Exact few-particle eigenstates in partially reduced QED

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We consider a reformulation of QED in which covariant Green functions are used to solve for the electromagnetic field in terms of the fermion fields. It is shown that exact few-fermion eigenstates of the resulting Hamiltonian can be obtained in the canonical equal-time formalism for the case where there are no free photons. These eigenstates lead to two- and three-body Dirac-like equations with electromagnetic interactions. Perturbative and some numerical solutions of the two-body equations are presented for positronium- and muonium-like systems, for various strengths of the coupling.

I. INTRODUCTION

It has been pointed out in previous publications [1] that various models in quantum field theory (QFT), including QED, can be reformulated using mediating-field Green functions in such a way that exact few-particle eigenstates of the resulting partially truncated Hamiltonian can be obtained. This approach was then applied to two-body eigenstates in the scalar Yukawa (Wick-Cutkosky) theory [2,3]. We implement such an approach to QED in this paper.

The Lagrangian of two fermion fields, $\psi(x)$ and $\phi(x)$, interacting electromagnetically is

$$
\begin{aligned}
\mathcal{L} = &\bar{\psi}(x)[i\gamma^\mu\partial_\mu - q_1\gamma^\mu A_\mu(x) - m_1]\psi(x) \\
&+ \bar{\phi}(x)[i\gamma^\mu\partial_\mu - q_2\gamma^\mu A_\mu(x) - m_2]\phi(x) \\
&- \frac{1}{4}[\partial_\alpha A_\beta(x) - \partial_\beta A_\alpha(x)][\partial^\alpha A^\beta(x) - \partial^\beta A^\alpha(x)].
\end{aligned}
$$

(1.1)

The corresponding Euler-Lagrange equations of motion are the coupled Dirac-Maxwell equations

$$(i\gamma^\mu\partial_\mu - m_1)\psi(x) = q_1\gamma^\mu A_\mu(x)\psi(x), \quad (1.2)$$

$$(i\gamma^\mu\partial_\mu - m_2)\phi(x) = q_2\gamma^\mu A_\mu(x)\phi(x), \quad (1.3)$$

and

$$\partial_\mu\partial^\mu A^\nu(x) - \partial^\nu\partial_\mu A^\mu(x) = j^\nu(x), \quad (1.4)$$

where

$$j^\nu(x) = q_1\bar{\psi}(x)\gamma^\nu\psi(x) + q_2\bar{\phi}(x)\gamma^\nu\phi(x). \quad (1.5)$$

Equations (1.2)–(1.4) can be decoupled in part by using the well-known [4,5] formal solution of the Maxwell equation (1.4), namely,

$$A_\mu(x) = A_\mu^0(x) + \int D_{\mu\nu}(x-x')j^\nu(x')d^4x', \quad (1.6)$$

where $D_{\mu\nu}(x-x')$ is a Green function (or photon propagator in QFT terminology), defined by

$$\partial_\alpha\partial^\alpha D_{\mu\nu}(x-x') - \partial_\mu\partial^\alpha D_{\alpha\nu}(x-x') = g_{\mu\nu}\delta^4(x-x'), \quad (1.7)$$

and $A_\mu^0(x)$ is a solution of the homogeneous (or "free field") equation (1.4) with $j^\mu(x) = 0$.

We recall, in passing, that Eq. (1.7) does not define the covariant Green function $D_{\mu\nu}(x-x')$ uniquely. For one thing, one can always add a solution of the homogeneous equation [Eq. (1.7) with $g_{\mu\nu}\rightarrow 0$]. This allows for a certain freedom in the choice of $D_{\mu\nu}$, as is discussed in standard texts (e.g., Refs. [4,5]). In practice, the solution of Eq. (1.7), like that of Eq. (1.4), requires a choice of gauge. However, we do not need to specify one at this stage.

Substitution of the formal solution (1.6) into Eqs. (1.2) and (1.3) yields the partially reduced equations

$$(i\gamma^\mu\partial_\mu - m_1)\psi(x)$$
$$= q_1\gamma^\mu\left(A_\mu^0(x) + \int d^4x' D_{\mu\nu}(x-x')j^\nu(x')\right)\psi(x) \quad (1.8)$$

and

$$(i\gamma^\mu\partial_\mu - m_2)\phi(x)$$
$$= q_2\gamma^\mu\left(A_\mu^0(x) + \int d^4x' D_{\mu\nu}(x-x')j^\nu(x')\right)\phi(x). \quad (1.9)$$

These are nonlinear coupled Dirac equations for two different fermion fields. To our knowledge, no exact (analytic or numeric) solutions of Eqs. (1.8) and (1.9) for classical fields have been reported in the literature, even for the case of a single-fermion field (say $\phi = 0$), though approximate (perturbative) solutions have been discussed by various authors, particularly by Barut and his co-workers (see Refs. [6,7] and references therein). However, our interest here is in the quantized field theory.

