Spin squeezing and entanglement in spinor condensates

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We analyze quantum correlation properties of a spinor-1 ($f=1$) Bose-Einstein condensate using the Gell-Mann realization of SU(3) symmetry. We show that previously discussed phenomena of condensate fragmentation and spin mixing can be explained in terms of the hypercharge symmetry. The ground state of a spinor-1 condensate is found to be fragmented for ferromagnetic interactions. The notion of two-bosonic-mode squeezing is generalized to the two-spin (U-V) squeezing within the SU(3) formalism. Spin squeezing in the isospin subspace (T) is found and is numerically investigated. We also provide results for the stationary states of spinor-1 condensates.

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I. INTRODUCTION

The availability of atomic Bose-Einstein condensates (BEC) with spin degrees of freedom has stimulated much recent interest because of their applications in quantum information physics. Atomic spinor-1 condensates (hyperfine spin $f=1$ for each atom) were first realized by transferring spin-polarized BEC prepared in a magnetic trap into a far-off-resonant optical trap [1] and more recently in an all optical trap [2]. Spinor-1 condensate can be treated as a three-component order parameter, one for each Zeeman component of the hyperfine manifold. Early theoretical studies have clarified rotationally invariant descriptions including elastic s-wave collisions [3–7]. Recent investigations reveal that such a system also possesses complex ground-state structures and can exhibit novel dynamical effects [7–9], such as fragmentation [10], spin mixing, and entanglement [3,7,11]. This paper contains further analysis of such quantum correlation properties of a spinor-1 condensate [11,12].

For a spin-half ($f=1/2$) atomic system, a rotationally invariant Hamiltonian is known not to induce spin squeezing as the total spin is conserved [13]. For a spinor-1 condensate, it was found that its Hamiltonian becomes rotationally invariant if the single (spatial)-mode approximation is made to its order parameters, i.e., assuming $\psi_{m_f}(\vec{r})=\phi(\vec{r})a_{m_f}$ with the same mode function $\phi(\vec{r})$ [3,7]. (a_{m_f} is the annihilation operator for atoms in Zeeman state $m_f=\pm,0$.) However, various nonlinear processes do occur within different subspaces of the full SU(3) structure of a spinor-1 condensate. In this paper, we verify the existence of spin squeezing in the isospin subgroup [12].

Historically, atomic squeezed states were first considered for a system of two-level atoms. Even in this simple SU(2) case, it was found that some operational definitions of spin squeezing can become system dependent. Depending on the context in which the concept of squeezing is applied, different definitions arise [14]. Squeezing in atomic variables was first introduced through reduced fluctuations in the atomic (Pauli) operators of the system [15] such that atomic resonance fluorescence in the far-field zone is squeezed. In this case, it is useful and convenient to define the atomic squeezing parameter according to

$$\xi_h=\Delta J_i/\sqrt{|\langle J_j\rangle/2|},\quad i\neq j\in(x,y,z). \tag{1}$$

This definition can be essentially read off from the Heisenberg uncertainty relation $\Delta J_i\Delta J_j\geqslant|\langle J_k\rangle/2|$ for the collective angular-momentum components of the two-level atomic system.

When in a squeezed state $\xi_h<1$, the quantum fluctuation of one collective angular-momentum component becomes lower than the Heisenberg limited value at the cost of increased fluctuation in the other component. The general family of two-level atomic states satisfying this criterion was found to be Bloch states, or SU(2) coherent states. These "squeezed states" [in the sense of Eq. (1)] are obtained by simply rotating the collective (Dicke) state $|J,\pm J\rangle$ in space [16]. When it comes to taking advantage of atomic squeezed states in spectroscopy, it turned out to be that it is possible to determine more useful set of states with new squeezing conditions [17]. In particular, for Ramsey oscillatory field spectroscopy, a new squeezing parameter [17],

$$\xi_R=\sqrt{2J}\Delta J_\perp/|\langle\vec{J}\rangle| \tag{2}$$

is called for with $J_\perp$ the angular-momentum component normal to the $\langle\vec{J}\rangle$, i.e., in the direction of the unit vector $\vec{n}$ along which $\Delta(\vec{n}\cdot\vec{J})$ is minimized. The squeezing condition $\xi_R<1$ is not straightforwardly determined by the Heisenberg uncertainty relation. Instead, it is defined by requiring the improvement of signal-to-noise ratio in a typical Ramsey spectroscopy. It was later shown that the same criterion is also applicable for improving the phase sensitivity of a Mach-Zehnder interferometer [14]. $\xi_R\equiv1$ for Bloch states or SU(2) coherent states.

An independent refinement of ξ_h was suggested by Kitagawa and Ueda [13] to make it independent of angular-momentum coordinate system or specific measurement schemes. They emphasized that collective spin squeezing should reflect quantum correlations between individual atomic spins and defined a squeezing parameter

$$\xi_q=\Delta J_\perp/\sqrt{J/2} \tag{3}$$

to measure such correlations. [The factor $J/2$ in the denominator represents the variance of a Bloch state, which comes

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from simply adding up the variance of each individual spin (1/2)]. When quantum correlations exist among different atomic spins, the variance of certain component of the collective spin can become lower than $J/2$. This leads naturally to the criterion $\xi_q<1$ for spin squeezing. Instead of simple rotation of Dicke states, more complicated "axis twisting" schemes are required to generate squeezed states satisfying this criteria [19]. We note that this definition is directly related to the spectroscopic definition as $\xi_R=(J/|\langle J_z\rangle|)\xi_q$.

More recently, a particular type of quantum correlation, namely the multiparticle entanglement, becomes important for quantum information physics. To relate atomic squeezing to entanglement, a more stringent criterion for atomic squeezing which combines the quantum correlation definition with the inseparability requirement of system density matrix is given by [18]

$$\xi_e^2=\frac{(2J)\Delta(\vec{n}_1\cdot\vec{J})^2}{\langle\vec{n}_2\cdot\vec{J}\rangle^2+\langle\vec{n}_3\cdot\vec{J}\rangle^2}<1, \tag{4}$$

with the $\vec{n}_i$ being mutually orthogonal unit vectors. ξ_e is in fact identical to ξ_R along the direction $\langle\vec{n}_1\cdot\vec{J}\rangle=0$. It was proven rigorously that when $\xi_e<1$, the total state of the N two-level atoms becomes inseparable, i.e., entangled in a general sense. All three definitions above apply to a two-component (two level) atomic system. Many complications arise when attempt is made to extend spin squeezing to a spinor-1 SU(3) system. Under certain restrictive conditions, ξ_e has been used recently to discuss a two-mode entanglement in a spinor-1 condensate [11].

