Three-qutrit correlations violate local realism more strongly than those of three qubits

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We present numerical data showing that three-qutrit correlations for a pure state, which is not maximally entangled, violate local realism more strongly than three-qubit correlations. The strength of violation is measured by the minimal amount of noise that must be admixed to the system so that the noisy correlations have a local and realistic model.

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The seminal paper of Greenberger, Horne, and Zeilinger [1] has initiated a completely new phase in the discussions regarding the Bell theorem [2]. Einstein-Podolsky-Rosen [3] elements of reality were suddenly ridiculed by a straightforward argumentation. The physics community immediately noticed that the increasing complexity of entangled systems does not lead to a less pronounced disagreement with the classical views, but just the opposite. Moreover, the disagreement exponentially grew with the number of qubits involved in the GHZ-type entangled states. Indeed, prior to the publication of Ref. [1], it was commonly perceived that everything regarding the Bell theorem is known. However, the new insight has renewed the interest in the Bell theorem and its implications.

Another widely shared perspective was that one cannot gain additional useful insight into the Bell theorem by increasing the dimensionality of the entangled systems. Some papers even suggested that in *N*-dimensional systems, increasing the dimension *N* effectively brings the system closer and closer to the classical realm. However, due to the fact that the $N>2$ dimensional systems can reveal the Kochen-Specker paradox [4], this view could be challenged. The advent of the quantum information theory created the awareness that such systems require much less entanglement to be nonseparable than qubits [5]. Certain strange features, such as bound entanglement [6] or inextensible product bases [7], suddenly emerged.

Recently, it was shown that higher dimensional entangled systems indeed may lead to stronger violations of local realism, even in straightforward experimental situations involving only the von Neumann–type experiments (with no sequential measurements, etc.).

In the early 1990s, the blueprints for straightforward Bell tests involving higher dimensional systems were given (for a summary see [8]). The idea was to use unbiased multiport beam splitters to define the local observables. Surprisingly, it turned out that such observables suffice to reveal the fact that a pair of entangled higher-dimensional systems violate local realism more strongly than qubits [9]. This result was obtained numerically by employing the linear optimization procedures to search for underlying local realistic joint probability distribution that would reproduce the quantum prediction (with some noise admixture). The results were confirmed analytically in Refs. [10] and [11]. Later in Ref. [12] it was shown that in the case of pairs of entangled higher-dimensional systems, violations of local realism are even stronger for *nonmaximally* entangled states. In a parallel research, it has been shown that higher dimensional systems can lead to the GHZ-like paradox without inequalities [13,14]. In view of all these facts, it is tempting to test the strength of violation of local realism by triples of higher-dimensional systems (starting of course with three qutrits), and that for nonmaximally entangled states.

Since Bell-type inequalities for three qutrit systems are unknown at the moment, it is necessary to invoke the numerical algorithm presented in [15]. As we shall see, some surprising results can be obtained in this way.

We show here the result of our numerical analysis.

(1) There is a strong violation of local realism (for the standard von Neumann–type measurements) for three qutrit systems in the symmetric GHZ state; however, it is not as strong as in the case of the three entangled qubits.

(2) Allowing nonmaximally entangled states, the situation changes. We find the three qutrit state which reveals correlations much more resistant to noise than those for entangled three qubits (maximally entangled three qubit states give maximal violation of local realism [16–18]).

In our numerical analysis, we consider a class of pure states of three qutrits in the form of

$$|\psi\rangle=\sum_{g,i,j=1}^{3} d_{gij}|g\rangle|i\rangle|j\rangle \tag{1}$$

with *real* coefficients d_{gij}. The kets $|g\rangle,|i\rangle,|j\rangle$ denote the orthonormal basis states for the first, second, and the third qutrit, respectively. Three spatially separated observers, Alice, Bob and Cecil, are allowed to perform the measurement of two alternative local noncommuting trichotomic observables on the state $|\psi\rangle$. We assume that they measure observables defined by unbiased symmetric three-port beam splitters [8]. In such a situation the kets in Eq. (1) represent spatial beams, in which the particles can propagate. The observers select the specific local observables by setting appropriate phase shifts in the beams leading to the entry ports of the beam splitters. The overall unitary transformation performed by such a device is given by