The partially reduced equations (1.8) and (1.9) are derivable from the stationary action principle

$$\delta S[\psi, \phi] = \delta \int \mathcal{L}_R d^4 x = 0, \qquad (1.10)$$

with the Lagrangian density

$$\mathcal{L}_R = \bar{\psi}(x)[i\gamma^\mu \partial_\mu - m_1 - q_1 \gamma_\mu A_0^\mu(x)]\psi(x)$$

$$+ \bar{\phi}(x)[i\gamma^\mu \partial_\mu - m_2 - q_2 \gamma_\mu A_0^\mu(x)]\phi(x)$$

$$- \frac{1}{2} \int d^4 x' j^\mu(x') D_{\mu\nu}(x-x') j^\nu(x), \qquad (1.11)$$

provided that the Green function is symmetric in the sense that

$$D_{\mu\nu}(x-x') = D_{\mu\nu}(x'-x) \quad \text{and}$$

$$D_{\mu\nu}(x-x') = D_{\nu\mu}(x-x'). \qquad (1.12)$$

One can proceed to use conventional covariant perturbation theory using the reformulated QED Lagrangian (1.11). The interaction part of Eq. (1.11) has a somewhat modified structure from that of the usual formulation of QED. Thus, there are two interaction terms. The last term of Eq. (1.11) is a ''current-current'' interaction, which contains the photon propagator sandwiched between the fermionic currents. As such, it corresponds to Feynman diagrams without external photon lines. The terms containing A_0^μ correspond to diagrams that cannot be generated by the term containing $D_{\mu\nu}$, particularly diagrams involving external photon lines (care would have to be taken not to double count physical effects). However, we shall not pursue perturbation theory in this work. Rather, we shall consider an approach that allows one to write down some unorthodox but exact eigenstates of a truncated model, in which terms involving A_0^ν are ignored.

The paper is organized as follows. In Sec. II we quantize the system using the canonical equal-time formalism in the Schrödinger picture. In Sec. III an unconventional ''empty'' vacuum state is used to construct exact one-, two-, and three-fermion eigenstates of the Hamiltonian, truncated to exclude states with free (physical) photons. In Sec. IV we show that the resulting two-fermion equation is the Breit equation in the Coulomb gauge, but that it is the Eddington-Gaunt equation in the Lorentz gauge. In Sec. V we demonstrate that the Breit equation can be obtained in the Lorentz gauge, provided that higher-order retardation effects are taken into account.

The reduction of the Breit equation to radial form is described briefly in Sec. VI. For states of zero total angular momentum ($J=0$), four coupled radial equations are shown to arise. The analytical structure of their solutions is studied in Sec. VII. Perturbative $O(\alpha^4)$ corrections to the Rydberg spectrum of $J=0$ states are obtained in Sec. VIII. In the case of equal rest masses, the $J=0^+$ state equations have no unusual singularities and can be solved numerically. Some of these results are presented and discussed in Sec. IX. The remainder of the paper is devoted to the study of $J>0$ states.

In the Sec. X the set of eight coupled radial first-order differential equations is reduced to four first-order ones and then to two second-order Schrödinger-like equations. They are solved perturbatively in Sec. XI and $O(\alpha^4)$ relativistic corrections to the nonrelativistic mass spectrum are obtained. A summary and concluding remarks are given in Sec. XII.

II. HAMILTONIAN IN THE CANONICAL, EQUAL-TIME FORMALISM

We consider this theory in the canonical, equal-time formalism. To this end we write down the Hamiltonian density corresponding to the Lagrangian (1.11),

$$\mathcal{H}_R = \psi^\dagger(x)(-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + m_1 \beta)\psi(x) + q_1 \bar{\psi}(x) \gamma_\mu A_0^\mu(x)\psi(x)$$

$$+ \phi^\dagger(x)(-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + m_2 \beta)\phi(x) + q_2 \bar{\phi}(x) \gamma_\mu A_0^\mu(x)\phi(x)$$

$$+ \frac{1}{2} \int d^4 x' j^\mu(x') D_{\mu\nu}(x-x') j^\nu(x), \qquad (2.1)$$

where we have not written out the Hamiltonian density for the free $A_0^\mu(x)$ field.

Equal-time quantization corresponds to the imposition of anticommutation rules for the fermion fields, namely,

$$\{\psi_\alpha(\boldsymbol{x},t), \psi_\beta^\dagger(\boldsymbol{y},t)\} = \{\phi_\alpha(\boldsymbol{x},t), \phi_\beta^\dagger(\boldsymbol{y},t)\} = \delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y}),$$

$$(2.2)$$

and all others vanish. In addition, if $A_0^\mu \neq 0$, there are the usual commutation rules for the A_0^μ field, and commutation of the A_0^μ field operators with the ψ and ϕ field operators.

