Efficient and robust entanglement generation in a many-particle system with resonant dipole-dipole interactions

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We propose and discuss a scheme for robust and efficient generation of many-particle entanglement in an ensemble of Rydberg atoms with resonant dipole-dipole interactions. It is shown that in the limit of complete dipole blocking, the system is isomorphic to a multimode Jaynes-Cummings model. While dark-state population transfer is not capable of creating entanglement, other adiabatic processes are identified that lead to complex, maximally entangled states, such as the *N*-particle analog of the Greenberger-Horne-Zeilinger (GHZ) state in a few steps. The process is robust, works for even and odd particle numbers and the characteristic time for entanglement generation scales linearly with *N*.

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Entanglement is one of the most distinct quantum features of many-particle systems and has only recently started to be studied in a more systematic way. It provides strong tests of quantum nonlocality [1] and is at the heart of quantum information science with numerous applications [2–5]. One of the open practical problems is to identify mechanisms for its robust and controlled generation. Recently Mølmer and Sørensen suggested a scheme to create the *N*-particle analog of the GHZ state in an ion trap without the need for a precise control over the collective vibrational modes of the ions [6]. Due to Kramers degeneracy [7] in the underlying nonlinear Hamiltonian, different unitary operations needed to be applied for even and odd number of particles. The optimum interaction time *T* scales linearly with the number of particles and extreme fine tuning of *T* is required. As a consequence, this method is highly sensitive to variations of external and internal parameters.

In the present paper we discuss a robust and efficient method to create complex entangled states like the *N*-particle analog of the GHZ state [8]

$$\frac{1}{\sqrt{2}}(|a,a,\ldots,a\rangle+|b,b,\ldots,b\rangle), \tag{1}$$

i.e., a superposition of all atoms in state $|a\rangle$ and all atoms in state $|b\rangle$, in a total interaction time which scales linear with *N*. The underlying interaction is the resonant dipole-dipole interaction and the associated blockade effect in an ensemble of frozen Rydberg atoms studied in Ref. [9]. It was shown in Ref. [9] that the dipole blockade can be used to generate any (symmetric) entangled many-particle state by applying a specific sequence of resonant pulses [10]. A substantial drawback is however the need of a large number of pulses (which scales linear in *N*) with well-defined pulse area. Hence the method is also highly sensitive to variations of external and internal parameters. We will show in the present paper that this drawback can be overcome by applying adiabatic techniques.

Due to the presence of decoherence, as e.g., spontaneous emission in the dipole-blockade scheme of Ref. [9] or heating in the ion-trap scheme of Ref. [11], it is important that the total interaction time *T* is kept as small as possible. For the Sørensen-Mølmer scheme as well as the π-pulse technique in the dipole-blockade system, *T* scales linearly with the number of particles *N*, which is in fact the optimum situation. Adiabatic processes, on the other hand, require large interaction times determined by the energy separation of the instantaneous eigenstates. Since in many-particle systems the typical energy separation is exponentially small in *N*, it is not clear whether an adiabatic method can give a polynomial or even linear scaling of *T* with *N* [12].

Following the proposal of Ref. [9] let us consider an ensemble of *N* atoms with two lower levels $|a\rangle$ and $|b\rangle$ both coupled to a Rydberg state $|r\rangle$ by coherent laser fields with (real) Rabi frequencies $\Omega_1(t)$ and $\Omega_2(t)$, respectively. Let us further assume that there are two additional Rydberg levels above and below $|r\rangle$ with equal energy splitting. In such a configuration there is a resonant energy transfer between two atoms in Rydberg levels, leading to a symmetric splitting of the doubly excited states. If the minimum splitting, given by the atoms of largest separation, exceeds the natural linewidth, resonant laser excitation into doubly excited or higher excited states is suppressed (dipole blockade). In this limit the dipole-dipole interaction can easily be modeled by treating atoms in the Rydberg state as fermions with annihilation and creation operators $\sigma,\sigma^\dagger(\{\sigma,\sigma^\dagger\}=1)$, while representing atoms in levels $|a\rangle$ and $|b\rangle$ by bosons with creation and annihilation operators $a,a^\dagger$, $b,b^\dagger$, $([a,a^\dagger]=[b,b^\dagger]=1)$. The interaction can then be described by a multimode Jaynes-Cummings Hamiltonian [13],

$$H(t)=\Delta_1 a^\dagger a+\Delta_2 b^\dagger b+[(\Omega_1 a+\Omega_2 b)\sigma^\dagger+\text{H.c.}]. \tag{2}$$

The detunings Δ_i have to be much smaller than the minimum splitting of the doubly excited manifold for the blockade limit to hold.