A related problem to spin squeezing is its efficient generation and detection. In accordance with their respective definitions for SU(2) systems, several physical mechanisms have been proposed along this direction. Kitegawa and Ueda considered a model Hamiltonian $H_{KU}=\hbar\chi J_z^2$ that can be realized via the Coulomb interaction between electrons in the two arms of an interferometer [13]. Barnett and Dupertuis suggested that spin squeezing can be achieved in a two-atom system described by $H_{BD}=i\hbar(g^*J_{1+}J_{2+}-\text{H.c.})$ [20]. The use of a pseudospin two-component atomic condensate system has also been suggested [18]. Recently, two different groups considered atomic (spin) squeezing and entanglement in a spinor-1 condensate, under the assumption that one of the components is highly populated such that quantum properties are important only among the remaining two sparsely populated components [11,12]. Our aim in this study is to remove such a restrictive condition, and consider the full quantum correlations within a spinor-1 condensate.

Let us consider a general three-component system labeled by $i,j\in\{+,-,0\}$. For spectroscopic and interferometric applications the observables of interest are the relative number of particles N_i-N_j (particle partitioning) and the corresponding phase differences $\phi_i-\phi_j$ with their measurements limited by noises $\delta N_{ij}=\langle[\Delta(N_i-N_j)]^2\rangle^{1/2}$ and $\delta\phi_{ij}=\langle[\Delta(\phi_i-\phi_j)]^2\rangle^{1/2}$. For a two-component system, the particle partitioning becomes the collective angular-momentum projection as $N_+-N_-=2J_z$ in the standard Schwinger representation; the relative phase becomes the corresponding azimuthal phase $\phi_z\equiv\phi_+-\phi_-$, which is conjugate to J_z. Quantum mechanically they satisfy $[J_z,\phi_z]=i$. Thus from $\delta J_z\delta\phi_z\geqslant 1$ and $\delta N_{+-}=2\langle\Delta J_z^2\rangle^{1/2}$, we find $\delta\phi_{+-}\approx\langle\Delta J_y^2\rangle^{1/2}/|\langle J_x\rangle|$ [19]. Therefore, for spin squeezed states, one achieves higher angular resolution and reduced particle partitioning noise. In a three-component system, we can similarly associate the three number difference N_i-N_j with three subspace pseudospins (each of spin-1/2) $\vec{U}$, $\vec{T}$, and $\vec{V}$ such that $N_+-N_-=2T_3$, $N_+-N_0=2V_3$, and $N_--N_0=2U_3$. The phase differences can then be similarly expressed in terms of components $U_{x,y}$, $V_{x,y}$, and $T_{x,y}$. When demanding noise reduction in such a SU(3) system, we need to consider squeezing in the three spin-1/2 subsystems. One may naively expect that results from the above discussed SU(2) squeezing can be applied to each of the three subsystems, and collectively, one can simply demand that $\xi_e<1$ to be satisfied simultaneously. In reality this does not work as the three spin-1/2 subsystems do not commute with each other. This is also the fundamental reason that makes it difficult to generate and detect quantum correlations in a full SU(3) system. Furthermore, due to the above noncommuting nature, the three SU(2) subspins cannot be squeezed independently of each other. Previous discussions of a spinor-1 condensate entanglement are always limited to just one SU(2) subspace, usually in the limit $N_0\sim N$, i.e., one mode is highly populated. Approximately, this limit destroys the underlying noncommutative algebra among (U,V,T) and simplifies the problem to that of a usual two-mode SU(2) system.

One of the major results of this paper is that the effective Hamiltonian of a general spinor-1 condensate can be decomposed as

$$H=\hbar\chi_{KU}T_3^2+\hbar\chi_{BD}(U_+V_++\text{H.c.}), \tag{5}$$

which involves both the Kitagawa-Ueda (KU) and Barnett-Dupertuis (BD) type of spin squeezing simultaneously. In other words, all three fictitious spins can indeed be found squeezed in a spinor-1 condensate, as the above two distinct nonlinearities commute with each other, and therefore squeeze all three SU(2) subspaces simultaneously. We find that the BD-type interaction dominates when N_0 is large; while in the opposite limit the KU-type squeezing governs. For intermediate values of N_0 it is necessary to consider a generalized spin squeezing for the three-mode spinor-1 system. To achieve this, we provide a criterion for the U-V two-spin squeezing based on reduced quantum fluctuations imposed by the BD-type nonlinearity. When such a condition is satisfied, the state of a spinor-1 condensate as a macroscopic-coherent quantum object becomes useful for three-mode spectroscopic and interferometric applications. We further show that this condition also corresponds to a two-mode entanglement in terms of the Holstein-Primakopf bosonic modes, and it reduces to previous results in the large N_0 limit [11,12]. Squeezing in T spin is particularly useful for quantum information applications based on collective (Dicke) states $|J,J_z\rangle$. These states are in fact stationary in a spinor-1 condensate and can be manipulated via external control fields [7]. Since $J_z = N_+ - N_- = 2T_3$, such T-squeezed states ensure well-defined Dicke states.

In our study of spin squeezing in a spinor-1 condensate as outlined in this paper, we present a systematic approach by recognizing the (U,V,T) pseudospin subspaces as the Gell-Mann (quark) realization of the SU(3) algebra [21]. Similar recognitions are found useful in the recent discussions of quantum and semiclassical dynamics of three coupled atomic condensates [22], where the BD-type two-spin squeezing nonlinearity was absent. Earlier investigations of three-level atomic systems also made efficient use of the density matrix and expressed atomic Hamiltonians in terms SU(3) generators [23,24]. The main difference between our approach (on spinor-1 BEC) and those of earlier studies are the enveloping Weyl-Heisenberg algebra of the bosonic operators, which leads to subsequently much larger Hilbert space of the system. In addition to spin squeezing, we also investigate other quantum correlation effects, e.g., condensate fragmentation with the theoretical framework. We show that previous theories based upon the SO(2) rotational symmetry group cannot give a decomposition of angular-momentum operator with nonlinearities that could easily be considered for spin squeezing.

II. SU(3) FORMULATION FOR SPINOR-1 BEC

We shall treat the spinor-1 BEC under the single-mode approximation [7]. It should be noted that the validity of it is now reasonably well understood [7,11,25], and for ferromagnetic interactions, it is in fact exact as shown recently [25]. Under the single-mode approximation [7], a spinor-1 condensate is described by the Hamiltonian

$$H = \mu N - \lambda_s' N(N+1) + \lambda_a'(L^2 - 2N), \tag{6}$$

where $N = n_+ + n_- + n_0$ is the total number of atoms and the collective angular momentum (L) with the familiar raising (lowering) operator $L_+ = \sqrt{2}(a_+^\dagger a_0 + a_0^\dagger a_-)$ $(L_- = L_+^\dagger)$, $L_z = n_+ - n_-$, and $n_i = a_i^\dagger a_i$. μ is the chemical potential. $\lambda_{s,a}'$ are renormalized interaction coefficients, related to various s-wave scattering lengths and $\phi(\vec{r})$ [3,7]. $a_{\pm,0}$ can form a similar Schwinger representation of SU(3) in the following manner:

$$T_+ = a_+^\dagger a_-, \quad T_3 = \frac{1}{2}(n_+ - n_-), \tag{7}$$

$$\begin{pmatrix} V_+ \\ U_+ \end{pmatrix} = a_\pm^\dagger a_0, \quad \begin{pmatrix} V_3 \\ U_3 \end{pmatrix} = \frac{1}{2}(n_\pm - n_0), \tag{8}$$

$$N = n_+ + n_- + n_0, \tag{9}$$

$$Y = \frac{1}{3}(n_+ + n_- - 2n_0). \tag{10}$$

The linear combinations $X_\pm \pm X_\mp$ $(X = T,U,V)$ together with T_3 and Y resemble the set of eight generators of SU(3) in spherical representation. $T_{\pm,3}$, $U_{\pm,3}$, and $V_{\pm,3}$ fulfill commutation relations $[X_+, X_-] = 2X_3$ and $[X_3, X_\pm] = \pm X_\pm$ of the su(2) algebra. Let us emphasize that we call T operators the isospin and Y operators the hypercharge only because of their formal resemblance [21]. U and V subalgebras will be called U and V spin, respectively. We then have $L_+ = \sqrt{2}(V_+ + U_-)$, $L_z = 2T_3$, and

