

Characterization of Gaussian operations and distillation of Gaussian states

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We characterize the class of all physical operations that transform Gaussian states to Gaussian states. We show that this class coincides with that of all operations that can be performed on Gaussian states using linear optical elements and homodyne measurements. For bipartite systems we characterize the processes that can be implemented by local operations and classical communication, as well as those that can be implemented using positive partial transpose preserving maps. As an application, we show that Gaussian states cannot be distilled by local Gaussian operations and classical communication. We also define and characterize positive (but not completely positive) Gaussian maps.

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I. INTRODUCTION

Many applications in the field of quantum information require the ability of preparing general states and of performing arbitrary transformations with them. However, there are physical systems where the set of states that can be generated, as well as the transformations that can be implemented are very restricted. For example, in quantum optical systems linear transformations involving beam splitters, phase plates, homodyne measurements and polarizers are readily implemented, whereas more general ones can only be performed with a low efficiency. Moreover, with these tools and a squeezer one can generate only a small class of states, the so-called Gaussian states. Despite this fact, in these systems a surprising richness of quantum information protocols has been found within the realm of linear optics: entanglement generation [1], teleportation [2], key distribution [3], quantum error correction [4], and cloning [5], some of which have already been implemented [6]. For the moment it is not known how (or whether) important operations, such as entanglement distillation or a (useful) depolarization of continuous variable states, can be implemented with linear optics [7–12]. This raises the question which transformations can be realized in general by the concatenation of such operations, i.e. the tools currently available in the lab.

A partial answer to this question is contained in a mathematical paper written in the seventies [13]. There a subclass of operations that transform Gaussian states into Gaussian states have been mathematically characterized, namely, those which are *trace preserving*. The relation of these maps to some of the experimentally feasible operations has been discussed by Eisert and Plenio in Ref. [12], and used by these authors to derive a criterion for the interconvertibility of two-mode Gaussian states under feasible local transformations. Unfortunately, the set of operations considered in Ref. [13] does not include measurements (that are not trace preserving). However, one of the main strengths of linear optics is the highly efficient measurement of the quadrature observables X and P , which homodyne detection affords. Furthermore, measurements followed by classical communication have been seen to be an essential ingredient in certain basic quantum information protocols such as quantum teleportation [14] or entanglement distillation [15]. Thus it is impor-

tant to find a mathematical formulation of the physical actions that can be applied to Gaussian states in which also not trace-preserving operations (measurements) are included [16].

In this work we give a full answer to this problem by providing a simple description of all operations of this sort. As an application we discuss the question of distillation of Gaussian states by local Gaussian operations and classical communication (LOGCC). In this context, it was recently shown [17] that the entanglement of a symmetric two-mode Gaussian state of two parties cannot be increased with the help of another copy of that state and a homodyne measurement. Here we give a proof that distillation is not possible for an arbitrary number of modes per site, general Gaussian states and general Gaussian operations.

This paper is organized as follows. In Sec. II we fix our notation for Gaussian states, setting the stage for the results given in the following sections. Section III contains the main results of this paper: we give a characterization of all completely positive maps that transform Gaussian states into Gaussian states (Gaussian operations). We show that they can all be implemented with the currently available means. We derive a simple, compact form of these maps. In Sec. IV we consider Gaussian operations on bipartite systems; we classify them with respect to their locality and separability properties. As an application of the methods introduced before, we show that Gaussian states cannot be distilled by using Gaussian LOCC. In Sec. V we describe positive Gaussian maps and characterize them completely. The Appendix contains some material on a new entanglement measure for Gaussian states introduced and used in Sec. IV.

II. GAUSSIAN STATES

We consider the Hilbert space of n -harmonic oscillators $\mathcal{H} = L^2(\mathbb{R}^n)$. A Gaussian state is described by a density operator ρ whose characteristic function [18] $\chi_\rho(x) := \text{tr}[\rho W(x)]$ is a Gaussian function in $x \in \mathbb{R}^{2n}$ (or, equivalently, by a Gaussian Wigner function, which is related to χ by a symplectic Fourier transformation). The operators

$$W(x) = \exp[-ix^T R], \quad (1)$$

are the *Weyl operators* (displacement operators) and $R = (X_1, P_1, X_2, \dots, P_n)$, with $[X_k, P_l] = i\delta_{kl}$. Since the density operator is bounded, we can write without loss of generality [19]

$$\rho = \pi^{-n} \int_{\mathbb{R}^{2n}} dx e^{-(1/4)x^T \gamma x + i d^T x} W(x), \quad (2)$$

where we have used $\text{tr}[W(x)] = \pi^n \delta(x)$ [19] to normalize ρ . Occasionally, we will denote ρ as in Eq. (2) by $\rho_{\gamma, d}$. The matrix $\gamma = \gamma^T \geq iJ_n$ is a $2n \times 2n$ real matrix called *correlation matrix* (CM) and d is a $2n$ -real vector called *displacement*. These two quantities fully characterize the Gaussian state ρ . The *symplectic matrix* J_n is

$$J_n = \bigoplus_{k=1}^n J_1, \quad J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (3)$$