The isomorphism to the multimode Jaynes-Cummings model has a number of interesting consequences. First of all it simplifies the analysis by allowing to employ angular momentum techniques. Second many known features of the Jaynes-Cummings dynamics, such as decay and revivals of

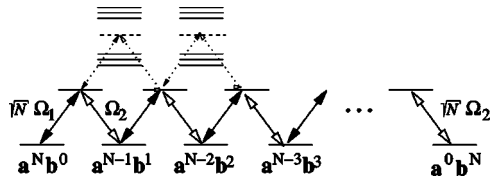

FIG. 1. Coupling scheme of collective N-atom states in limit of dipole blockade, shown here for $\Delta_1=\Delta_2=0$. Individual atoms have two lower states $|a\rangle$ and $|b\rangle$ coupled to Rydberg state $|r\rangle$ with Rabi frequencies Ω_1 and Ω_2, respectively. $|\mathbf{a}^{N-m}\mathbf{b}^m\rangle$ denotes symmetric superposition of $N-m$ atoms in state $|a\rangle$ and m atoms in state $|b\rangle$.

oscillations [14], squeezed-state generation, and quantum state transfer between different modes [15] can be anticipated in the dipole-blocking system.

The blockade of double and higher excitations results in a chainwise coupling between symmetric collective states as shown in Fig. 1. This coupling with an odd total number of levels suggests the application of dark-state Raman adiabatic passage techniques [16]. To analyze adiabatic passage in such a system it is convenient to introduce dark- and bright-state boson operators

$$D=a\cos\theta-b\sin\theta,\quad B=a\sin\theta+b\cos\theta,$$

with $\tan\theta=\Omega_1/\Omega_2$. In terms of these variables the interaction Hamiltonian reads

$$H=\frac{\Delta_1+\Delta_2}{2}(D^\dagger D+B^\dagger B)+\frac{\Delta_1-\Delta_2}{2}(D^\dagger D-B^\dagger B)\cos 2\theta + \frac{\Delta_1-\Delta_2}{2}(D^\dagger B+B^\dagger D)\sin 2\theta+\Omega_0(B\,\sigma^\dagger+B^\dagger\,\sigma), \tag{3}$$

with $\Omega_0=\sqrt{\Omega_1^2(t)+\Omega_2^2(t)}$.

Under conditions of two-photon resonance, i.e., $\Delta_1=\Delta_2$ the dark-state subspace decouples from the remaining system. Its dynamics has however SU(2) symmetry and factorized states remain factorized. Hence dark-state adiabatic transfer is not suitable for entanglement generation. This result can easily be understood physically. Since the dark state does not contain any excited-state population, the presence of dipole-dipole interactions and the resulting dipole blockade are irrelevant. Nevertheless, as will be shown in the following, adiabatic techniques can be used to create entanglement if other than the zero-eigenvalue state are involved.

To this end we consider here a situation opposite to the two-photon resonance when $\Delta=\Delta_1=-\Delta_2$. We first discuss the case of a substantial delay between the two pulses Ω_1 and Ω_2, corresponding to small values of $\sin 2\theta$ except in a very small intermediate time interval, such that the coupling between the dark and bright components is negligible. In this approximation the Hamiltonian (3) can be expressed in the simple form

$$H=\frac{\Delta}{2}\sigma_z\cos 2\theta+\Omega_0(B\sigma^\dagger+B^\dagger\sigma^-), \tag{4}$$

with $\sigma_z=\sigma^\dagger\sigma-\sigma\sigma^\dagger$. Here irrelevant terms containing the constants of motion $D^\dagger D$ (number of particles in the dark-state manifold) and $B^\dagger B+\sigma^\dagger\sigma$ (number of particles in the bright-state–Rydberg manifold) have been omitted. The corresponding Schrödinger equation can be solved analytically in the adiabatic limit, i.e., when the mixing angle $\theta(t)$ changes sufficiently slowly in time.

