Deconstructing dense coding

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The remarkable transmission of two bits of information via a single qubit entangled with another at the destination is presented as an expansion of the unremarkable classical circuit that transmits the bits with two direct qubit-qubit couplings between the source and destination.

Quantum dense coding [1] enables Alice to communicate two bits of classical information by sending Bob a single physical qubit, which is maximally entangled with another qubit already in his possession. She does the trick by applying one of four unitary transformations to her member of the entangled pair, thereby converting the state of the pair into one of four mutually orthogonal two-qubit states. Bob can learn which state it is after receiving the second member of the pair.

What is surprising is that Alice appears to act on only a single qubit, thereby providing Bob with *two* bits of information by sending him only *one* appropriately prepared qubit. But this way of telling the tale downplays a second interaction that takes place before the curtain rises on the official story. That earlier interaction is required to create the entanglement between the qubits that Alice and Bob initially share.

The full story remains surprising even with this added prologue, but the real surprise is that the entangling interaction, essential for the transmission of the two bits, can take place *before* Alice has even chosen the bits she wishes to communicate to Bob. What the story really demonstrates is the remarkable ability of entangled states to store interaction in a highly fungible form that need not be cashed in until the need arises.

I have made a similar point [2] about quantum teleportation, showing explicitly how the missing interaction that makes the difference between a routine classical circuit and a quantum ''miracle'' is buried in the interaction that produces a crucial shared entangled pair before the state to be teleported need even have been formed. Because teleportation and dense coding both exploit preexisting shared entanglement to facilitate communication with surprisingly little additional interaction, one might expect there to be a similar circuit-theoretic deconstruction of dense coding. But because there is no direct mapping from one protocol to the other — teleportation involves three qubits and dense coding four — it is not obvious from the expansion in Ref. [2] of classical state swapping into quantum teleportation, how dense coding might arise from an expansion of the classical [3] circuit that communicates two bits of information by means of two direct qubit-to-qubit interactions.

In this paper, I show how to do this. The construction is given in Fig. 1. The generalization from qubits to *d*-state systems is given in Fig. 2 and Eqs. (1)–(3).

If the initial state of the four qubits in Fig. 1(a) is $|x\rangle|y\rangle|0\rangle|0\rangle$ (reading from top to bottom on the left) then the effect of the two **cX** (Controlled-NOT) gates is to transform it into $|x\rangle|y\rangle|x\rangle|y\rangle$. This automates a classical procedure by which Alice, who possesses the upper two qubits, can communicate two classical bits of information to Bob, who possesses the lower two.

To go from this undramatic classically transparent procedure to quantum dense coding we first expand the **cX** on the right into quantum components, beginning with the fact [Fig. 1(b)] that [4] **X**=**HZH**. This is useful because we wish to eliminate, or at least disguise, the direct coupling on the left between Bob's lower qubit and Alice. Because the operator **Z** is diagonal in the computational basis, it is immaterial whether **Z** acts on a control qubit immediately before or immediately after a **cX**. So since **cX** is its own inverse we can expand Fig. 1(b) to Fig. 1(c), and then move the paired **cX** and Hadamard gates to the extreme left and right, as shown in Fig. 1(d). The goal of eliminating the direct coupling between Bob's lower qubit and Alice can now be achieved by noting that the two **cX** gates on the left of Fig. 1(d) are equivalent to the three **cX** gates on the left of Fig. 1(e), since both sets, acting on the computational basis, leave the control qubits unaltered, while applying **X** to the lowest qubit if and only if the states of the two control qubits differ. But since Bob's qubits both start on the left in the state $|0\rangle$, and **X** acts as the identity on $\mathbf{H}|0\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$, the leftmost **cX** in Fig. 1(e) always acts as the identity and can be dropped from the circuit.

The result, Fig. 1(f), is an automated dense-coding circuit. The two gates on the left convert $|0\rangle|0\rangle$ to the maximally entangled state $1/\sqrt{2}(|0\rangle|0\rangle + |1\rangle|1\rangle)$. The upper member of the entangled pair is then acted on by **X,Z,ZX** or no transformation at all, depending on whether the state of the upper two qubits is $|0\rangle|1\rangle$, $|1\rangle|0\rangle$, $|1\rangle|1\rangle$, or $|0\rangle|0\rangle$. The two gates on the extreme right then transform the resulting entangled state of the two lower qubits (one of the four states of the ''Bell basis'') back to whichever computational basis state of the upper two qubits gave rise to it.