$$U_{j'j}=\frac{1}{\sqrt{3}}\exp\left(\frac{i2\pi}{3}(j'-1)(j-1)\right)\exp(\phi_j), \qquad (2)$$

where j denotes an input beam to the device, and j' an output one, and ϕ_j are the three phases that can be set by the local observer (for a more detailed description see [8]). Please note that the actual physics of the device is irrelevant for our theoretical discussion here, thus it suffices just to assume that the observers perform their von Neumann measurements in the basis which is related to the "computational" basis of the initial state (1) by the transformation (2). It is interesting that the unitary transformation for all phase settings leads to a new basis for the local qutrit, which is unbiased with respect to the "computational" one.

Let us denote Alice's local unitary transformations associated with her device by $U_A(\vec{\phi}_0),U_A(\vec{\phi}_1)$, Bob's by $U_B(\vec{\chi}_0),U_B(\vec{\chi}_1)$, and Cecil's by $U_C(\vec{\delta}_0),U_C(\vec{\delta}_1)$, where the three component vectors $\vec{\phi}_k,\vec{\chi}_l,\vec{\delta}_m(k,l,m=0,1)$ denote the set of the phases defining the appropriate observables. The measurement of each observable can yield three possible results which we denote by a for Alice, b for Bob, and c for Cecil $(a,b,c=1,2,3)$. The probability $P_{QM}(a_k,b_l,c_m)$, that Alice, Bob, and Cecil obtain the specific results after performing the unitary transformations $U_A(\vec{\phi}_k)$, $U_B(\vec{\chi}_l)$, and $U_C(\vec{\delta}_m)$, respectively, is given by the following formula:

$$\begin{aligned}P_{QM}(a_k,b_l,c_m)&=|\langle a_k|\langle b_l|\langle c_m|U_A(\vec{\phi}_k)U_B(\vec{\chi}_l)U_C(\vec{\delta}_m)|\psi\rangle|^2\\&=\frac{1}{27}+\frac{1}{27}\sum_{g'i'j'\neq gij}d_{g'i'j'}d_{gij}\\&\quad\times\cos\left(\frac{2\pi}{3}[(a_k-1)(g-g')+(b_l-1)(i-i')\right.\\&\quad\left.+(c_m-1)(j-j')]+\phi_k^g-\phi_k^{g'}+\chi_l^i-\chi_l^{i'}+\delta_m^j-\delta_m^{j'}\right),\end{aligned} \qquad (3)$$

where, for instance, ϕ_k^g denotes the gth component of $\vec{\phi}_k$.

In the presence of random noise, in order to describe the system one has to introduce the mixed state $\rho_F=(1-F)|\psi\rangle\langle\psi|+F\rho_{noise}$, where $\rho_{noise}=\frac{1}{27}I$, and I is the identity operator. The non-negative parameter F specifies the amount of noise present in the system. In such a case, the quantum probabilities read

$$P_{QM}^F(a_k,b_l,c_m)=(1-F)P_{QM}(a_k,b_l,c_m)+\frac{F}{27}.$$

The hypothesis of local realism assumes that there exists some joint probability distribution $P_{LR}(a_0,a_1;b_0,b_1;c_0,c_1)$ that returns quantum probabilities $P_{QM}^F(a_k,b_l,c_m)$ as marginals, e.g.,

$$P_{QM}^F(a_0,b_0,c_0)=\sum_{a_1=1}^{3}\sum_{b_1=1}^{3}\sum_{c_1=1}^{3}P_{LR}(a_0,a_1;b_0,b_1;c_0,c_1). \qquad (4)$$

Please note that a concise notation of the full set of such conditions can be given by

$$P_{QM}^F(a_k,b_l,c_m)=\sum_{a_{k+1}=1}^{3}\sum_{b_{l+1}=1}^{3}\sum_{c_{m+1}=1}^{3}\times P_{LR}(a_0,a_1;b_0,b_1;c_0,c_1), \qquad (5)$$

where $k+1,l+1,m+1$ are understood as modulo 2. For each pure state $|\psi\rangle$, one can find the threshold F_{thr} (the *minimal* value of F) above which such a joint probability distribution satisfying Eq. (5) exists (obviously, for any separable state $F_{thr}=0$; however, this may hold also for some nonseparable states).