To obtain a convenient closed form of the solution we introduce angular momentum operators $J_1=(\sigma^\dagger B+\sigma B^\dagger)/2\sqrt{M}, J_2=i(\sigma B^\dagger-\sigma^\dagger B)/2\sqrt{M}$, and $J_3=\sigma_z/2$, where $M=B^\dagger B+\sigma^\dagger\sigma$ is the constant particle number in the bright-state–Rydberg manifold. In terms of these operators the Hamiltonian reads $H=\overline{\Omega}e^{-i\beta J_2}J_3e^{i\beta J_2}$, where $\overline{\Omega}\equiv\sqrt{4M\Omega_0^2(t)+\Delta^2\cos^2 2\theta}$ and $\tan\beta(t)=2\sqrt{M}\Omega_0(t)/\Delta\cos 2\theta$.

Let us now consider the case of all N atoms being initially in $|a\rangle$. If an intuitive pulse sequence is applied, i.e. if Ω_1 is switched on and off before Ω_2, one has: $\beta=0\to\pi$ and the system starts from a bright state

$$|\Psi_0\rangle=|\mathbf{a}^N\rangle\equiv\frac{1}{\sqrt{N!}}(a^\dagger)^N|0\rangle=\frac{1}{\sqrt{N!}}(B^\dagger)^N|0\rangle. \tag{5}$$

The unitary evolution operator then reads

$$W=-2iJ_2\exp\left[-iJ_3\int_{-\infty}^{+\infty}dt\,\overline{\Omega}(t)\right].$$

One-time application of W generates a symmetric collective state containing a single Rydberg excitation and all other atoms are in $|b\rangle$,

$$|\Psi_1\rangle=W|\Psi_0\rangle=\frac{1}{\sqrt{(N-1)!}}(b^\dagger)^{N-1}\sigma^\dagger|0\rangle,$$

corresponding to

$$|\mathbf{a}^N\rangle\overset{W}{\to}|\mathbf{b}^{N-1}\mathbf{r}\rangle. \tag{6}$$

Applying W twice, generates the W -state of Ref. [17]

$$|\mathbf{a}^N\rangle\overset{W^2}{\to}|\mathbf{a}^{N-1}\mathbf{b}\rangle,$$

where $|a^{N-m}b^m\rangle$ denotes symmetric superposition of $N-m$ atoms in the state $|a\rangle$ and m atoms in state $|b\rangle$. On the other hand starting from an initial state with all atoms in $|b\rangle$ corresponds to a pulse sequence in counterintuitive order and leads to the transfer

$$|\mathbf{b}^N\rangle\overset{W}{\to}|\mathbf{a}^N\rangle. \tag{7}$$

Iterative applications of the *same* operator W allows one to reach any state in the $2N+1$ dimensional manifold of symmetric many-particle excitations with at most one Rydberg atom. The W operation is based on adiabatic evolution and is robust against variations of parameters as long as the

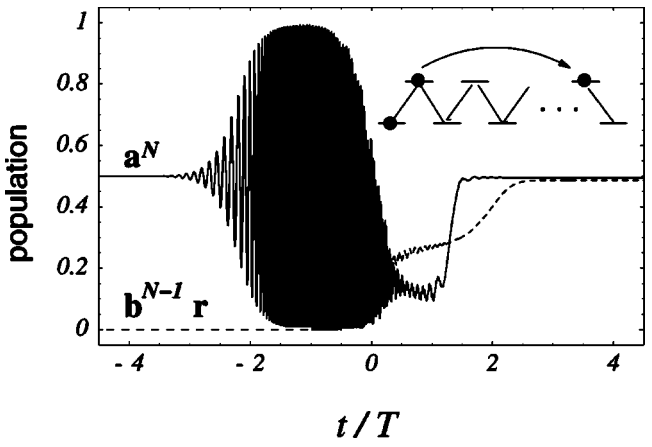

FIG. 2. Temporal evolution of population in $|\mathbf{a}^N\rangle$ and $|\mathbf{b}^{N-1}\mathbf{r}\rangle$ form initial state $|\Psi_0'\rangle$ for $N=5$. The laser pulses are Gaussian $\Omega_{1,2}(t)=\Omega_m \exp[-(t\pm\tau)^2/T^2]$, the delay is $\tau=0.5T$ the pulse area $\Omega_m T=125$, and $\Delta T=50$. Ω_m is the peak value of Rabi frequencies.