(We will omit the index whenever there is no risk of confusion.) Note that the CM usually contains all the interesting information about the properties of the state that are useful for quantum information in general; in particular, the entanglement properties of a Gaussian state is solely determined by its CM. Thus, in some parts of this paper, in order to simplify the notation we will omit the displacement from our discussions when it does not play a relevant role.

Of course, Gaussian states can also be defined for composite systems, e.g., those whose density operators are in $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$. An important example of a Gaussian state is the maximally entangled state Φ [20]. The state Φ is the limit $r \rightarrow \infty$ of Gaussian states (n -identical two-mode squeezed states) with CM:

$$\gamma(r) = \begin{pmatrix} A_r & C_r \\ C_r & A_r \end{pmatrix}, \quad (4)$$

where $A_r = \cosh r \mathbb{1}$ and $C_r = \sinh r \Lambda$ are $2n \times 2n$ square matrices, and

$$\Lambda = \text{diag}(1, -1, 1, -1, \dots, -1). \quad (5)$$

The density operator of Φ is a projector on the improper state vector $|\Phi\rangle_{12} \propto \sum_{k \geq 0} |k\rangle_1 |k\rangle_2$.

III. GAUSSIAN OPERATIONS

Physical actions are mathematically characterized in terms of completely positive (CP) maps acting on the corresponding density operators. The best way of characterizing them is using the isomorphism between CP maps (physical actions) and positive operators (unnormalized states) [21].

A. General form of Gaussian operations

Given a CP map $\mathcal{E}$ acting on bounded operators $\mathcal{B}(\mathcal{H})$, we define the positive operator $E \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$ as follows:

$$E_{12} = (\mathcal{E} \otimes \mathbb{1})(|\Phi\rangle_{12} \langle \Phi|). \quad (6)$$

This equation has a direct physical meaning. It tells us that given $\mathcal{E}$, we can always obtain the state E by preparing a maximally entangled state and acting with the map on the second subsystem. Conversely, given the state $E \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$ and a state $\rho \in \mathcal{B}(\mathcal{H})$, if we measure the second subsystem of E and ρ in the “Bell basis,” i.e., an orthonormal basis of maximally entangled states containing $|\Phi\rangle$, and obtain the result corresponding to the state $|\Phi\rangle$, then the resulting state is $\mathcal{E}(\rho)$. Thus, given the state E we can always implement (probabilistically) the map $\mathcal{E}$ provided we can perform Bell measurements. This can be viewed as “teleporting ρ through the gate $\mathcal{E}$ ” [22]. In formulas, we have

$$\mathcal{E}(\rho) \propto \text{tr}_2[E_{12}^T \rho_2] = \text{tr}_{23}(E_{12} \rho_3 |\Phi\rangle_{23} \langle \Phi|). \quad (7)$$

Thus, given a CP map we can generate the corresponding state and given the state we can physically implement $\mathcal{E}$ (probabilistically) [23].

Now we define a Gaussian completely positive (GCP) map, $\mathcal{G}$ by the properties that both $\mathcal{G}$ and $\mathbb{1} \otimes \mathcal{G}$ map Gaussian states to Gaussian states. With the help of the isomorphism it is straightforward to characterize them. First, we use the fact that the state $|\Phi\rangle$ appearing in Eq. (6) is Gaussian, so that the corresponding operator G must be Gaussian [24]. We write it as

$$G = \int_{\mathbb{R}^{4n}} dx e^{-(1/4)x^T \Gamma x + i D^T x - C} W(x). \quad (8)$$

Clearly $G \geq 0$ if Γ, D, C are real and $\Gamma \geq iJ$; that is, $\mathcal{G}$ is a GCP map if the Gaussian operator isomorphic to $\mathcal{G}$ is described by a proper CM Γ and, conversely, to each such operator corresponds a GCP map $\mathcal{G}$. Since all Gaussian states can be generated (e.g., from the vacuum state) by unitary Gaussian operations and discarding subsystems [25] this shows that all Gaussian operations can be implemented by these means plus Bell measurements (homodyne detection).