There is a well-defined mathematical procedure called linear programming that allows us to find the threshold F_{thr} for the given state $|\psi\rangle$ and for the given set of observables. We should stress that F_{thr} found in this way gives us sufficient and necessary conditions for violation of local realism. The procedure works as follows.

The computation of the threshold F_{thr} is equivalent to finding the joint probability distribution $P_{LR}(a_0,a_1;b_0,b_1;c_0,c_1)$, i.e., the set of 3^6 of positive numbers summing up to one and fulfilling $8\times27=216$ conditions given by Eq. (5) such that F is minimal. Therefore, F and $P_{LR}(a_0,a_1;b_0,b_1;c_0,c_1)$ can be treated as variables lying in a (3^6+1)-dimensional real space. The set of linear conditions (5) and the condition that $0\leqslant F\leqslant 1$ defines a convex set in this space.

Next, we define a linear function, whose domain is the convex set defined above so that it returns the number F. The task of finding F_{thr} is then equivalent to a search for the minimum of this function. As the domain of the function is very complicated, the procedure can only be done numerically (we have used the numerical procedure HOPDM 2.30, see [19]).

It is obvious that the F_{thr} depends on the observables measured by Alice, Bob, and Cecil (which in turn depend on the set of phases) as well as on the state $|\psi\rangle$ (indeed, for some unfortunate choices of the observables, or the states or both, one can have $F_{thr}=0$). Let us clarify that the task of the linear optimization procedure is to find the *minimal* F each time for which the relation (5) can be satisfied by some positive probabilities on its right-hand side. However, the left-hand side of Eq. (5) depends on the chosen states and observables, and we are interested in the case when getting the local realistic model requires a maximal possible admixture of noise, therefore we search for such states and observables, for which the minimal F_{thr} has *the largest possible value*. There are two possible interesting scenarios. We can fix the state $|\psi\rangle$ and maximize F_{thr} over the observables. In this way we find the best violation of local realism for this given state. Alternatively, we can maximize F_{thr} over the coefficients defining the state, as well as over the observables. This procedure allows us to find the optimal state, and optimal observables measured on this state, which can yield the best possible violation of local realism by the class of pure states with real coefficients (1). Of course, we do not have to limit ourselves to pure states with real coefficients,

TABLE I. The expansion coefficients of the nonmaximally entangled state for which one requires $F_{thr}=0.571$.

Basis	$\|000\rangle$	$\|001\rangle$	$\|002\rangle$	$\|010\rangle$	$\|011\rangle$	$\|012\rangle$	$\|020\rangle$	$\|021\rangle$	$\|022\rangle$
Coeff	+0.186	+0.076	+0.230	+0.218	+0.046	+0.112	+0.172	+0.033	+0.247
Basis	$\|100\rangle$	$\|101\rangle$	$\|102\rangle$	$\|110\rangle$	$\|111\rangle$	$\|112\rangle$	$\|120\rangle$	$\|121\rangle$	$\|122\rangle$
Coeff	+0.216	+0.050	+0.110	+0.160	+0.049	+0.236	+0.204	+0.055	+0.235
Basis	$\|200\rangle$	$\|201\rangle$	$\|202\rangle$	$\|210\rangle$	$\|211\rangle$	$\|212\rangle$	$\|220\rangle$	$\|221\rangle$	$\|222\rangle$
Coeff	−0.078	+0.406	−0.029	−0.023	+0.385	+0.035	−0.123	+0.393	−0.128

nor even to pure states but then in these cases the number of parameters over which we have to optimize becomes too large for our computers to handle.