The Hamiltonian (2.1) contains an interaction term that is nonlocal in time, which can complicate the transition to a quantized theory. We shall avoid this problem by working in the Schrödinger picture with $t=0$ in the expressions for the field operators and currents, that is, $\psi(x) = \psi(\boldsymbol{x}, t=0)$, $j^\mu(x) = j^\mu(\boldsymbol{x}, t=0)$, etc. in Eq. (2.1). This corresponds to neglecting higher-order retardation effects. Thereupon we obtain the result

$$\int dt' D_{\mu\nu}(x-x') = G_{\mu\nu}(\boldsymbol{x}-\boldsymbol{x}'), \qquad (2.3)$$

where

$$G_{\mu\nu}(\boldsymbol{x}) = \int \frac{d^3 k}{(2\pi)^3} G_{\mu\nu}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \quad \text{and}$$

$$G_{\mu\nu}(\boldsymbol{k}) = D_{\mu\nu}(k^\mu = (0,\boldsymbol{k})). \qquad (2.4)$$

For example, in the Lorentz gauge ($\partial_\mu A^\mu = 0$), we have

$$G_{\mu\nu}(\boldsymbol{x}) = g_{\mu\nu} \frac{1}{4\pi|\boldsymbol{x}|}. \qquad (2.5)$$

Thus, in the Schrödinger picture, the third term of the Hamiltonian density (2.1) takes on the form

$$\mathcal{H}_I(\boldsymbol{x}) = \frac{1}{2} \int d^3 x' \, j^\mu(\boldsymbol{x}') G_{\mu\nu}(\boldsymbol{x}-\boldsymbol{x}') j^\nu(\boldsymbol{x}). \qquad (2.6)$$

In the remainder of this paper, we shall consider a simplified model without the interaction terms in Eq. (2.1) that contain A_μ^0. Such a model is suitable for describing few-fermion states interacting via virtual photon exchange, but without decay or annihilation involving free (physical) photons. In short, in all that follows we consider the field theory based on the Hamiltonian density of Eq. (2.1) but with $A_0^\mu(x) = 0$. An attractive feature of this model is that exact few-fermion eigenstates of the Hamiltonian can be obtained.

III. ONE, TWO, AND THREE FERMION EIGENSTATES

We consider now the model for which the Hamiltonian, in the Schrödinger picture with $t=0$, is given by the expression

$$H_R = H_\psi + H_\phi + H_I, \qquad (3.1)$$

where

$$H_\psi = \int d^3 x \, \psi^\dagger(\boldsymbol{x},0)(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+m_1\beta)\psi(\boldsymbol{x},0), \qquad (3.2)$$

$$H_\phi = \int d^3 x \, \phi^\dagger(\boldsymbol{x},0)(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+m_2\beta)\phi(\boldsymbol{x},0), \qquad (3.3)$$

and $H_I = \int d^3 x \mathcal{H}_I(\boldsymbol{x})$, where $\mathcal{H}_I(\boldsymbol{x})$ is given in Eq. (2.6). Note, again, that the terms in A_0^μ have been suppressed, so that processes in which free (physical) photons are emitted or absorbed are not accommodated.

The Hamiltonian H_R has the same structure as the Coulomb-QED (CQED) Hamiltonian, that is, the Hamiltonian of QED in the Coulomb gauge, but with the transverse-photon part (that contains $\boldsymbol{\alpha}\cdot\boldsymbol{A}$) turned off. Indeed H_R would be identical to H_{CQED} if the indices μ and ν took on only the value 0 in Eq. (2.6) [as it is, $\mu,\nu = 0,1,2,3$ in Eq. (2.6)]. It has been shown earlier [8] that exact two-fermion eigenstates of H_{CQED} can be written down if we use an unconventional (or "empty") vacuum state, $|\tilde{0}\rangle$, defined by

$$\psi_\alpha(\boldsymbol{x},0)|\tilde{0}\rangle = \phi_\alpha(\boldsymbol{x},0)|\tilde{0}\rangle = 0. \qquad (3.4)$$

The same is true of the present more realistic model, as we point out below.

The unconventional empty vacuum definition (3.4) means that $\psi(\boldsymbol{x})$ is interpreted as a (free) Dirac-particle annihilation operator, while $\psi^\dagger(\boldsymbol{x})$ is, correspondingly, a Dirac-particle creation operator. By "Dirac-particle" we mean one described by the full Dirac spinor, including positive- and negative-frequency components. (Recall that in the conventional approach, i.e., using a Dirac "filled negative-energy sea" vacuum, which is annihilated by the positive-frequency component of ψ, it is only the negative-frequency component of ψ that is an antiparticle creation operator, and the positive-frequency component of $\psi^\dagger$ that is the particle creation operator).