Now, we determine the action of the map $\mathcal{G}$ on a general Gaussian state ρ in terms of G . Apart from normalization, we have

$$\mathcal{G}: \rho_{\gamma, d} \mapsto \rho_{\gamma', d'}, \quad (9)$$

and we find for γ', d' :

$$\gamma' = \tilde{\Gamma}_1 - \tilde{\Gamma}_{12} \frac{1}{\tilde{\Gamma}_2 + \gamma} \tilde{\Gamma}_{12}^T, \quad (10a)$$

$$d' = D_1 + \tilde{\Gamma}_{12} \frac{1}{\tilde{\Gamma}_2 + \gamma} (D_2 + d), \quad (10b)$$

where we have denoted

$$\Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_{12} \\ \Gamma_{12}^T & \Gamma_2 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \quad (11)$$

and

$$\tilde{\Gamma} = (\mathbb{1} \oplus \Lambda) \Gamma (\mathbb{1} \oplus \Lambda). \quad (12)$$

Thus, we have that all GCP maps on $\mathcal{B}(\mathcal{H})$ are characterized by a correlation matrix $\Gamma \geq iJ_{2n}$ and a displacement vector D .

In order to derive Eqs. (10), we just use Eq. (7), replacing E_{12} by G and ρ_2 by a Gaussian state $\rho_{\gamma,d}$ as in Eq. (2). In evaluating the trace we use the commutation relation $W(x)W(y) = e^{i/2x^T J y} W(x+y)$ [18] and $\text{tr}[W(x)] \propto \delta(x)$ to obtain

$$\rho_{\gamma',d'} \propto \text{tr}_2(G^{T_2} \rho_{\gamma,d}), \quad (13)$$

with γ', d' as in Eqs. (10).

B. Examples

How to interpret the operation described by (Γ, D) ? To better understand what actions (Γ, D) describe, we now briefly discuss how the familiar Gaussian operations are contained in our formalism. To do this, we apply these operations to the first subsystem of the maximally entangled Gaussian state Φ with CM $\Gamma = \lim_{r \rightarrow \infty} \gamma(r)$, with $\gamma(r)$ as in Eq. (4).

Obviously, the identity operation corresponds to the maximally entangled state Φ , i.e., to $\Gamma = \lim_r \gamma(r), D = 0$. Now, performing a displacement operation on Φ leaves the CM unchanged, but produces a displacement $D = (D_1, 0)$. Now we turn to the trace-preserving GCP maps considered in Ref. [13]. These describe all actions that can be performed on ρ by first adding ancillary systems in Gaussian states, then performing unitary Gaussian transformations on the whole system, and finally discarding the ancillas. On the level of CMs these operations were shown to be described by $\gamma \mapsto M^T \gamma M + N$. A Weyl operator $W(x)$ is mapped to $(\det M)^{-1} e^{-1/4x^T (M^{-1})^T N M^{-1} x} W(M^{-1}x)$ by these operations. It then follows that the Gaussian operator that corresponds to this operation has the CM:

$$\Gamma = \lim_{r \rightarrow \infty} \begin{pmatrix} M^T A_r M + N & M^T C_r \\ C_r M & A_r \end{pmatrix}.$$

Using formulas Eq. (10) gives $\gamma' = \lim_r M^T A_r M + N - M^T C_r \Lambda (\Lambda A_r \Lambda + \gamma)^{-1} \Lambda C_r M$. For $r \rightarrow \infty$ we have $(\Lambda A_r \Lambda + \gamma)^{-1} \rightarrow A_r^{-1} - A_r^{-1} \gamma A_r^{-1} + o(\cosh r)^{-3}$ that yields the desired result [26].

Finally, we consider an example of Gaussian measurements. The typical measurement is homodyne detection, which realizes the von Neumann measurement of the operator X . It has been shown before [27] that with the use of an ancillary system and a beam splitter, homodyne measurements may be used to realize the generalized measurement corresponding to the positive-operator-valued measure (POVM) $\{|\alpha\rangle\langle\alpha|, \alpha \in \mathbb{C}\}$, where $|\alpha\rangle$ is a coherent state, i.e., in the language of CMs a state with CM, $\gamma = 1$ and displacement $d = (\text{Re } \alpha, \text{Im } \alpha)$. Since every other pure Gaussian state can be obtained from $|\alpha\rangle$ by Gaussian unitaries, this implies that all POVMs of the form $\{|\gamma, d\rangle\langle\gamma, d| : d \in \mathbb{R}^{2n}\}$ can be performed with homodyne detection and suitable preprocessing.