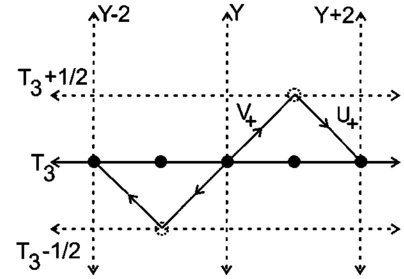

FIG. 1. The action of G_Y in T_3-Y space. Any point is coupled only to its next-nearest neighbors along T_3 axis through a two step process V_+U_+ on the Y line with T_3 unchanged in the end. Note that V_+ and U_+ commute with each other and the conjugate process is also shown.

$$L^2 = 4T_3^2 + \frac{1}{2}(N - \epsilon_+)(N - \epsilon_-) - 2(Y - Y_0)^2 + G_Y, \tag{11}$$

$$G_Y = 2(V_+U_+ + \text{H.c.}), \tag{12}$$

with $\epsilon_\pm = -3/2 \pm \sqrt{2}$, and $Y_0 = -N/6 - 1/4$. For $N \gg 1$ this gives $Y_0 \approx -N/6$, which corresponds to $n_0 = N/2$. Using $[T_3, V_\pm] = \pm V_\pm/2$ and $[T_3, U_\pm] = \mp U_\pm/2$, we find $[T_3, G_Y] = 0$ consistent with $[H, L_z] = 0$. Hence the Hamiltonian (6) separates into three commuting parts $H = H_N[N] + H_T[T_3] + H_Y[Y]$. To our knowledge, this decomposition has not been discussed before. In Ref. [22], a model of $H = \chi(T_3^2 + 3Y^2)$ has been considered for both the quantum and the semiclassical dynamics of Y as well as for SU(3) coherent states. We note that the decomposition (11) differs from the Casimir relation for the two-mode case [26]. In fact, with the spin singlet pair operator $A = (a_0^2 - 2a_+a_-)/\sqrt{3}$ as defined by Koashi and Ueda [8], we find $L^2 = N(N+1) - A^\dagger A$. We note that, neither $H \propto L^2 - 2N$ [7] nor $H \propto N(N-1) - A^\dagger A$ [4] can lead to any simple recognition of the nonlinear coupling among the various spin components.

By denoting the simultaneous eigenstates of commuting operators (N, Y, T_3) as $|N, T_3, Y\rangle$, we find

$$\begin{pmatrix} V_+ \\ U_+ \end{pmatrix} |N, T_3, Y\rangle = \sqrt{\left(\frac{N}{3} - Y\right)\left(\frac{N}{3} \pm T_3 + \frac{Y}{2} + 1\right)} \times \left| N, T_3 \pm \frac{1}{2}, Y + 1 \right\rangle,$$

i.e., G_Y only couples next-nearest neighbors along the Y axis through off-axial hopping as depicted in Fig. 1 [27]. Perhaps it is not surprising that operators $T_\pm$, $U_\pm$, and $V_\pm$ are simply the off-diagonal elements of the single-particle density

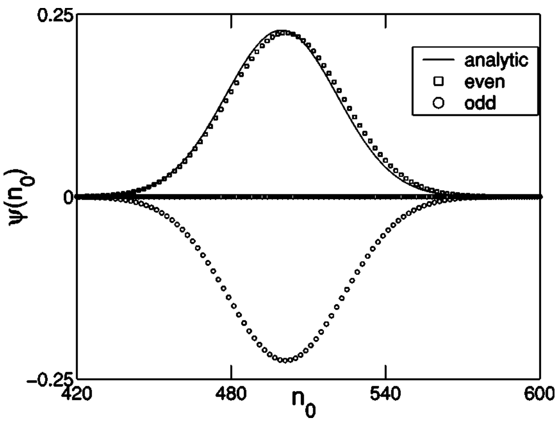

FIG. 2. The ground-state expansion coefficients $\psi(Y)\equiv\psi(n_0)$ as a function of n_0 for $N=1000$ atoms in the $T_3=0$ block. The solid curve is the approximate analytical result Eq. (15) while the other curves are obtained by an exact diagonalization procedure.

operator $\rho_{\mu\nu}=a_\mu^\dagger a_\nu$, while N, T_3, and Y are related to the diagonal elements. $T_\pm$, $V_\mp$, and $U_\pm$ all raise and lower the T_3 value by (1 or 1/2).

As an example, we consider the simple case of the $T_3=0$ block along the line in Fig. 1 of the Hamiltonian (6) for $\lambda_a'<0$, the ferromagnetic case (as in ^{87}Rb [2]). The polar case of $\lambda_a'>0$ (as in ^{23}Na BEC [1]) has been discussed in Ref. [3]. Dropping the constant $H_N[N]$ and writing in the units of $|\lambda_a'|$, the Hamiltonian becomes

$$H=2(Y-Y_0)^2-2t_Y(|Y+2\rangle\langle Y|+\text{H.c.}), \tag{13}$$

where $t_Y=(N/3-Y)(N/3+Y/2+1)$. Following the tight-binding procedure for the restricted two-mode case (+ and −) discussed in Ref. [9], its eigenstates can be found by determining the $\psi(Y)$ of $|\psi\rangle=\Sigma_Y\psi(Y)|Y\rangle$ through a difference equation

$$E\psi(Y)=2(Y-Y_0)^2\psi(Y)-2[t_{Y-2}\psi(Y-2)+t_Y\psi(Y+2)]. \tag{14}$$

In the continuum limit and up to the first order in $O(|Y|/N)$, the equivalent differential form becomes

$$\left(\frac{E}{8Y_0^2}+1\right)\psi=-2\frac{\partial^2\psi}{\partial Y^2}-\frac{1}{Y_0}\frac{\partial\psi}{\partial Y}+\frac{(Y-Y_0)^2}{4Y_0^2}\psi. \tag{15}$$