With these conventions, we write the normal-ordered Hamiltonian

$$;H_R; = H_\psi + H_\phi + \frac{1}{2}\int d^3 x \, d^3 x' \, G_{\mu\nu}(\boldsymbol{x}-\boldsymbol{x}')$$

$$\times [q_1^2 \bar{\psi}\gamma^\mu(\bar{\psi}'\gamma^\nu\psi')\psi + q_1 q_2 \bar{\phi}\gamma^\mu(\bar{\psi}'\gamma^\nu\psi')\phi$$

$$+ q_2 q_1 \bar{\psi}\gamma^\mu(\bar{\phi}'\gamma^\nu\phi')\psi + q_2^2 \bar{\phi}\gamma^\mu(\bar{\phi}'\gamma^\nu\phi')\phi],$$

$$(3.5)$$

where $\psi = \psi(\boldsymbol{x})$ and $\bar{\phi}' = \bar{\phi}(\boldsymbol{x}')$, etc. The normal ordering is achieved by using the anticommutation rules (2.2) as usual; but note that it is not identical to the conventional normal ordering because of the unconventional empty vacuum that is being used, and the unconventional definition of ψ and ϕ as annihilation operators and $\psi^\dagger, \phi^\dagger$ as creation operators. To underscore this unconventional procedure we use the notation $;H_R;$ rather than $:H_R:$ in Eq. (3.5).

We note that the state defined by

$$|1\rangle = \int d^3 x \, \psi^\dagger(\boldsymbol{x}) F(\boldsymbol{x})|\tilde{0}\rangle, \qquad (3.6)$$

where $F(\boldsymbol{x})$ is a 4×1 c-number coefficient vector, is an eigenstate of $;H_R;$ [Eq. (3.5)] provided that $F(\boldsymbol{x})$ satisfies the equation

$$(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+m_1\beta)F(\boldsymbol{x}) = EF(\boldsymbol{x}), \qquad (3.7)$$

which is the usual time-independent one-particle Dirac equation (with positive- and negative-energy solutions), so that $F(\boldsymbol{x})$ is a Dirac spinor. Therefore, we refer to $|1\rangle$ as a one-Dirac-fermion state.

Similarly, the two-Dirac-fermion state

$$|2\rangle = \int d^3 x \, d^3 y \, F_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y}) \psi_\alpha^\dagger(\boldsymbol{x}) \phi_\beta^\dagger(\boldsymbol{y})|\tilde{0}\rangle \qquad (3.8)$$

(summation on $\alpha,\beta = 1,2,3,4$ implied) is an eigenstate of $;H_R;$ [Eq. (3.5)], provided that the 4×4 eigenmatrix F satisfies the equation

$$h_{m_1}(\boldsymbol{x})F(\boldsymbol{x},\boldsymbol{y}) + [h_{m_2}(\boldsymbol{y})F^T(\boldsymbol{x},\boldsymbol{y})]^T$$

$$+ q_1 q_2 G_{\mu\nu}(\boldsymbol{x}-\boldsymbol{y})\tilde{\gamma}^\mu F(\boldsymbol{x},\boldsymbol{y})(\tilde{\gamma}^\nu)^T = EF(\boldsymbol{x},\boldsymbol{y}),$$

$$(3.9)$$

where $h_{m_j}(\boldsymbol{x}) = -i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}+m_j\beta$, $\tilde{\gamma}^\mu = \gamma_0\gamma^\mu = (1,\boldsymbol{\alpha})$, and the superscript T indicates the transpose of the matrix in question.

The detailed form of the interaction matrix $G_{\mu\nu}(\boldsymbol{x}-\boldsymbol{x}')$ depends on the choice of gauge. Equation (3.9) is a two-fermion Dirac-like, or Breit-like, equation with positive- and negative-energy solutions, and is, in this respect, different from those obtained in the conventional approach [9–15], in which the negative-energy solutions do not arise.

We note, in passing, that if the interaction is turned off in Eq. (3.9) (i.e., $q_1 = q_2 = 0$), then the solution can be written as

$$F(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{x}) g^T(\boldsymbol{y}), \tag{3.10}$$

where $f(\boldsymbol{x})$ and $g(\boldsymbol{y})$ are solutions of the one-body Dirac eigenvalue equation (3.7). This indicates that in $F = [F_{ij}]$, the index i corresponds to particle 1 (with coordinates $\boldsymbol{x}$) while j corresponds to particle 2 (with coordinates $\boldsymbol{y}$).

In the rest frame of the two-fermion system (i.e., when $|2\rangle$ is taken to be an eigenstate of the momentum operator for this QFT, with eigenvalue 0), Eq. (3.9) reduces to an analogous equation in the single relative coordinate $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$:

$$h_{m_1}(\boldsymbol{r}) F(\boldsymbol{r}) + [h_{m_2}(-\boldsymbol{r}) F^T(\boldsymbol{r})]^T + q_1 q_2 G_{\mu\nu}(\boldsymbol{r}) \tilde{\gamma}^\mu F(\boldsymbol{r}) (\tilde{\gamma}^\nu)^T$$

$$= E F(\boldsymbol{r}). \tag{3.11}$$

It can, therefore, be reduced to a set of ordinary, coupled, first-order differential equations for states of given J^P. Such equations can, at the very least, be solved numerically. This is a straightforward, though somewhat tedious, problem [8,16] which we address below.