To see which CM Γ corresponds to the measurement of $|\gamma, d\rangle\langle\gamma, d|$ we apply it to Φ . In order to get an interesting

result, we perform only a partial measurement, i.e., we consider an $(n+m)$ -mode system and measure only the last m modes. The corresponding operator is

$$E = \langle\gamma, d|\Phi|\gamma, d\rangle,$$

where $|\gamma, d\rangle$ describes a pure Gaussian state of m modes. Expressing Φ in terms of Weyl operators, replacing $\langle\gamma, d|W(x)|\gamma, d\rangle$ by $e^{-1/4x^T \gamma x - id^T x}$ and integrating over the modes measured we obtain a CM

$$\Gamma = \begin{pmatrix} A_r & C_r & 0 \\ C_r & A_r & 0 \\ 0 & 0 & \gamma^{-1} \end{pmatrix} \quad (14)$$

and a displacement $D = (0, 0, d)^T$. Note that the first row corresponds to system “1,” while the second and third row refer to system “2.” This represents a straightforward generalization of the situation considered in Sec. III to maps that decrease the number of modes present. Evaluating Eq. (7) for a $n+m$ mode Gaussian state with CM γ :

$$\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix},$$

we obtain [26]

$$\gamma' = A - C \frac{1}{B + \gamma_p} C^T, \quad (15a)$$

$$d' = \frac{1}{2} C \frac{1}{B + \gamma_p} d, \quad (15b)$$

which corresponds to the change in CM derived in Ref. [17] for projections into pure Gaussian states. Homodyne detection itself represents the limiting case in which the CM γ_p becomes infinitely squeezed. In this limit, the inverse in Eqs. (15) is to be understood as the pseudoinverse (inverse on the range).

In general, *noise-free Gaussian operations* (unitaries and von Neumann measurements) correspond to *pure state* CMs Γ and noise added to the CM describing the operation directly translates into noise added to the output state, i.e., we have that $\Gamma' = \Gamma + P \geq \Gamma$ implies that $G_{\Gamma'}(\gamma) \geq G_{\Gamma}(\gamma) \forall \gamma \geq iJ$. To see this, consider the operation $\mathcal{G}_{\Gamma'}$ and write G in Eq. (13) as a mixture of states with CM Γ . This shows that the state $\mathcal{G}_{\Gamma'}(\rho_{\gamma,d})$ is a Gaussian mixture of states $\mathcal{G}_{\Gamma}(\rho_{\gamma,d+d_2})$ displaced by d_1 , where $x = (d_1, d_2)$ are distributed according to a probability distribution proportional to $\exp(-1/4x^T P^{-1}x)$. But since a displacement d_2 of the input state does only affect the displacement of the output state [cf. Eqs. (10)], it follows that $\mathcal{G}_{\Gamma'}(\rho_{\gamma,d})$ is nothing but a Gaussian mixture of states $\mathcal{G}_{\Gamma}(\rho_{\gamma,d})$ displaced by some value y which is distributed according to a Gaussian distribution with covariance depending on P and γ . Thus the operation $\mathcal{G}_{\Gamma'}$ could be realized (for known γ) by first performing $\mathcal{G}_{\Gamma}$ and then performing random displacements to add the appropriate noise, which proves the assertion. Since displacements can

be done locally, this becomes particularly useful in the discussion of entanglement distillation with Gaussian means below.

C. Deterministic operations

In general, the transformation Eq. (9) is not trace preserving. This is related to the fact that we have considered only one of the possible Bell measurements in Eq. (7). The projector $|\Phi\rangle\langle\Phi|$ can be extended to a POVM by considering all displacements $W(x)|\Phi\rangle$, $x \in \mathbb{R}^{4n}$. Using the second relation in Eq. (7) it is easy to see that if $\mathcal{G}_{\Gamma,0}(\rho) = \text{tr}_{23}(E_{12}\rho_3|\Phi\rangle_{23}\langle\Phi|)$ then $\mathcal{G}_{\Gamma,D}(\rho) = \text{tr}_{23}[E_{12}\rho_3 W(-D)|\Phi\rangle_{23}\langle\Phi|W(D)]$, i.e., D can be understood as the (continuous) output of the Bell-measurement implementing $\mathcal{G}$.

Note that D has no influence on the CM of the resulting state. Hence, provided γ and Γ are known, $\mathcal{G}$ can be turned into a trace-preserving operation by postprocessing: conditional on the measurement result (D_1, D_2) the corresponding displacement $D_1 + \tilde{\Gamma}_{12}(\tilde{\Gamma}_2 + \gamma)^{-1}D_2$ can be undone, leading to a *deterministic* transformation that maps every Gaussian state $\rho_{\gamma,d}$ to $\rho_{\gamma',\tilde{\Gamma}_{12}(\tilde{\Gamma}_2 + \gamma)^{-1}d}$ with certainty. Note that this is true even if γ is a state on a multipartite system, since displacements can be done locally. It is a curious feature of Gaussian operations that even measurements do not change the CM nondeterministically.