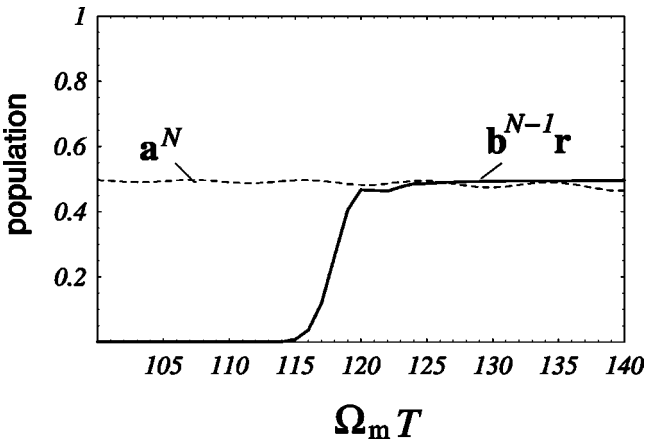

FIG. 3. Final population of states $|\mathbf{a}^N\rangle$ and $|\mathbf{b}^{N-1}\mathbf{r}\rangle$ under conditions of Fig. 2 as function of $\Omega_m T$ for $\tau=0.5T$.

condition $\gamma T \ll 1$ is fulfilled, with γ being the decay rate of the Rydberg levels and T is the time of the pulsed fields.

Although the application of W leads to an entangled state whose creation would require many π pulses, $O(N)$ steps are needed for the generation of complex states like the N-particle analog of the GHZ state (1). We will now show that a small modification of the W operation can achieve this goal in a few steps.

For this we assume that the system is initially in an equal superposition of atoms being in $|a\rangle$ and the symmetric state containing a single Rydberg excitation,

$$|\Psi_0'\rangle = \frac{1}{\sqrt{2}}(|\mathbf{a}^N\rangle + |\mathbf{a}^{N-1}\mathbf{r}\rangle). \tag{8}$$

$|\Psi_0'\rangle$ can easily be created out of $|\Psi_0\rangle$ in a robust way e.g., by sweeping the frequency of Ω_1 through resonance (rapid adiabatic passage) [18]. We now apply the W operation discussed above with a smaller time delay between the two pulses. In this process, denoted as $\tilde{W}$, the dark-bright state coupling in the Hamiltonian (3) proportional to $\sin 2\theta$ needs to be taken into account. Furthermore it is assumed that $|\Omega_0| \gg |\Delta|$. Under these conditions the Schrödinger equation can no longer be solved analytically. However numerically evaluating the equations (for N up to 30), we found the population behavior shown in Fig. 2.

The mechanism can qualitatively be understood as follows: Due to the nonvanishing detuning Δ and the chosen intuitive pulse order, the state amplitude in $|\mathbf{a}^N\rangle$ undergoes an adiabatic return process [18] and ends up in the same state as it started from. At the same time the chosen pulse order is counter-intuitive for the state $|\mathbf{a}^{N-1}\mathbf{r}\rangle$ and hence its amplitude undergoes Raman adiabatic passage to $|\mathbf{b}^{N-1}\mathbf{r}\rangle$ through a chain of successive V-type transitions. Since the fields are not in N-photon resonance, it is essential that $|\Omega_m| \gg |\Delta|$, where Ω_m is the peak value of Rabi frequencies.

While the state transfer from $|\mathbf{a}^{N-1}\mathbf{r}\rangle$ to $|\mathbf{b}^{N-1}\mathbf{r}\rangle$ is STIRAP-like [18] and thus also faithfully maps the phases of the initial to the target state, the adiabatic return process is associated with a dynamical phase $\exp\{i\phi\}$. Very similar to the situation in adiabatic quantum computation and Berry-phase measurements [19] this phase needs to be compensated. This can be achieved in the following way: After creation of the superposition $\sim(e^{i\phi}|\mathbf{a}^N\rangle + |\mathbf{b}^{N-1}\mathbf{r}\rangle)$ with unknown dynamical phase ϕ, $\tilde{W}$ is applied again with a reversed time order of the pulses. This doubles the dynamical phase accumulated in $|\mathbf{a}^N\rangle$ and returns the population from $|\mathbf{b}^{N-1}\mathbf{r}\rangle$ to $|\mathbf{a}^{N-1}\mathbf{r}\rangle$. Applying $\tilde{W}$ for a third time, however, with $\Delta \to -2\Delta$ and $\Omega \to 2\Omega$ leads to an exact cancellation of the unknown dynamical phase in $|\mathbf{a}^N\rangle$, while transferring the excited state amplitude to $|\mathbf{b}^{N-1}\mathbf{r}\rangle$.