We have applied the procedure described above for the fixed state $|\psi\rangle$, which we have chosen to be a symmetric GHZ state, i.e., $|\psi\rangle=(1/\sqrt{3})(|111\rangle+|222\rangle+|333\rangle)$. Running the program we have found that the threshold amount of noise that has to be admixed to the symmetric GHZ state, so that the correlations generated by it, for *any* sets of pairs of local settings of the phases, become describable in a local and realistic way, is $F_{thr}=0.4$. The optimal observables form the point of view of violations of local realism, i.e., exactly those for which the noise admixture must be maximal to get a local realistic model, are defined by the following sets of phases $\vec{\phi}_0=(0,0,\frac{2}{3}\pi)$, $\vec{\phi}_1=(0,0,0)$; $\vec{\chi}_0=(0,0,\pi)$, $\vec{\chi}_1=(0,0,\frac{5}{3}\pi)$; $\vec{\delta}_0=(0,\frac{1}{3}\pi,0)$, $\vec{\delta}_1=(0,\pi,0)$. We can therefore say that the violation of local realism in this case is stronger than for two qutrits in the symmetric GHZ state, in which case the threshold amount of noise is only 0.304. However, it is weaker than the violation by three entangled qubits, for which the threshold amount of noise is 0.5.

Naturally, one should check whether one can obtain better violations for nonmaximally entangled states. Therefore we have taken the predictions for Eq. (1), and used a procedure for the maximization of F_{thr} over the parameters d_{gij} as well as the observables.

We have found that there exists a nonmaximally entangled state, and a certain set of local observables, for which one requires $F_{thr}=0.571$ noise admixture for the correlations to have a local realistic description. The expansion coefficients of the state are given Table I, whereas the phases defining the optimal observables will not be presented here, as they are not easily interpretable. However, for very close local settings given by $\vec{\phi}_0=(0,\frac{2}{3}\pi,-\frac{5}{9}\pi)$, $\vec{\phi}_1=(0,\frac{2}{3}\pi,0)$; $\vec{\chi}_0=(0,\frac{17}{18}\pi,-\frac{1}{18}\pi)$, $\vec{\chi}_1=(0,0,0)$; $\vec{\delta}_0=(0,\pi,\frac{23}{36}\pi)$, $\vec{\delta}_1=(0,\frac{7}{36}\pi,-\frac{2}{3}\pi)$, there is a state for which the threshold noise equals 0.570.

In summary, we have shown that for the symmetric GHZ state three entangled qutrits violate local realism stronger than two entangled qutrits (the threshold amount of noise 0.304, see [9]). The threshold amount of noise to get local realistic correlations is 0.4. This violation is not as strong as for three entangled qubits for which one has to admix 50% of noise to make the system describable by local realistic theories. However, we can obtain a much stronger violation for the nonmaximally entangled states. In this case there exists a nonmaximally entangled state (see Table I) for which $F_{thr}=0.57$, i.e., we have to add 57% of noise before we enter the region in which the state admits local and realistic description.

We must stress that although for the state given in Table I the threshold amount of noise $F_{thr}=0.57$ gives the necessary and sufficient conditions for the existence of local realism, for the measurement of the observables given by unbiased symmetric three-port beam splitters it does not mean that with a different choice of observables, or by allowing complex coefficients in the state (1), one cannot increase F_{thr}.

Moreover, it is reasonable to expect that for four or higher numbers of entangled qutrits the difference between the robustness against noise (i.e., the resistance of quantum correlations to classical description) of symmetric GHZ states and nonmaximally entangled ones will still increase. Note that an optimal nonmaximally entangled state of two qutrits (for which the threshold amount of noise is 0.3139) is around 3% more resistant to noise than the symmetric GHZ state (for which the threshold amount of noise is 0.3038). In the case of three entangled qutrits the difference between the threshold amount of noise for the nonmaximally entangled state (0.571) and for the maximally entangled state (0.4) is about 40%.

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