, for Neumann boundary conditions, the coupling can enhance the exponential growth. This contrasts with the case of Dirichlet boundary conditions, in which the coupling always suppresses the exponential growth. These facts may be relevant for an eventual experimental verification of the dynamical Casimir effect.

All the above considerations assume ideal conditions, such as perfectly conducting plates, exact parametric resonant condition $\Omega = 2\omega_{\mathbf{k}}$, arbitrary large Q factor for the cavity (no leakage of photons), no thermal noise, etc. Some of these conditions were relaxed in our previous paper [13], where we analyzed, for Dirichlet boundary conditions, the enhancement of photon creation due to finite temperature effects, slightly off-resonance situations, the case of three coupled modes, etc. The generalization of these findings to the electromagnetic case is not too complicated, and we expect similar conclusions. Given our results for the dynamical behavior of TE and TM modes, it is also possible to study the full electromagnetic problem in three dimensional leaky cavities along the lines of [20].

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