Its ground state is therefore

$$|\psi\rangle=\sum_Y \frac{\exp\left(-\frac{\sqrt{2}(Y-Y_0)^2}{8|Y_0|}+\frac{Y-Y_0}{4|Y_0|}\right)}{(\pi|Y_0|/\sqrt{2})^{1/4}}|Y\rangle, \tag{16}$$

which gives a diagonal $\langle\rho_{\mu\nu}\rangle$ with $\langle n_0\rangle=N/2$ and $\langle n_\pm\rangle=N/4$, i.e., a fragmented state [9,10]. To check the validity of this approximate analytical result, we also solved the same problem for $N=10^3$ within the $T_3=0$ block by an exact diagonalization procedure. The results are compared in Fig. 2. We see that the analytical result agrees well with the exact numerical result. Since the Hamiltonian is block diagonal in even n_0 and odd n_0 spaces, we find two degenerate ground states with even and odd n_0 components, respectively. These ground-state pairs have opposite phases as seen in Fig. 2. The approximate result here applies for a value of even N, which leads to an even n_0 within the $T_3=0$ block. Hence, only the even n_0 block of the Hamiltonian should be considered when compared with the Fig. 2. We will also show below that under the single-mode approximation, the exact ground state for $N\gg 1$ is generally a fragmented state with $\langle n_0\rangle=N/2$. We note the structural resemblance to the Schrödinger cat state (superposition of two macroscopic quantum states) separated in the angular momentum L_z, found in two-component BECs with Josephson-type coupling [28,29].

In a spinor-1 condensate as considered here, we find that the dynamical behavior can be characterized by $Y=N/3-n_0$, which can be expressed as $Y=2(U_3+V_3)/3$. Since the azimuthal phases are conjugate to angular-momentum z projection operators, the catlike ground states predicted here resemble the angular-momentum cat states of a two-component condensate in its conjugate phase spaces. Finally, we note that the symmetry point Y_0 can be adjusted by external control fields that contribute terms proportional to $n_{\pm,0}$ to the Hamiltonian, and can be absorbed into the $(Y-Y_0)^2$ term through a new Y_0 as $n_\pm=N/3+Y/2\pm T_3$ and $n_0=N/3-Y$.

We have now seen that the Hamiltonian of the system effectively describes a one-dimensional dynamics along the Y axis, similar to that of a diffusive random walk process but now with an attractor (for $\lambda_a'<0$) Y_0. Hence, we expect Y_0 to influence population dynamics in a similar manner as it affects the fragmentation. For $T_3=0$, it is known that populations oscillate around time-averaged values $n_0=N/2$ and $n_\pm=N/4$, which are the same as the results we found for the fragmented ground states. We conclude that steady-state values of population oscillations as well as fragmentation is determined by the hypercharge symmetry point Y_0, which can be shifted by external fields.

III. TWO-SPIN AND ISOSPIN SQUEEZING

The form of G_Y suggests the existence of two-mode squeezing as was also noted recently by Duan *et al.* [11], who studied a spinor-1 condensate initially prepared in the Fock state with only $m_f=0$ state populated. During the time when the total number of excitations into states $m_f=\pm 1$ are negligible, the spin-mixing term (G_Y) in the Hamiltonian simply reduces to a two-mode squeezing nonlinearity via $\langle n_0\rangle(a_+^\dagger a_-^\dagger+\text{H.c.})$. This creates a continuous variable-type entanglement, or mode entanglement in the second quantization form. In order to relate continuous variable-type entanglement to measurable spectroscopic spin squeezing and particle entanglement, Ref. [11] first showed that in the low excitation limit, the two-mode entanglement criterion can also be expressed in terms of spin squeezing parameters for $L_{x,y}$.

In order to use the two-level SU(2) definition for spin squeezing of $L_{x,y}$, new pseudospins $J_\pm$ were introduced within the the two-level subsystems $|+1\rangle\pm|-1\rangle$ and $|0\rangle$. They found that when $L_z\approx 0$, the system Hamiltonian becomes effectively $H=\lambda_a'(L^2-2N)\approx\lambda_a'(L_x^2+L_y^2)\sim(J_{+x}^2+J_{-y}^2)$, which causes each spin-1/2 subsystem to be squeezed via the single-axis twisting scheme. For the independent single-axis twisting scheme to work efficiently in achieving substantial spin squeezing, the commutator $[J_{+x/y},J_{-x/y}]=(T_- - T_+)/4\propto T_y$, needs to be small. Hence, squeezing in the isospin is essential to achieve this two-mode squeezing goal. Without it, large quantum fluctuations in T_y would destroy the two-mode squeezing.

Unfortunately, both the relationship between the two-mode squeezing and spin squeezing as well as the interpretation in terms of a dual single-axis twisting fails to be adequate for higher excitations under more realistic situations. Indeed, for the extreme opposite case of $n_0\ll n_\pm\sim N/2$, the Hamiltonian describes a single-mode amplitude squeezing as it reduces to $G_Y\sim(a_0^{\dagger 2}+\text{H.c.})$. Anywhere in between these two extreme limits, we propose a different type of squeezing, two-spin squeezing, as a generalization of single-spin squeezing by taking into account quantum correlations for mode-entanglement applications.

We first note that the two extreme types of squeezing in G_Y can be handled at arbitrary levels of excitation by introducing a new two-spin squeezing operator via

$$K_+=V_+U_+\sim\begin{cases} a_+^\dagger a_-^\dagger, & n_0\gg n_\pm \\ a_0^2, & n_0\ll n_\pm, \end{cases} \tag{17}$$

$$K_-=K_+^\dagger, \tag{18}$$

with $[K_-,K_+]=2K_3$. The squeezing mechanism in Hamiltonian (6) is now understood to be a generalized BD-type squeezing via the $V_+U_+ +\text{H.c.}$ nonlinearity in G_Y. This is significantly more complicated than the two-bosonic-mode squeezing as the two spins U and V have a noncommuting algebra. The mode entanglement of approximate bosonic modes $a_\pm'=a_\pm a_0^\dagger/\sqrt{\langle n_0\rangle}$ of Ref. [11] can in fact be generalized to mode entanglement between exactly bosonic Holstein-Primakoff modes [30], $a_{x=u,v}$ defined through $X_+=a_x^\dagger\sqrt{S_x-N_x}$ and $X_3=N_x-S_x/2$, in the spin $S_x/2$ realization of corresponding su(2) algebras of U and V spins with $N_x=a_x^\dagger a_x$. The squeezing treatment with the exact bosonic modes a_U and a_V remains to be more complicated than the usual two-bosonic-mode squeezing as it also suffers from the underlying noncommutative algebra. This representation reduces to the usual SU(1,1) two-mode squeezing or amplitude squeezing in the appropriate n_0 limits. At low excitations when $n_0\approx N$, we have $X_3\approx -N/2$, $S_x\approx\langle n_0\rangle$, and $N_x\approx 0$. In this case, $X_-\approx\sqrt{\langle n_0\rangle}a_x$ and $G_Y=2\langle n_0\rangle(a_v^\dagger a_u^\dagger+\text{H.c.})$ demonstrates the two-mode [SU(1,1)] squeezing as in Ref. [11]. In the large n_0 scheme of Ref. [11], such modes are sparsely populated, since $a'^\dagger_+a'_+=n_+(1+n_0)/n_0$. In the opposite case of large $n_\pm$, we are in the strong excitation regime with $N_v\approx 1$, $S_{v/u}\approx\langle n_\pm\rangle$, which gives effective modes to be $a_\pm^\dagger a_0/\sqrt{n_\pm}$ with large occupations.