A generalization of the dense-coding protocol from qubits to *d*-state systems has recently been given by Liu *et al.* [5]. In the corresponding generalization of the circuit-theoretic derivation, the **cX** operator becomes the controlled shift

$$\mathbf{cX}: |x\rangle|y\rangle \rightarrow |x\rangle|y \oplus x\rangle, \quad 0 \leq x, y < d, \qquad (1)$$

where $\oplus$ denotes addition modulo *d*, the Hadamard transformation **H** becomes the quantum Fourier transform
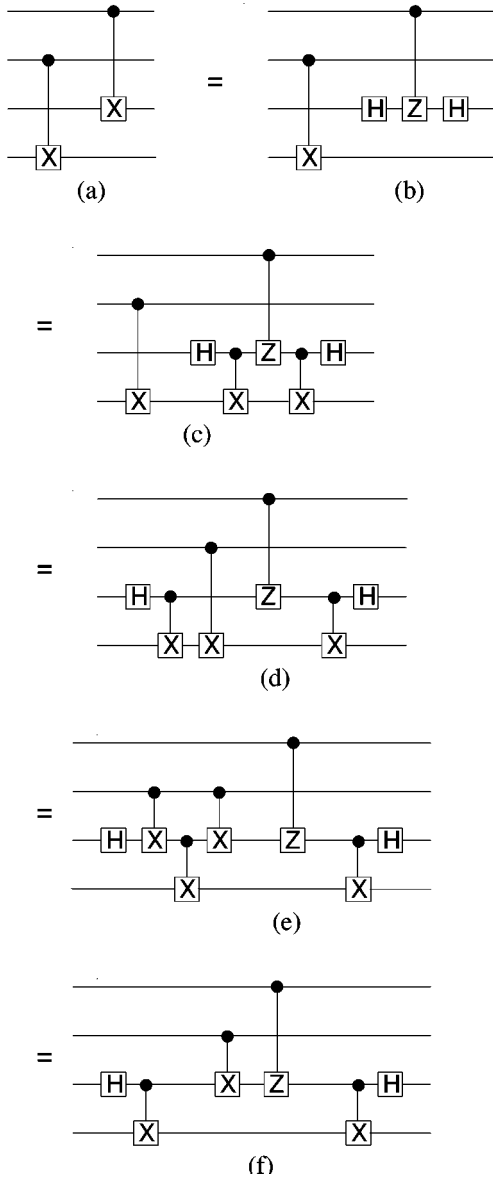
FIG. 1. How to transform the classical circuit (a) that takes $|xy00\rangle$ to $|xyxy\rangle$ by direct couplings within two pairs of qubits, into the quantum dense-coding circuit (f) that begins with preparation of an entangled state and ends with a transformation of the Bell basis into the computational basis.

$$\mathbf{H}:|y\rangle \rightarrow \frac{1}{\sqrt{d}} \sum_{0 \leqslant z < d} e^{2\pi i z y/d}|z\rangle, \qquad (2)$$

and the controlled-Z operation becomes

$$\mathbf{cZ}:|x\rangle|y\rangle \rightarrow e^{-2\pi i x y/d}|x\rangle|y\rangle. \qquad (3)$$

One easily verifies that

$$(\mathbf{H})_2(\mathbf{cX})_{12} = (\mathbf{cZ})^{\dagger}_{12}(\mathbf{H})_2 \qquad (4)$$