The structure of the Hamiltonian H_R [Eq. (3.5)] is such that generalizations to systems of more than two fermions are readily obtained. Thus, the three-fermion state, corresponding to a system such as $|e^- e^- \mu^+\rangle$, defined by

$$|3\rangle = \int d^3x_1 d^3x_2 d^3x_3 F_{\alpha_1 \alpha_2 \alpha_3}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \psi^\dagger_{\alpha_1}(\boldsymbol{x}_1)$$

$$\times \psi^\dagger_{\alpha_2}(\boldsymbol{x}_2) \phi^\dagger_{\alpha_3}(\boldsymbol{x}_3) |\tilde{0}\rangle, \tag{3.12}$$

is an exact eigenstate of H_R with eigenvalue E, provided that the $4^3 = 64$ coefficient functions $F_{\alpha_1 \alpha_2 \alpha_3}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ satisfy the three-body Dirac-like equation

$$[h_{m_1}(\boldsymbol{x}_1)]_{\alpha_1 \alpha} F_{\alpha \alpha_2 \alpha_3}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$$

$$+ [h_{m_1}(\boldsymbol{x}_2)]_{\alpha_2 \alpha} F_{\alpha_1 \alpha \alpha_3}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$$

$$+ [h_{m_2}(\boldsymbol{x}_3)]_{\alpha_3 \alpha} F_{\alpha_1 \alpha_2 \alpha}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) + q_1 q_2 G_{\mu\nu}(\boldsymbol{x}_3 - \boldsymbol{x}_1)$$

$$\times (\tilde{\gamma}^\mu)_{\alpha_1 \alpha} (\tilde{\gamma}^\nu)_{\alpha_3 \beta} F_{\alpha \alpha_2 \beta}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) + q_1 q_2 G_{\mu\nu}(\boldsymbol{x}_3 - \boldsymbol{x}_2)$$

$$\times (\tilde{\gamma}^\mu)_{\alpha_2 \alpha} (\tilde{\gamma}^\nu)_{\alpha_3 \beta} F_{\alpha_1 \alpha \beta}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) + q_1^2 G_{\mu\nu}(\boldsymbol{x}_1 - \boldsymbol{x}_2)$$

$$\times (\tilde{\gamma}^\mu)_{\alpha_1 \alpha} (\tilde{\gamma}^\nu)_{\alpha_2 \beta} F_{\alpha \beta \alpha_3}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$$

$$= E F_{\alpha_1 \alpha_2 \alpha_3}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3), \tag{3.13}$$

where summation on repeated spinor indices is implied. In the rest frame of the three-body system, Eq. (3.13) reduces to an equation in two independent vectors only. Nevertheless, the reduction of the equation for states of given J^P is more formidable than in the two-body case. Even then, one is left

with the full complexity of a relativistic three-body system. We shall not consider solutions of the three-body equation (3.13) in this paper.

IV. TWO-BODY EQUATION IN THE COULOMB GAUGE

At this stage we must specify the choice of the Green function, that is, the choice of gauge. We could use any gauge, in principle, but we shall use the Coulomb gauge, as this avoids nonphysical degrees of freedom and the need to take account of auxiliary conditions (such as $\partial_\mu A^\mu = 0$ in the Lorentz gauge). We use the relation

$$D_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} D_{\mu\nu}(k), \tag{4.1}$$

and note that

$$D_{00}(k) = \frac{1}{\boldsymbol{k}^2}, \quad D_{0j}(k) = 0, \quad D_{ij}(k) = \frac{1}{\boldsymbol{k}^2} \left(\delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2} \right) \tag{4.2}$$

in the Coulomb gauge [17]. Therefore, if we use Eq. (2.3) and the identity [[18], Eq. (39.8)]

$$\int \frac{d^3k}{(2\pi)^3} e^{ik \cdot x} \frac{\boldsymbol{a} \cdot \boldsymbol{k} \, \boldsymbol{b} \cdot \boldsymbol{k}}{(\boldsymbol{k}^2)^2} = \frac{1}{4\pi} \frac{1}{2r} \left(\boldsymbol{a} \cdot \boldsymbol{b} - \frac{\boldsymbol{a} \cdot \boldsymbol{r} \, \boldsymbol{b} \cdot \boldsymbol{r}}{r^2} \right), \tag{4.3}$$

we obtain the coordinate-space representation of the Coulomb-gauge Green function,

$$G_{00}(\boldsymbol{r}) = \frac{1}{4\pi r}, \quad G_{0i} = 0, \quad G_{ij}(\boldsymbol{r}) = -\frac{1}{8\pi r} \left(\delta_{ij} + \frac{x_i x_j}{r^2} \right), \tag{4.4}$$

where $r = |\boldsymbol{r}|$ and $\boldsymbol{r} = (x_1, x_2, x_3)$. Consequently, in the Coulomb gauge, Eq. (3.11) becomes

$$h_{m_1}(\boldsymbol{r}) F(\boldsymbol{r}) + [h_{m_2}(-\boldsymbol{r}) F^T(\boldsymbol{r})]^T$$

$$+ V(r) \left[F(\boldsymbol{r}) - \frac{1}{2} \left(\boldsymbol{\alpha} F(\boldsymbol{r}) \cdot \boldsymbol{\alpha}^T + \frac{1}{r^2} \boldsymbol{r} \cdot \boldsymbol{\alpha} F(\boldsymbol{r}) \boldsymbol{r} \cdot \boldsymbol{\alpha}^T \right) \right]$$

$$= E F(\boldsymbol{r}), \tag{4.5}$$

where $V(r) = (q_1 q_2 / 4\pi)(1/r)$.