In order to define two-spin squeezing introduced via the K operators in a similar way to the two-mode bosonic squeezing, we introduce Hermitian quadrature operators

$$X_u^\alpha=(e^{i\alpha}U_- +\text{H.c.})/\sqrt{2}, \tag{19}$$

$$X_v^\alpha=(e^{i\alpha}V_- +\text{H.c.})/\sqrt{2}, \tag{20}$$

$$Q_+^\alpha=(X_v^\alpha+X_u^\alpha)/2, \tag{21}$$

$$Q_-^{\alpha+\pi/2}=(X_v^{\alpha+\pi/2}-X_u^{\alpha+\pi/2})/2. \tag{22}$$

From $\vec{J}_\pm=\vec{V}-\vec{U}$ and $J_{\pm 3}=(3Y/2\pm T_x)/2$, we find $Q_\pm^\alpha=\vec{n}(\alpha)\cdot\vec{J}_\pm$ with $\vec{n}(\alpha)=(\cos\alpha,\sin\alpha,0)$. If U and V were not correlated, their respective quantum noises would contribute to that of $J_\pm$ additively. Existence of quantum correlations between the U and V spins would reduce the quantum fluctuation in $J_\pm$. Thus, the $(J_\pm)$ spin squeezing is achieved by two-spin (U-V) squeezing. From $[U_-,V_-]=0$, we find that

$$(\Delta Q_+^\alpha)^2+(\Delta Q_-^{\alpha+\pi/2})^2=\sum_{a=u,v}[(\Delta X_a^\alpha)^2+(\Delta X_a^{\alpha+\pi/2})^2]+\mathcal{C}_{uv}^\alpha, \tag{23}$$

with the U-V correlation function

$$\mathcal{C}_{uv}^\alpha=e^{-2i\alpha}\langle V_+,U_+\rangle/2+\text{c.c.} \tag{24}$$

denote the correlations among U-V spins to the quadrature noise, which reduces the uncertainty bound when two-spin squeezing occurs. We find a lower bound for the quadrature noise $\Sigma_{a=u,v}[(\Delta X_a^\alpha)^2+(\Delta X_a^{\alpha+\pi/2})^2]$, by noting that $[X_v^\alpha,X_v^{\alpha+\pi/2}]=2iV_3$, $[X_u^\alpha,X_u^{\alpha+\pi/2}]=-2iU_3$, and $|\langle U_3\rangle|+|\langle V_3\rangle|\geq|\langle U_3+V_3\rangle|=3|\langle Y\rangle|/2$. We finally find

$$(\Delta Q_+^\alpha)^2+(\Delta Q_-^{\alpha+\pi/2})^2\geq 3|\langle Y\rangle|/4+\mathcal{C}_{uv}^\alpha. \tag{25}$$

Therefore, taking into consideration the important spin-spin correlation between different particles similar to the spin-1/2 case [19], we can introduce the U-V squeezing condition as

$$\xi_{uv}^\alpha=\frac{\Delta(Q_+^\alpha)^2+(\Delta Q_-^{\alpha+\pi/2})^2}{|\langle Y\rangle|}<3/4, \tag{26}$$

similar to the continuous variable system [31]. This is the central result of this paper on the two-spin squeezing in a spinor-1 condensate.

The significance of spin-spin correlation function to spin squeezing and entanglement for a two-mode system was previously discussed in Ref. [32], where they showed that a negative, finite correlation parameter causes spin squeezing and entanglement of the atomic states. With the Holstein-Primakoff relations, it is straightforward to show this condition contracts into $(\xi_+^\alpha)^2+(\xi_-^{\alpha+\pi/2})^2<2$ when $n_0\to N$. Thus Eq. (26) generalizes the two-mode entanglement criterion $(\xi_+^\alpha)^2+(\xi_-^{\alpha+\pi/2})^2<2$ at low excitations [11] to arbitrary levels of excitation for two-spin squeezing. For completeness, we note the squeezing parameter for $J_\pm$ spins are [11]

$$(\xi_\pm^\alpha)^2 = \frac{N\langle(\Delta Q_\pm^\alpha)^2\rangle}{\langle Q_\pm^{\alpha+\pi/2}\rangle^2 + \langle J_{\pm 3}\rangle^2}, \tag{27}$$

while the Heisenberg uncertainty relation gives $\Delta Q_\pm^\alpha \Delta Q_\pm^{\alpha+\pi/2} \geqslant |\langle J_{\pm 3}\rangle|/2$. For many-particle entanglement of three-level atoms, the criterion is given by either $\xi_\pm < 1$.

The *U*-*V* squeezing discussed above displays existence of nonlinear interactions within/among *T*, *U*, and *V* subspace of Eq. (6). One may also contemplate for a one-axis isospin twisting (through T_3^2) of the particular form of L^2 (11). However, the dynamics of spinor-1 BEC becomes considerably more complicated because of off-axis hopping processes along the hypercharge axis (as in Fig. 1). Due to the noncommutativity of subspin systems (U,V,T), squeezing and entanglement appear even without essentially any axis twisting. In fact, even when $T_3=0$, squeezing within the isospin subgroup can still happen as the *U*-*V* two-spin squeezing interaction would redistribute the noise also for the isospin subspace, in addition to the *U*-*V* spin space. To appreciate this fact, let us consider the rotation operator involving only *U*-*V* spins and employ the SU(2) disentangling theorem to obtain

$$R[\zeta] = e^{\zeta L_+ - \zeta^* L_-} = e^{\eta L_+}(1+|\eta|^2)^{L_z} e^{-\eta^* L_-}, \tag{28}$$

with $\eta = \zeta \tan \zeta/|\zeta|$. Using $[V_+, U_-] = T_+$ and $[V_+, T_+] = [U_-, T_+] = 0$, we find

$$e^{\eta L_+} = e^{\eta\sqrt{2}V_+} e^{\eta\sqrt{2}U_-} e^{-\eta T_+/\sqrt{2}}. \tag{29}$$

Hence, we arrive at

$$R[\zeta] = e^{\sqrt{2}\eta V_+} e^{\sqrt{2}\eta U_-} R_T[\eta] e^{-\sqrt{2}\eta^* U_+} e^{-\sqrt{2}\eta^* V_-}, \tag{30}$$

with a rotation operator within isospin space via $R_T = \exp(-\eta T_+/\sqrt{2})(1+|\eta|^2)^{2T_3}\exp(-\eta^* T_-/\sqrt{2})$. This result reflects the nature of Euler-angle rotations in three dimensions for a spin-1 system. We thus conclude that squeezing in $\vec{J}_\pm$ through redistributing the noise via rotations is always accompanied by a redistribution of the noise in the isospin subspace.