IV. BIPARTITE SYSTEMS. APPLICATIONS

In this section we consider Gaussian maps $\mathcal{G}$ on bipartite systems. In this situation it is interesting to distinguish whether $\mathcal{G}$ can be implemented with *local operations* on the subsystems A and B (possibly enhanced by classical communication, LOCC) or whether interaction between the system is necessary. Our formalism yields a very convenient form for any local Gaussian operations, and allows us to determine the nonlocal properties for any given Gaussian map.

As an application we use our formalism to show that entanglement distillation is not possible with Gaussian means. This extends the results of Ref. [17] to any number of modes, all kinds of Gaussian operations, and all kinds of Gaussian states. Note that, there are other means of performing distillation that do not require Gaussian maps, as long as a Kerr nonlinearity or photodetection are available [7,9]. However, at the moment these protocols still pose considerable experimental challenges and none has been implemented to date.

A. Local Gaussian maps assisted by classical communication

To determine whether $\mathcal{G}$ can be implemented with *local operations* on the subsystems A and B (possibly enhanced by classical communication, LOCC) or whether interaction between the systems is necessary we can use the ideas of Ref. [28] (that extend the Jamiołkowski isomorphism to bipartite systems). This allows us to read off the answer to this question from the Gaussian state G isomorphic to $\mathcal{G}$. If G is separable, then it can be generated by local action and classical communication. Note that following the discussion at

the end of the previous section this implementation can be done *deterministically* provided the CM of the state on which we act is known.

If G is entangled, two cases can be distinguished: is G has positive partial transpose (PPT), the corresponding map can be implemented with a so-called PPT-preserving channel, otherwise full-fledged quantum interaction between A and B is needed.

The separability criterion for Gaussian states [29,30] allows us to decide for every given map, whether it is separable [30], PPT-preserving [29] or neither. Moreover, the characterization of separable CMs given in Ref. [31], namely that γ is separable if $\exists \gamma_A, \gamma_B \geq iJ$ such that $\gamma \geq \gamma_A \oplus \gamma_B$ implies that—except for added correlated noise—all Gaussian LOCC operations are of product form.

B. Gaussian states cannot be distilled with Gaussian local operations and classical communication

Entanglement distillation is a process in which two separate parties A and B transform a large number of copies of a bipartite mixed entangled state ρ_{AB} (jointly written as $\rho_{AB}^{\otimes n}$) into a state $\psi_{AB}^{(n)}$, which, as n goes to infinity approaches a pure maximally entangled state. To this end, A and B are allowed to perform arbitrary local operations (correlated by classical communication) on their respective part of $\rho_{AB}^{\otimes n}$. In the following we show that such a process is not possible when ρ_{AB} is Gaussian and only Gaussian operations are allowed.

We consider a bipartite system composed of subsystems A and B and a partially entangled Gaussian state with CM γ_{AB} and want to check if a separable Gaussian map can increase the entanglement. To this end we define a simple function $V(\gamma)$ to quantify the entanglement of a general bipartite CM. Let $V(\gamma_{AB})$ be the largest value $p \leq 1$ such that $\gamma_{AB} \geq p(\gamma_A \oplus \gamma_B)$, for some CMs γ_A and γ_B . Note that for a maximally entangled Gaussian state $V=0$ [this and further properties of $V(\gamma)$ which are used in the following are proved in Appendix], and therefore the goal of a Gaussian distillation protocol would be to decrease $V(\gamma_{AB})$. In general one would allow an arbitrary number copies of the state with CM γ_{AB} , i.e., a state with the CM $\oplus_{k=1}^n \gamma_{AB}$. As shown in the Appendix, V does not change when adding more copies of the same state. The question then is, whether there is a local Gaussian operation $\mathcal{G}$ that produces from these states an output state γ'_{AB} with $V(\gamma'_{AB}) < V(\gamma_{AB})$ or even allows us to reach $V \rightarrow 0$ in the limit of infinitely many copies.

In the following we show that with local Gaussian operations it is impossible to decrease V at all. Our proof makes no assumptions on the size or type of the entangled state considered or the local Gaussian operations performed. In particular, it covers any number of copies of a $(n \times n)$ -mode input state.

We consider a separable Gaussian completely positive map acting on two systems A and B . As discussed above, the action of such a map on the correlation matrix γ_{AB} is completely characterized by another correlation matrix Γ acting on an extended space of systems A, A', B , and B' . The fact that the map is separable implies that $\Gamma = \Gamma_{AA'} \oplus \Gamma_{BB'} + P$,

where P is a positive matrix. In light of the discussion at the end of Sec. III B this means that Γ can be implemented by first performing the (completely uncorrelated) operation corresponding to $\Gamma_{AA'} \oplus \Gamma_{BB'}$ and then performing (classically correlated) random displacements of the resulting state according to a probability distribution depending on P and the CM γ_{AB} of the input state. Since these displacements do not increase the entanglement, we can concentrate on the effect of the product transformation $\Gamma_{AA'} \oplus \Gamma_{BB'}$.