In a final step the inverse of W, Eqs. (6) and (7), leads to the N-particle GHZ state (1). This corresponds to the overall few-step adiabatic process:

$$\begin{aligned} |\mathbf{a}^N\rangle &\to \frac{1}{\sqrt{2}}(|\mathbf{a}^N\rangle + |\mathbf{a}^{N-1}\mathbf{r}\rangle) \\ &\to \frac{1}{\sqrt{2}}(|\mathbf{a}^N\rangle + |\mathbf{b}^{N-1}\mathbf{r}\rangle) \\ &\to \frac{1}{\sqrt{2}}(|\mathbf{b}^N\rangle + |\mathbf{a}^N\rangle). \end{aligned} \tag{9}$$

The transfer is in all parts robust. It does not depend on the exact pulse form of Ω_1 and Ω_2, nor does it require an extreme control of the delay time τ or the pulse length T. Furthermore, the mechanism works for even and odd numbers of atoms in the same way. In Fig. 3 we have shown the dependence of the final population in the states $|\mathbf{a}^N\rangle$ and $|\mathbf{b}^{N-1}\mathbf{r}\rangle$ for $N=5$ as function of the pulse area $\Omega_m T$. It can be seen that the mechanism is—above some critical limit—robust against small variation of the pulse area. It should be mentioned that for very large values of the pulse area the

populations decrease again, since then the term $\Delta \sin 2\theta$ in Eq. (3) is negligible and there is a transfer $(1/\sqrt{2})(|\mathbf{a}^N\rangle + |\mathbf{a}^{N-1}\mathbf{r}\rangle) \rightarrow (1/\sqrt{2})(|\mathbf{b}^N\rangle + |\mathbf{b}^{N-1}\mathbf{r}\rangle)$.

An important question is how the time T for generating the GHZ state scales with the number of particles. To estimate T it is sufficient to discuss the chainlike STIRAP transfer in the second step. For this we consider the equivalent transfer between $|\mathbf{a}^{N-1}\rangle$ and $|\mathbf{b}^{N-1}\rangle$ in an $N-1$ particle system and assume $\Delta = 0$. The adiabatic energies of the Hamiltonian (4) are in this case $E_n = \pm \Omega_0 \sqrt{n}$ $(n = 0,1,\ldots,N-1)$, with the corresponding eigenstates $|E_0\rangle \sim (D^\dagger)^{N-1}|0\rangle$, $|E_{\pm 1}\rangle \sim (B^\dagger \pm \sigma^\dagger)(D^\dagger)^{N-2}|0\rangle$, etc. The initial state is the dark state of the $N-1$ particle system. To ensure adiabaticity it is necessary that $T \gg |\langle E_{\pm 1}|(d/d\theta)H|E_0\rangle|/|E_{\pm 1} - E_0|^2$. One easily verifies that $dH/d\theta = \Omega_0(D\sigma^\dagger + D^\dagger\sigma)$ (where one has to take into account that D and B depend on θ). With this one finds $\Omega_0 T \gg \sqrt{(N-1)/2} \sim \sqrt{N}$. On the other hand the maximum value of Ω_0 is limited by the dipole-blocking condition. If κ denotes the frequency splitting of the doubly excited Rydberg manifold, we have $\sqrt{N}\Omega_0 \ll \kappa$. Combining these conditions one arrives at

$$T \gg \frac{N}{\kappa}. \tag{10}$$

Thus we see that in the present scheme adiabatic transfer is possible in a total interaction time which scales linearly with the number of particles.

In conclusion, we have proposed an efficient and robust method to generate complex entanglement structures, such as the N-particle GHZ state in a many-particle system with resonant dipole-dipole interactions. The method is robust against variations of parameters since for all steps adiabatic transfer processes are used. Although dark-state adiabatic passage is not suitable for entanglement generation, as it does not involve population of the interacting Rydberg levels, other adiabatic processes are identified that allow e.g., for the generation of the N-particle GHZ state (1) in a few steps. The suggested method works for even and odd number of particles. Exact knowledge of the number of particles is not required, making the method robust against number fluctuations. As opposed to the proposal of Ref. [6] no extreme fine tuning of the interaction time is needed and the minimum interaction time scales linearly with the number of particles. Finally it should be mentioned that similar ideas can be applied to other many-particle systems, e.g., ions in a trap.

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