Squeezing and many-particle entanglement via the isospin can be checked using the usual spin squeezing criterion, which for both *T* squeezing and the above derived *U*-*V* squeezing are independent of their respective initial conditions. Hence, we have now greater freedom to consider a suitably prepared spinor-1 condensate to achieve many-particle and/or mode entanglement for quantum information applications as well as various type of spin squeezing for atom interferometry and spectroscopy applications in the long-time limit with more macroscopic populations in all $f=1$ three-component states can occur. In the limiting case discussed before either $n_0 \sim N$ or $n_\pm \sim N$ is required to be large, the quantum states (modes) of interest are always sparsely populated. More generally, one can use Raman coupled laser pulses on a spinor-1 condensate to generate states with arbitrary populations in each mode and with arbitrary initial phases. This allows then for the consideration of stationary states in the fully quantum-mechanical framework for their use in squeezing-entanglement applications.

IV. RESULTS AND DISCUSSIONS

We now present some results on the numerical investigation of isospin squeezing. If the condensate atom number *N* is fixed, a generic state $|\psi(0)\rangle = (\alpha_0 a_0^\dagger + \alpha_- a_-^\dagger + \alpha_+ a_+^\dagger)^N |0,0,0\rangle/\sqrt{N!}$, can be prepared with Raman pulses [1], where $|0,0,0\rangle$ is the vacuum in the Fock basis $|n_0, n_-, n_+\rangle$ and $\alpha_j = |\alpha_j| e^{i\delta_j}$ complex. Using $m = n_+ - n_-$, we can write

$$|\psi(0)\rangle = \sum_{mk} \psi_{Nmk}(\vec{\alpha}) \left| 2k, \frac{N-m}{2} - k, \frac{N+m}{2} - k \right\rangle,$$

where $\vec{\alpha} = (\alpha_0, \alpha_-, \alpha_+)$, $k = 0,1,\ldots,(N-|m|)/2$ for even $N+m$, $2k = 1,3,\ldots,(N-|m|)$ for odd $N+m$, and

$$\psi_{Nmk} = \sqrt{C_N^{2k} C_{N-2k}^{[(N-m)/2]-k}}\, \alpha_0^{2k} \alpha_-^{[(N-m)/2]-k} \alpha_+^{[(N+m)/2]-k},$$

where $C_n^m = \binom{n}{m}$ denotes the binomial coefficient. The basis transformation coefficients between angular momentum and Fock states are available from Ref. [35], written in more compact forms as

$$|lm\rangle = \sum_k G_{lmk} \left| 2k, \frac{N-m}{2} - k, \frac{N+m}{2} - k \right\rangle, \tag{31}$$

with

$$G_{lmk} = 2^k s_l \sum_r \frac{(-1)^r}{4^r} \begin{bmatrix} Nlm \\ kr \end{bmatrix}$$

and the symbolic notation

$$\begin{bmatrix} Nlm \\ kr \end{bmatrix} = \sqrt{C_{2r}^r C_{2k}^{2r} C_l^{2k-2r} C_{N-2k}^{l-2k+2r} C_{N-l-2r}^{(N-l)/2-r}} \times C_{l-2k+2r}^{(l-m)/2-k+r} / \sqrt{C_{N-2k}^{(N-m)/2-k} C_{2l}^{l-m}}. \tag{32}$$

We note $l = N, N-2, \ldots, N-2[N/2]$ with $[n] = n, n-1/2$ for $n=$ even, odd, and $r = \max[0, k-(l-|m|)/2], \ldots, \min[k,(N-l)/2]$, $m = 0, \pm 1, \ldots, \pm l$. $k = 0,1,\ldots,(N-|m|)/2$ for $l+m=$ even and $2k = 1,3,\ldots,(N-|m|)$ for $l+m=$ odd. The normalization is given by

$$\frac{1}{s_l^2} = \sum_{j=0}^{(N-l)/2} \frac{1}{4^j} C_{2j}^j C_{(N+l)/2-j}^l. \tag{33}$$

Simpler analytic results exist for special cases, e.g.,

$$G_{Nmk} = 2^k \sqrt{C_N^{2k} C_{N-2k}^{(N-m)/2-k}} / \sqrt{C_{2N}^{N-m}}, \tag{34}$$

which takes an asymptotic form $G_{N0k} \simeq \sqrt{C_N^{2k}/2^{N-1}}$ when $N \gg 1$. We also find

$$\langle n_0 \rangle = \frac{1}{2^{N-1}} \sum_{k=0}^{N/2} C_N^{2k}(2k) = N/2, \tag{35}$$

the same result as obtained previously from Eq. (16).

These expressions allow us to express the initial state as $|\psi(0)\rangle = \Sigma_{lm}\psi_{lm}(0)|lm\rangle$, with

$$\psi_{lm}(0) = \Lambda_{Nm}(\alpha_-,\alpha_+)\sum_k \eta^k(\vec{\alpha}) G_{Nmk} G_{lmk}, \tag{36}$$

$$\Lambda_{Nm} = \sqrt{C_{2N}^{N-m}}\,\alpha_-^{(N-m)/2}\alpha_+^{N+m/2}, \tag{37}$$

and $\eta = \alpha_0^2/(2\alpha_-\alpha_+)$. When $\eta = 1$, it becomes an eigenstate of Hamiltonian (6) as $\Sigma_k G_{Nmk} G_{lmk} \to \delta_{Nl}$. This generalizes the stationary state of Ref. [7] into the quantum regime. The condition for stationarity becomes $2\delta_0 = \delta_- + \delta_+$ and $|\alpha_-| + |\alpha_+| = 1$ (since $|\alpha_0|^2 + |\alpha_-|^2 + |\alpha_+|^2 \equiv 1$). For the special case of $\delta_j = 0$ and $\alpha_- = \alpha_+$, we obtain $\alpha_0^2 = 1/2$, in complete agreement with earlier results [7]. By defining $P_j = |\alpha_j|^2$ as spin component populations, we find that stationary states require $P_0 = 1/2$ whenever $P_- = P_+$. This is, however, not sufficient without establishing the phase constraint found above, which becomes particularly useful as it provides for more freedom in state preparation using Raman coupled laser fields. As an example, we now consider isospin squeezing with the same form of initial states as in Ref. [7] for $\alpha_0 = \sqrt{P_0}e^{i\theta/2}$ and $\alpha_\pm = \sqrt{1-P_0}$. This gives $\langle T_x(0)\rangle = N(1-P_0)/2$ as the only nonvanishing isospin component (at $t=0$). The population in the $m_f = 0$ component then acts as a knob between the two extreme squeezing-type discussed earlier as well as between the G_Y and T_3 terms. In the special case of within the $T_3 = 0$ block, we find that the dynamics of the system is determined only along the hypercharge (Y) axis. Previous study in Ref. [11] with initial state $|0,N_0,0\rangle$ results in spin-mixing dynamics, due to which N_0 was found to quickly reduce to some value without further oscillations or recovery. In our scheme, we find $n_{0,\pm}$ all exhibits collapse and revival patterns, so does Y as $Y = N/3 - n_0$. Even for the $T_3 = 0$ block, we have seen redistribution of noise among the U-V components affects fluctuations in the isospin as well. The squeezing parameter

$$\xi_\phi^2 = \frac{N\langle \Delta(T_y')^2\rangle}{\langle T_x'\rangle^2 + \langle T_z'\rangle^2}, \tag{38}$$

is analogous to ξ_p^α (27) but for isospin $T' = R[\phi]T$ after rotated around x axis by an angle ϕ. Isospin squeezing is then characterized by $\xi_\phi < 1$. At $\phi = 2\pi/3$, this occurs after a very short time (see Fig. 3). It is especially interesting to note that ξ_ϕ exhibits collapse and revival patterns. The optimal angle $\phi_{\min}$ for maximal squeezing (minimal ξ_ϕ) [18,26,33] is shown in Fig. 4. It oscillates around its time-averaged value $\approx 2\pi/3$. In general, we find ξ_ϕ achieves its minimum sooner and the minimum is smaller with decreasing values of θ or increasing values of P_0.