Now let

$$\Gamma_{AA'} = \begin{pmatrix} A_1 & C_A \\ C_A^T & A_2 \end{pmatrix}, \quad \Gamma_{BB'} = \begin{pmatrix} B_1 & C_B \\ C_B^T & B_2 \end{pmatrix}. \quad (16)$$

Let us denote by γ'_{AB} the correlation matrix of A and B after the action of the map $\Gamma_{AA'} \oplus \Gamma_{BB'}$. Then we have $\gamma'_{AB} \geq R_A \oplus R_B$, where

$$R_A = \tilde{A}_1 - \tilde{C}_A \frac{1}{\tilde{A}_2 + p\gamma_A} \tilde{C}_A^T \geq \tilde{A}_1 - \tilde{C}_A \frac{1}{\tilde{A}_2 + piJ} \tilde{C}_A^T, \quad (17)$$

and similarly for R_B . Now, we use that $\Gamma_{AA'} \geq 0$, $i(J \oplus J)$ since Γ is a CM and therefore, $\tilde{\Gamma}_{AA'} \geq p[iJ \oplus (-iJ)]$. This implies that the right-hand side of Eq. (17) is $\geq ipJ$ and we immediately obtain that $\gamma'_A \equiv 1/p R_A$ is a correlation matrix. From this follows that $\gamma'_{AB} \geq p(\gamma'_A \oplus \gamma'_B)$ and therefore, $V(\gamma'_{AB}) \geq V(\gamma_{AB})$.

What does this imply for distillation? First, it proves that the maximally entangled state cannot be approached even asymptotically (i.e., in the limit when initially infinitely many copies of γ_{AB} are available). This follows directly from the fact (cf. Appendix) that $V(\gamma_{AB} \oplus \gamma_{AB}) = V(\gamma_{AB})$, i.e., V is invariant when adding more copies of the same resource. So entanglement distillation of Gaussian states with Gaussian means is impossible. More generally, $V(\gamma_{AB})$ puts a bound on all state transformations that can be achieved by Gaussian LOCC [and even Gaussian LOCC supplemented by an unlimited amount of auxiliary entangled Gaussian states of $V(\rho_{aux}) \geq V(\gamma_{AB})$].

However, the result still leaves room for interesting entanglement transformations with Gaussian means. The best thing that could happen—respecting the bound set by $V_{min} = V(\gamma_{AB})$ —is to have the *pure* entangled state with $V(\rho_{pure}) = V_{min}$. Our proof does not rule out the possibility of “entanglement purification,” i.e., of transforming a large number of Gaussian mixed entangled states into a (asymptotically) pure entangled state with the same value of V with Gaussian means. First calculations indicate that this might indeed be possible. These results will be reported elsewhere.

V. GAUSSIAN POSITIVE MAPS

In this section we show how to extend the approach presented in Sec. III to include Gaussian *positive* but not completely positive maps. To this end, we first define the set of Gaussian operators, generalizing Gaussian density matrices to not self-adjoint operators. Every operator $A \in \mathcal{B}(\mathcal{H})$ is

completely determined by $\chi_A(x) := \text{tr}[AW(x)]$ [18]. It follows that A may be written in terms of χ_A as [19]

$$A = \pi^{-n} \int_{\mathbb{R}^n} dx \chi_A(x) W(-x). \quad (18)$$

We define the set of Gaussian operators on $\mathcal{H}$ by

$$\mathcal{Q}(\mathcal{H}) := \left\{ A : A = \int_{\mathbb{R}^{2n}} dx e^{-(1/4)x^T \gamma x + ib^T x - c} W(x) \right\}, \quad (19)$$

where $\gamma^T = \gamma \in M_{2n}(\mathbb{C})$, $b \in \mathbb{C}^{2n}$, and $c \in \mathbb{C}$. It is straightforward to check that A is (i) *bounded* iff $\text{Re } \gamma > 0$; (ii) *self-adjoint* iff $\text{Im } \gamma = 0$, $\text{Im } b = 0$, $\text{Im } c = 0$; and (iii) *positive* iff self-adjoint and $\gamma \geq iJ$; and (4) $\text{tr}(A) = e^{-c}$, where $\text{tr}[W(x)] = \pi^n \delta(x)$, $x \in \mathbb{R}^{2n}$ was used. To prove (3), consider one mode, $\gamma = g1_2$. The corresponding operator is $\forall g \geq 0$ diagonal in the number basis (for $g \geq 1$ it describes the well-known thermal states of a field mode) and the eigenvalues $\langle n|A_g|n \rangle$ are seen to be all positive iff $g \geq 1$, otherwise odd numbers correspond to negative eigenvalues. Finally, recall that all self-adjoint Gaussian operators can be transformed into (a tensor product of Gaussian operators of) that form by quasifree unitaries, performing the normal mode decomposition of ρ or, equivalently, the symplectic diagonalization of γ , which concludes the proof.