and therefore

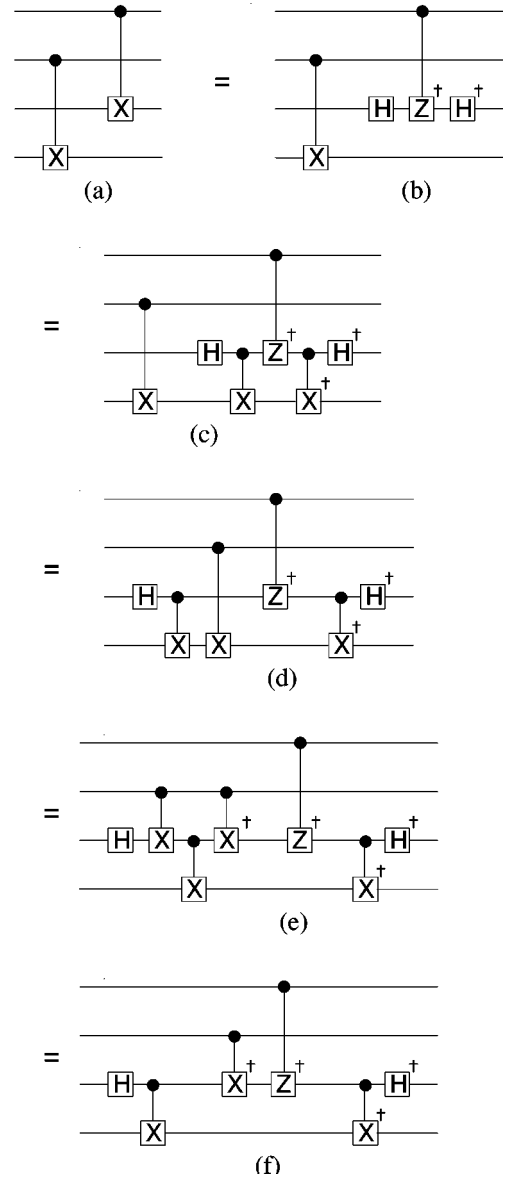$$\mathbf{cX}_{12} = (\mathbf{H})^{\dagger}_2(\mathbf{cZ})^{\dagger}_{12}(\mathbf{H})_2. \qquad (5)$$

FIG. 2. The generalizations of **cX**, **cZ**, and **H** to d-state systems are no longer their own inverses, but otherwise the extraction of d-state dense coding from the trivial classical circuit is exactly as in Fig. 1.

Figure 2 extends the identities of Fig. 1 to d-state systems. The only difference in the diagrams is that the unitary gates are no longer self-inverse, and must be distinguished from their adjoints. Figure 2(a) shows two direct couplings by controlled shifts (1) that take $|x\rangle|y\rangle|0\rangle|0\rangle$ into $|x\rangle|y\rangle|x\rangle|y\rangle$, $0 \leqslant x,y < d$. Figure 2(b) introduces [6] the identity (5). A controlled shift and its compensating inverse are introduced in Fig. 2(c). The replacement of the two **cX** gates on the left of Fig. 2(d) by the two **cX** and one **cX**† gates on the left of Fig. 2(e) is clearly valid for controlled shifts, and the **cX** gate on the left of Fig. 2(e) can be dropped since **H**$|0\rangle$ is invariant under arbitrary shifts.

Figure 2(f) is the d-state version of dense coding. The two gates on the left produce the entangled state

$$\frac{1}{\sqrt{d}} \sum_{0 \leq z < d} |z\rangle |z\rangle. \tag{6}$$

The two gates in the middle transform (6) by the action (or inaction) of the $\mathbf{cX}^{\dagger}$ and $\mathbf{cZ}^{\dagger}$ gates on the member of the entangled pair in Alice's possession. The two gates on the right act on the pair after both members are in Bob's possession, transforming its state into that product of Alice's two computational-basis states that governed the two controlled operations in the middle.

These circuit-theoretic deconstructions of dense coding (and the corresponding deconstructions of teleportation in Ref. [2]) back into elementary classical circuits, illustrate the role of entanglement as interaction-in-advance-of-need, by explicitly tracing its origin back to a direct classical interaction. They have the pedagogical virtue of requiring no algebraic scratchwork whatever [except for the confirmation of Eq. (5) for d-state systems] to verify that the quantum circuits act as advertized.

[1] Charles H. Bennett, and Stephen J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).

[2] N. David Mermin, Phys. Rev. A **65**, 012320 (2001); e-print quant-ph/0105117.

[3] I call a circuit classical if every gate transforms classical states into other classical states, i.e., if every gate acts as a permutation on states of the computational basis.

[4] I use the standard quantum computational nomenclature: $\mathbf{X} = \sigma_x$, $\mathbf{Z} = \sigma_z$, and $\mathbf{H} = 1/\sqrt{2}(\mathbf{X} + \mathbf{Z})$ is the Hadamard gate. The two-qubit controlled gates $\mathbf{cX}$ and $\mathbf{cZ}$ act with $\mathbf{X}$ or $\mathbf{Z}$ on their target qubits if their control qubits (black dot) are in the state $|1\rangle$ and as the identity if their control qubits are in the state $|0\rangle$.

[5] Bei Zeng, Xiao-Shu Liu, Yan-Song Li, and Gui Lu Long, e-print quant-ph/0104102, (unpublished).

[6] Note the deplorable but pervasive convention that the operator on the left acts first in a circuit diagram even though the operator on the right acts first in an equation.