This effect is clearly unique to three-mode systems. In usual population spectroscopy (e.g., Ramsey type) or in interferometry (e.g., of Mach-Zehnder type) for a two-mode

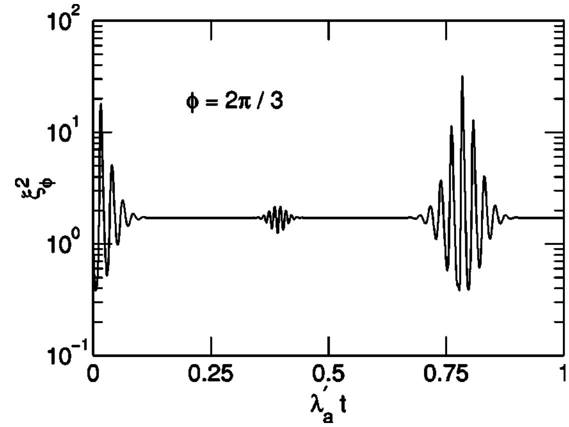

FIG. 3. Time-dependent squeezing parameter at $\phi = 2\pi/3$ for $N = 100$ atoms, $P_0 = 1/3$, and $\theta = \pi/2$.

system, particle partitioning noise and phase sensitivity can only be controlled by the modes involved directly. Here, the $m_f = 0$ mode actually does not belong to the isospin group, yet it still influences the isospin noise properties. In contrast to the two-mode result $N_\pm = J \pm J_z = N/2 \pm 2T_3$, a three-mode system has $N_\pm = N/3 + (U_3 + V_3 \pm 2T_3)/2$. A direct measurement of N_+ or N_- will uncover all noise terms due to quantum correlations among the various spin components. A measurement of $N_+ - N_-$, on the other hand, is similar to the two-mode case as the result it is only affected by the noise in the isospin. When $T_3 = 0$, the influence of the $m_f = 0$ mode population is reflected in the two-spin squeezing interaction between the U and V spins, which in turn also redistributes the noise in isospin.

In Fig. 5, results of two-spin squeezing are shown for various initial Fock states $|N_-,N_0,N_+\rangle$ of a spinor-1 condensate. The lack of oscillations in Fig. 1(a) is due to nonoscillatory behavior of n_0 for the particular initial conditions used here. The solid curves are for the two-mode entanglement criterion of Ref. [11], valid only when $N_0 \gg N_\pm$. We see that when the initial states are such that $N_\pm$ modes are not near empty, the achievable two-spin or two-mode squeezing essentially diminishes. However, there is a

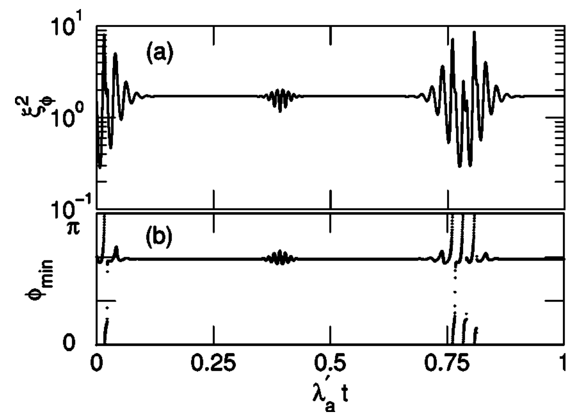

FIG. 4. (a) The same as in Fig. 2 but for the optimized squeezing parameter; (b) the optimal angle $\phi_{\min}$ that maximizes squeezing as in (a).

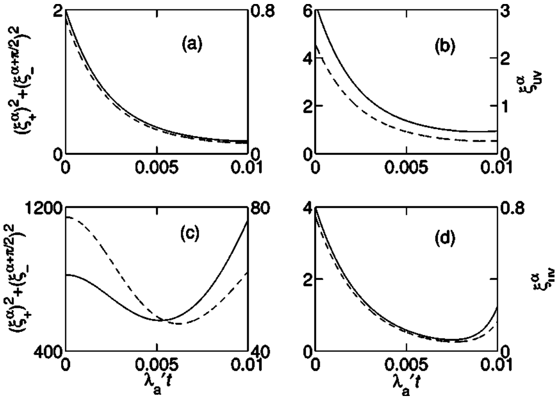

FIG. 5. Time-dependent *U*-*V* squeezing parameter (dashed curve) and two-mode entanglement criterion (solid curve) for $N=100$ atoms initially prepared in a Fock state of $\psi(0)=|N_-,N_0,N_+\rangle$: $|0,100,0\rangle$ in (a), $|1,98,1\rangle$ in (b), $|25,50,25\rangle$ in (c), and $|50,0,50\rangle$ in (d).

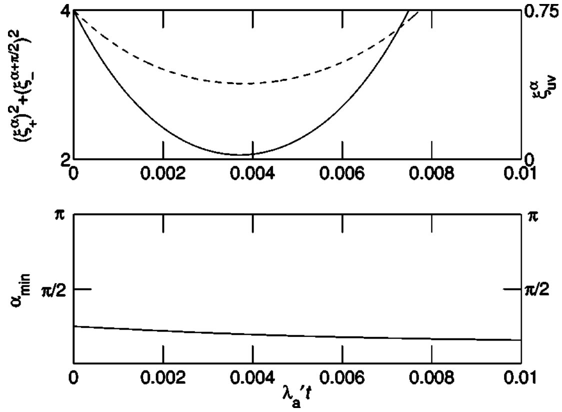

FIG. 6. Same as Fig. 5 but now for the initial state $\psi(0)=|25,0,75\rangle$.

also turning point, when squeezing is again recovered if $N_\pm$ becomes significantly populated. Hence, we have found a squeezing regime when the initial conditions are such that $N_\pm \gg N_0$. The results are almost equivalent to the case $N_0 \gg N_\pm$ considered in Ref. [11]. This new initial condition generates the two-mode entanglement via two-spin squeezing between the *U*-*V* spin modes, i.e., between the Holstein-Primakoff bosons. It should be noted that the two-mode entanglement criterion in terms of spin squeezing parameters $(\xi_\pm^\alpha)^2$ has been derived for $N_0 \gg N_\pm$ in Ref. [11]. We show here that this criterion is also satisfied in the opposite case of $N_0 \ll N_\pm$. This observation emphasizes that the *U*-*V* squeezing criterion and the corresponding mode entanglement can be sought for other initial conditions when the criterion of Ref. [11] is no longer applicable. For that aim, we consider an initial state $|25,\ 0,\ 75\rangle$ as shown in Fig. 6, where the *U*-*V* squeezing is indeed found.