Keeping in mind the notation, as explained below Eq. (3.10), we see that Eq. (4.5) is nothing other than the Breit equation [19], written in the rest frame of the two-fermion system.

The fact that the coefficient matrix F of the two-particle eigenstate (3.8) is a solution of the Breit equation means that the two-fermion eigenenergies that follow from Eq. (4.5) cannot be valid beyond $O(\alpha^4)$ (where $\alpha = |q_1 q_2|/4\pi$). It is well known that the Breit equation does not include radiative effects [18]. This limitation is a consequence of our neglect of retardation effects beyond the first order [cf. Eq. (2.3) above], the use of the empty vacuum [cf. Eq. (3.4)], and the

neglect of the terms containing the free photon field A_0^μ [cf. Eq. (2.1)] in the Hamiltonian (3.1).

If we had used the Lorentz-gauge form of the Green function (2.5), Eq. (3.11) would take on the form

$$h_{m_1}(\mathbf{r})F(\mathbf{r})+[h_{m_2}(-\mathbf{r})F^T(\mathbf{r})]^T+V(r)[F(\mathbf{r})-\boldsymbol{\alpha}F(\mathbf{r})\cdot\boldsymbol{\alpha}^T]$$

$$=EF(\mathbf{r}), \tag{4.6}$$

which is recognized to be the Eddington-Gaunt equation [20,21]. The Gaunt equation, unlike the Breit equation, does not contain even lowest-order retardation effects (see also Sec. V). Therefore, it will yield energy eigenvalues that will differ from those of the Breit equation already at $O(\alpha^4)$.

Note also that Eqs. (4.5) and (4.6), with the terms involving $\boldsymbol{\alpha}$ left out, become identical to the Coulomb-QED model discussed earlier [8].

V. TWO-BODY EQUATION IN THE LORENTZ GAUGE

Although we shall use the Coulomb gauge in this paper, it is instructive to see how, in the Lorentz gauge, one needs to keep retardation effects at least to the lowest nonvanishing order, in order to achieve the same results. The Green function of the d'Alembert equation in the Lorentz gauge has the form

$$D_{\mu\nu}(x)=g_{\mu\nu}D(x), \quad D(x)=\frac{1}{4\pi}\delta(x^2). \tag{5.1}$$

The reduced Lagrangian $\mathcal{L}_R$ in this case reads [with $A_0(x)=0$]

$$\mathcal{L}_R=\mathcal{L}_\psi+\mathcal{L}_\phi+\mathcal{L}_I=\bar{\psi}(x)(i\gamma^\mu\partial_\mu-m_1)\psi(x)$$

$$+\bar{\phi}(x)(i\gamma^\mu\partial_\mu-m_2)\phi(x)-\frac{1}{2}A^\mu(x)j_\mu(x), \tag{5.2}$$

where the potential of electromagnetic interparticle interaction is

$$A^\mu(x)=\int d^4x'D(x-x')j^\mu(x')$$

$$\equiv\int d^3x'\int dt'D[(t-t')^2-(\mathbf{x}-\mathbf{x}')^2]j^\mu(t',\mathbf{x}'). \tag{5.3}$$

Thus, the Lagrangian (5.2) is nonlocal in time. Because of this, the standard Hamiltonization procedure is not applicable.

In order to employ the canonical Hamiltonian formalism, it is necessary to convert this Lagrangian to a single-time form. We shall do so by employing a procedure that takes into account the retardation effects approximately.

Using the substitution $t'=t+\lambda$ in Eq. (5.3) and expanding the current j in a Taylor series in λ, we obtain the result

$$j(\mathbf{x}',t')=j(\mathbf{x}',t+\lambda)$$

$$=j(\mathbf{x}',t)+\lambda\dot{j}(\mathbf{x}',t)+\frac{1}{2}\lambda^2\ddot{j}(\mathbf{x}',t)+\cdots, \tag{5.4}$$

which reduces the potential (5.3) to the form

$$A^\mu(\mathbf{x},t)=\int d^3x'\left\{G(r)j^\mu(\mathbf{x}',t)+\frac{1}{2}G_1(r)\ddot{j}^\mu(\mathbf{x}',t)+\cdots\right\}, \tag{5.5}$$

where $r=|\mathbf{r}|=|\mathbf{x}-\mathbf{x}'|$. The functions

$$G(r)=\int d\lambda\, D(\lambda^2-r^2)=\frac{1}{4\pi r} \tag{5.6}$$

and

$$G_1(r)=\int d\lambda\, D(\lambda^2-r^2)\lambda^2=\frac{r}{4\pi} \tag{5.7}$$

satisfy the relation

$$G_1'(r)=rG(r). \tag{5.8}$$

The terms of odd power in λ in the expansion (5.4) vanish because D is an even function of λ [cf. Eqs. (5.6) and (5.7)]. As a result, the interaction Lagrangian $\mathcal{L}_I$ (up to surface terms) takes the following single-time form:

$$\mathcal{L}_I\approx\mathcal{L}_I^{(0)}+\mathcal{L}_I^{(1)}+\mathcal{L}_I^{(2)}, \tag{5.9}$$

where

$$\mathcal{L}_I^{(0)}=-\frac{1}{2}\int d^3x'\,G(r)j^0(\mathbf{x})j^0(\mathbf{x}'), \tag{5.10}$$

$$\mathcal{L}_I^{(1)}=\frac{1}{2}\int d^3x'\,G(r)\mathbf{j}(\mathbf{x})\cdot\mathbf{j}(\mathbf{x}'), \tag{5.11}$$

$$\mathcal{L}_I^{(2)}=\frac{1}{4}\int d^3x'\,G_1(r)\dot{j}^0(\mathbf{x})\dot{j}^0(\mathbf{x}') \tag{5.12}$$

(hereafter we omit the common time argument t).

The quantized theory based on the Lagrangian $\mathcal{L}^{(0)}=\mathcal{L}_\psi+\mathcal{L}_\phi+\mathcal{L}_I^{(0)}$ (the Coulomb QED mentioned in Sec. III) was discussed in [8]. It takes into account the relativistic kinematics of the fermion fields exactly, but describes their electromagnetic interaction with the transverse-photon part turned off. The terms $\mathcal{L}_I^{(1)}$ and $\mathcal{L}_I^{(2)}$ can be treated as first-order corrections to $\mathcal{L}^{(0)}$, thus providing the approximate single-time form $\mathcal{L}_S=\mathcal{L}^{(0)}+\mathcal{L}_I^{(1)}+\mathcal{L}_I^{(2)}$ for the reduced nonlocal Lagrangian $\mathcal{L}_R$ [cf. Eq. (1.11)]. Other terms following from the expansion are corrections of higher order. They will not be considered in the present paper.

The Lagrangian $\mathcal{L}_S$ leads to Euler-Lagrange equations which are second order in time derivatives, because of the term $\mathcal{L}_I^{(2)}$. Thus, it describes the system with twice as many

degrees of freedom as $\mathcal{L}^{(0)}$ does, because $\psi^\dagger$ are no longer the conjugate momenta of ψ. This changes completely the dynamical content of the fields ψ and ϕ. Since the second-order time derivatives occur in small terms only, they should be eliminated by means of the Euler-Lagrange equations of a lower-order approximation. But the resulting field equations are then not necessarily the Euler-Lagrange equations of a known Lagrangian. Thus the transition to the Hamiltonian and to a canonical quantum description becomes unclear.

To avoid this difficulty, it is tempting to eliminate the time derivative of the charge density j^0 directly in $\mathcal{L}_I^{(2)}$ by taking into account the conservation law

$$\partial_\mu j^\mu = 0, \quad \text{i.e.,} \quad \dot{j}^0 + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0. \tag{5.13}$$

This conservation law is a consequence of the Euler-Lagrange equations, which follow from the reduced Lagrangian $\mathcal{L}_R$ as well as from the truncated one $\mathcal{L}^{(0)}$. However, the direct use of the equations of motion (or their consequences) in the Lagrangian is not a correct procedure: it changes the equations of motion themselves. This fact was first emphasized in the case of the Golubenkov-Smorodinskii Lagrangian [22,23] and then subsequently discussed in the literature [24–28]. Instead, one can use the method of "double zero," used in Refs. [24,27]. In our case this consists of the following modification of the Lagrangian:

$$\mathcal{L}_S \to \bar{\mathcal{L}}_S = \mathcal{L}_S + \mathcal{L}_I^{(3)}, \tag{5.14}$$

where

$$\mathcal{L}_I^{(3)} = -\frac{1}{4} \int d^3 x' \, G_1(r) \{ \dot{j}^0(\boldsymbol{x}) + \boldsymbol{\nabla} \cdot \boldsymbol{j}(\boldsymbol{x}) \} \{ \dot{j}^0(\boldsymbol{x}') + \boldsymbol{\nabla}' \cdot \boldsymbol{j}(\boldsymbol{x}') \}. \tag{5.15}$$

It is easy to see that the term $\mathcal{L}_I^{(3)}$ possesses the property

$$\delta \int d^3 x \mathcal{L}_I^{(3)} \big|_{\delta \int d^3 x \mathcal{L}^{(0)} = 0} = 0, \tag{5.16}$$

so that it does not change the variational problem to the accuracy required. On the other hand, it cancels those terms of $\mathcal{L}_S$ which are quadratic in the time derivatives of the fields. Thus the modified Lagrangian $\bar{\mathcal{L}}_S$ yields equations of motion, which are first order in the time derivatives of the particle fields ψ, ϕ.