Now we turn to linear maps on $\mathcal{B}(\mathcal{H})$. Generalizing Sec. III we define a *Gaussian map* as a linear map $\mathcal{G} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}')$ that maps Gaussian operators to Gaussian operators, i.e., $\mathcal{G}[\mathcal{Q}(\mathcal{H})] \subset \mathcal{Q}(\mathcal{H}')$. Again, we can use the isomorphism of Ref. [21] to show that all Gaussian maps on $\mathcal{B}(\mathcal{H})$ correspond to Gaussian operators on $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$, i.e., they may be described by a matrix Γ , a vector D , and a phase/normalization constant C . Then we can use Eq. (7) to calculate how the Gaussian map $\mathcal{G}$ corresponding to (Γ, D, C) acts on the Weyl operator $W(x)$. One finds

$$\mathcal{G}(W(x)) = \int e^{-(1/4)\begin{pmatrix} x \\ y \end{pmatrix}^T \tilde{\Gamma} \begin{pmatrix} x \\ y \end{pmatrix} + i\tilde{D}^T \begin{pmatrix} x \\ y \end{pmatrix} - C} W(y) dy,$$

where $\tilde{\Gamma}, \tilde{D}$ belong to the partial transpose of the operator G isomorphic to $\mathcal{G}$.

One quickly convinces oneself that the map $\mathcal{G}$ is self-adjoint, that is $\mathcal{G}(A^\dagger)^\dagger = \mathcal{G}(A)$ iff G is self-adjoint, i.e., iff $\text{Im } \Gamma = 0$, $\text{Im } D = 0$, $\text{Im } C = 0$.

A Gaussian map $\mathcal{G}$ is called a *Gaussian positive map* (GP map) iff it maps $\mathcal{Q}(\mathcal{H})_+$ to $\mathcal{Q}(\mathcal{H})_+$. In terms of matrices this means that $\mathcal{G}$ is Gaussian positive (g-positive) iff $\mathcal{G}(\gamma) \geq iJ \forall \gamma \geq iJ$. Expressing the action of G through its matrix Γ this is equivalent to the condition

$$\tilde{\Gamma}_1 - \tilde{\Gamma}_{12} \frac{1}{\tilde{\Gamma}_2 + \gamma} \tilde{\Gamma}_{12}^T \geq iJ \quad \forall \gamma \geq iJ.$$

Inverted this inequality it is seen to be equivalent to $\tilde{\Gamma}_1 \geq iJ$ and

$$\underbrace{\gamma + \tilde{\Gamma}_2 - \tilde{\Gamma}_{12}^T \frac{1}{\tilde{\Gamma}_1 - iJ} \tilde{\Gamma}_{12}}_M \geq 0 \quad \forall \gamma \geq iJ, \quad (20)$$

which can now be written in a γ -independent way as

$$\min_{z \in \mathbb{C}^{2n}} \max\{z^\dagger(M + iJ)z, z^\dagger(M - iJ)z\} \geq 0. \quad (21)$$

To see that this is equivalent to condition (20) note that for any $z = z_r + iz_i \in \mathbb{C}^{2n}$ there exists a symplectic map S such that $z^\dagger S S^T z = |iz_r^\dagger J z_i| = 2|z_r^\dagger J z_i| = c^2$; this can be seen immediately by extending $\{s_1 = z_r/c, s_2 = z_i/c\}$ to a symplectic basis $\{s_k\}$ and defining S by $S s_k = e_k$, where e_k refers to the canonical basis. Therefore, if condition (20) is to hold for all γ , it holds in particular for $\gamma = S^T S$.

The minimum of these maxima can for given Γ be efficiently sought numerically, thus providing a practical characterization of all positive Gaussian maps. We emphasize, that such a practical characterization of positive maps is commonly not available for general maps on d -level systems.

Condition (21) says that the matrix Γ of a positive map can be such that neither $\tilde{N} := \Gamma - iJ \geq 0$ nor $N := \tilde{\Gamma} - iJ \geq 0$, but there may be no vector $x \in \mathbb{C}^{2n}$ such that both $x^T \tilde{N} x$ and $x^T N x$ are negative. Examples are (i) CP maps ($\Gamma \geq iJ$, cf. Sec. III); (ii) maps for which $\Gamma \not\geq iJ$ but $\tilde{\Gamma} \geq iJ$. These are *decomposable* positive maps (such as transposition); (iii) Gaussian maps for which neither N nor $\tilde{N}$ is positive can also be constructed; these and the case of *nondecomposable* GP maps, i.e., those that are not derived from transposition and the relation of g -positivity to the usual notion of positivity will be discussed elsewhere [32].