V. CONCLUSION

We have provided a comprehensive treatment of quantum correlations in a spinor-1 condensate. Although no nonlinear interaction is apparent in the spinor condensate Hamiltonian when single-mode approximation is made, interesting quantum correlations do develop within subgroups of the SU(3) system. We have analyzed a spinor-1 condensate in terms of its *T*-, *U*-, and *V*-spin components. We have found and characterized squeezing within one particular subgroup, similar to that of the isospin structure and we have numerically investigated its dynamics in terms of collapses and revivals. We have developed the *U*-*V* spin squeezing as a generalization of the often adopted spin-1/2 squeezing [19] to two-spin squeezing. Its relation to mode entanglement [11] in the Holstein-Primakoff representation is also pointed out. We have presented results for condensate fragmentation and spin-mixing phenomena in terms of the hypercharge symmetry and provided general phase-amplitude conditions for stationary states in the full quantum regime.

In a typical experiment, a small magnetic-field gradient may be available [34], which results in an effective Hamiltonian [9] $H_B=\alpha(T_+ + T_-)+\beta T_3^2-\gamma_B T_3$, instead of Eq. (6), with α, β, and γ_B various renormalized parameters. In this case isospin squeezing still occurs through the one-axis twisting nonlinearity [19].

Spin squeezing parameters can be measured directly by the interferometry or Ramsey spectroscopy [19]. Alternatively, the isospin (T) squeezing in spinor-1 condensate can also be observed experimentally with light scattering. Using Raman coupled laser fields, an interaction of the type $H_R = g(T_+ J_- + \mathrm{H.c.})$ can be engineered [35,36], where $J_- = \sqrt{2}(a_L a_S^\dagger + a_L^\dagger a_A)$ is an angular-momentum operator, with a_A, a_S, a_L the annihilation operators for anti-Stokes, Stokes, and pump photons. The interaction H_R allows for the mapping of spin correlations into photon correlations as the total angular momentum T_3+J_z is conserved. The solutions for $J_-(t)$ depend on the initial conditions $J_-(0)$ and $T_-(0)$ [35,36]. Therefore, the quadrature operators of scattered photons are directly related to initial condensate spin quadratures and a homodyne measurement for Stokes parameters of the Raman field can reveal isospin squeezing [37].

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[1] D. M. Stemper-Kurn *et al.*, Phys. Rev. Lett. **80**, 2027 (1998).
[2] M. Barrett, J. Sauer, and M. S. Chapman, Phys. Rev. Lett. **87**, 010404 (2001).
[3] T.-L. Ho, Phys. Rev. Lett. **81**, 742 (1998).
[4] T. Ohmi and K. Machida, J. Phys. Soc. Jpn. **67**, 1822 (1998).
[5] W. Zhang and D. F. Walls, Phys. Rev. A **57**, 1248 (1998).
[6] E. Goldstein and P. Meystre, Phys. Rev. A **59**, 3896 (1998).
[7] C. K. Law, H. Pu, and N. P. Bigelow, Phys. Rev. Lett. **81**, 5257

(1998); H. Pu, C. K. Law, and N. P. Bigelow, Physica B **280**, 27 (2000).
[8] M. Koashi and M. Ueda, Phys. Rev. Lett. **84**, 1066 (2000).
[9] T.-L. Ho and S. K. Yip, Phys. Rev. Lett. **84**, 4031 (2000).
[10] P. Nozieres and D. Saint James, J. Phys. (Paris) **43**, 1133 (1982).
[11] L.-M. Duan, J. I. Cirac, and P. Zoller, e-print quant-ph/0107055.
[12] H. Pu and P. Meystre, Phys. Rev. Lett. **85**, 3987 (2000); L.-M. Duan *et al.*, *ibid.* **85**, 3991 (2000).
[13] M. Kitagawa and M. Ueda, Phys. Rev. A **47**, 5138 (1993); Phys. Rev. Lett. **67**, 1852 (1991).
[14] D. J. Wineland, J. J. Bollinger, W. M. Itano, and D. J. Heinzen, Phys. Rev. A **50**, 67 (1994).
[15] D. F. Walls and P. Zoller, Phys. Rev. Lett. **47**, 709 (1981).
[16] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A **6**, 2211 (1972).
[17] D. J. Wineland, J. J. Bollinger, W. M. Itano, F. L. Moore, and D. J. Heinzen, Phys. Rev. A **46**, R6797 (1992).
[18] A. Sǿrensen, L.-M. Duan, J. I. Cirac, and P. Zoller, Nature (London) **409**, 63 (2001).
[19] D. Ulam-Orgikh and M. Kitagawa, Phys. Rev. A **64**, 052106 (2001).
[20] S. M. Barnett and M.-A. Dupertuis, J. Opt. Soc. Am. B **4**, 505 (1987).
[21] M. Gell-Mann, Phys. Rev. **92**, 833 (1953); T. Nakuno and K. Nishijima, Prog. Theor. Phys. **10**, 581 (1953).
[22] K. Nemoto and B. C. Sanders, J. Phys. A **34**, 2051 (2001); K. Nemoto, C. A. Holmes, G. J. Milburn, and W. J. Munro, Phys. Rev. A **63**, 013604 (2000).
[23] F. T. Hioe, Phys. Rev. A **28**, 879 (1983).
[24] R. J. Cook and B. W. Shore, Phys. Rev. A **20**, 539 (1979).
[25] S. Yi, O. E. Müstecaplioğlu, C. P. Sun, and L. You, e-print cond-mat/0201173.
[26] Kristian Helmerson and L. You, Phys. Rev. Lett. **87**, 170402 (2001).
[27] Tin-Lun Ho and C. V. Ciobanu, e-print cond-mat/0011095.
[28] S. Raghavan, H. Pu, P. Meystre, and N. P. Bigelow, Opt. Commun. **188**, 149 (2001).
[29] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).
[30] Lu-Ming Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).
[31] G. S. Agarwal and R. R. Puri, Phys. Rev. A **49**, 4968 (1994); A. R. U. Devi, K. S. Mallesh, M. A. A. Sbaih, K. B. Nalini, and G. Ramachandran, e-print quant-ph/0204019.
[32] Y. Wu, Phys. Rev. A **54**, 4534 (1996).
[33] K. Molmer and A. Sǿrensen, Phys. Rev. Lett. **82**, 1835 (1999).
[34] J. Stenger *et al.*, Nature (London) **396**, 345 (1998).
[35] L. Wang, R. R. Puri, and J. H. Eberly, Phys. Rev. A **46**, 7192 (1992).
[36] A. S. Shumovsky and O. E. Müstecaplioğlu, Phys. Rev. Lett. **80**, 1202 (1998); Phys. Rev. B **60**, 3970 (1999).