VI. CONCLUSIONS

We have characterized all the physical actions that can be performed using linear optics, squeezers, and homodyne measurements. We have also characterized those that can be implemented with LOCC and those that can be implemented using PPT-preserving maps. We have used the methods developed in the preceding sections to show that Gaussian states cannot be distilled by local Gaussian operations and classical communication.

Finally we have extended the definitions given before to general linear maps that map Gaussian states to Gaussian states and provided a complete characterization of *positive* Gaussian maps. This emphasizes that Gaussian states are worth studying not only because of their experimental relevance (which will reduce as non-Gaussian states become more accessible), but also on mathematical grounds that this class of states is simple enough to derive strong results while being large enough to encompass most (if not all) aspects of entanglement.

Note added: Upon completion of this work we learned that Jaromír Fiurášek [34] independently arrived at a similar description of general Gaussian operations and, in particular, of Gaussian LOCCs.

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APPENDIX: PROPERTIES OF $V(\gamma)$

In this section we collect a number of useful properties of the quantity $V(\gamma)$ introduced in Sec. IV B. V is defined for bipartite CMs γ (or, equivalently, for Gaussian states $\rho_{\gamma,d}$) as

$$V(\gamma) := \max_{\gamma_A, \gamma_B \geq iJ} \{p \leq 1 : \gamma \geq p(\gamma_A \oplus \gamma_B)\}. \quad (A1)$$

(i) V for more than one state.

$$V(\gamma \oplus \gamma') = \min\{V(\gamma), V(\gamma')\}, \quad (A2)$$

that is, V does not decrease when several entangled states are joined together. Rather, V of the combined state is given by the smallest V of the individual states.

To see this, let $v = V(\gamma), v' = V(\gamma')$. Clearly, $V(\gamma \oplus \gamma') \geq \min\{v, v'\}$ since by definition of V we have $\gamma \oplus \gamma' \geq v(\gamma_A \oplus \gamma_B) \oplus v'(\gamma'_A \oplus \gamma'_B) \geq \min\{v, v'\}(\gamma_A \oplus \gamma_B \oplus \gamma'_A \oplus \gamma'_B)$. On the other hand $V(\gamma \oplus \gamma') \leq \min\{v, v'\}$ since $\gamma \oplus \gamma' \geq V(\gamma \oplus \gamma')(\gamma_{AA'} \oplus \gamma_{BB'})$ also holds for the reduced states with subsystems AB or $A'B'$ traced out. More generally, it follows that $V(\oplus_k \gamma_k) = \min\{V(\gamma_k)\}$.

(ii) An upper bound for V .

$$V(\rho_\gamma) \leq \min\{\lambda_{\min}(\gamma), 1\}, \quad (A3)$$

where $\lambda_{\min}(\gamma)$ is the smallest symplectic eigenvalue (smaller than 1) of the CM $\tilde{\gamma}$ of the partially transposed state ρ_γ^{TA} . $\lambda_{\min} < 1$ is necessary and sufficient for the corresponding state to have a nonpositive partial transpose.

(iii) V for the maximally entangled state $|\Phi\rangle$.

$$V(\Phi) = \lim_{r \rightarrow \infty} V(\gamma(r)) = 0, \quad (A4)$$

since $\lambda_{\min}(\gamma(r)) = e^{-r}$.

(iv) V and negativity [33]. For $1 \times N$ systems (i.e., in systems where no PPT-entanglement exists [31]) we also have $V(\rho) \geq \lambda_{\min}$ since in that case $\gamma/\lambda_{\min}$ has positive partial transpose and therefore is separable. This shows that $V(\rho)$ is related to the negativity measure of entanglement [33], and for $1 \times N$ systems $-\log_2[V(\rho_\gamma)]$ coincides with the log negativity (up to a factor). In contrast, for PPT-entangled Gaussian states [31] $V(\gamma)$ is strictly smaller than 1, while the negativity of such states is zero ($\lambda_{\min} \geq 1$).

(v) V as a Gaussian measure of entanglement. We have seen that $V(\gamma)$ does not decrease under local Gaussian operations. Hence it can be considered a measure of

entanglement for Gaussian states. In view of Eq. (4), we see that V does quantify both NPT- and PPT-Gaussian entanglement—in contrast to most other measures of entanglement calculated to date.

(vi) V is computable. It is worth pointing out that $V(\gamma)$ is also computable, as one can use the separability criterion derived in Ref. [30] to find the largest p for which γ/p is separable.

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- is very useful. See, e.g., R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 1987).
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