Next, we perform the following transformation of the field variables:

$$\psi \to \underline{\psi} = (1 - iq_1 W) \psi \approx e^{-iq_1 W} \psi,$$

$$\bar{\psi} \to \underline{\bar{\psi}} = (1 + iq_1 W) \bar{\psi} \approx e^{iq_1 W} \bar{\psi},$$

$$\phi \to \underline{\phi} = (1 - iq_2 W) \phi \approx e^{-iq_2 W} \phi,$$

$$\bar{\phi} \to \underline{\bar{\phi}} = (1 + iq_2 W) \bar{\phi} \approx e^{iq_2 W} \bar{\phi}, \tag{5.17}$$

where

$$W(\boldsymbol{x}) = \frac{1}{2} \int d^3 x' \, G_1(r) \boldsymbol{\nabla}' \cdot \boldsymbol{j}(\boldsymbol{x}'). \tag{5.18}$$

The transformation (5.17) can be regarded as an approximate U(1) gauge transformation, which, however, is not canonical due to the dependence of W on the fields. This transformation removes time derivatives from the interaction part of $\bar{\mathcal{L}}_S$. To the accuracy required, and up to surface terms, the Lagrangian $\bar{\mathcal{L}}_S$ can be written in the form

$$\bar{\mathcal{L}}_S = \underline{\mathcal{L}}^{(0)} + \underline{\mathcal{L}}_I^{(1)} + \frac{1}{4} \int d^3 x' \, G_1(r) [\boldsymbol{\nabla} \cdot \underline{\boldsymbol{j}}(\boldsymbol{x})][\boldsymbol{\nabla}' \cdot \underline{\boldsymbol{j}}(\boldsymbol{x}')], \tag{5.19}$$

where the notations $\underline{\mathcal{L}}^{(0)}$, $\underline{j}$, etc. mean that the fields ψ, ϕ are replaced by $\underline{\psi}$, $\underline{\phi}$.

Integrating the last term of $\bar{\mathcal{L}}_S$ by parts, omitting surface terms, and using Eqs. (5.6) and (5.8), we reduce the Lagrangian $\bar{\mathcal{L}}_S$ to the form

$$\bar{\mathcal{L}}_S = i(\underline{\psi}^\dagger \underline{\dot{\psi}} + \underline{\phi}^\dagger \underline{\dot{\phi}}) - \underline{\mathcal{H}}_S, \tag{5.20}$$

where

$$\begin{aligned} \underline{\mathcal{H}}_S = {} & \underline{\psi}^\dagger(\boldsymbol{x})(-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + m_1 \beta) \underline{\psi}(\boldsymbol{x}) \\ & + \underline{\phi}^\dagger(\boldsymbol{x})(-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + m_2 \beta) \underline{\phi}(\boldsymbol{x}) \\ & + \frac{1}{8\pi} \int d^3 x' \frac{1}{r} \left\{ \underline{j}^0(\boldsymbol{x}) \underline{j}^0(\boldsymbol{x}') - \frac{1}{2} \underline{\boldsymbol{j}}(\boldsymbol{x}) \cdot \underline{\boldsymbol{j}}(\boldsymbol{x}') \right. \\ & \left. - \frac{1}{2r^2} [\boldsymbol{r} \cdot \underline{\boldsymbol{j}}(\boldsymbol{x})][\boldsymbol{r} \cdot \underline{\boldsymbol{j}}(\boldsymbol{x}')] \right\}. \end{aligned} \tag{5.21}$$

This formulation allows us to treat the variables $\underline{\psi}^\dagger$ and $\underline{\phi}^\dagger$ as the canonical conjugates of $\underline{\psi}$ and $\underline{\phi}$, respectively. That is, we impose the anticommutation relations (2.2) on the underscored fields, and not on the original ones, when performing the quantization. Thereafter, since the Hamiltonian $\underline{H}_S = \int d^3 x \underline{\mathcal{H}}_S$ is formally identical to the Coulomb gauge Hamiltonian [cf. Eq. (2.1) with Eq. (4.4)], calculations such as those of Sec. IV lead to the Breit equation (4.5).

VI. TWO-BODY EQUATIONS IN BLOCK COMPONENT FORM

Although two-fermion equations have been around since the 1920s, their full reduction to the radial form is of more recent vintage (see, for example, [29,6]). The reduction of Eq. (4.5) to radial form is essentially the same as presented in Ref. [8], hence all the details shall not be repeated here. As shown in [8], we note that Eq. (4.5) has the Schrödinger equation as a nonrelativistic limit, and the Dirac equation as a one-body limit if either m_1 or m_2 tends to ∞.

For the two-fermion state $|2\rangle$, Eq. (3.8), to be simultaneously an eigenstate of $\boldsymbol{J}^2$, J_3, and parity, the "bispinor" $F = [F_{ij}]$ must be of the form

$$F(\mathbf{r}) = \frac{1}{r}\begin{bmatrix} is_1(r)\varphi^A(\hat{\mathbf{r}}) + is_2(r)\varphi^0(\hat{\mathbf{r}}), & t_1(r)\varphi^-(\hat{\mathbf{r}}) + t_2(r)\varphi^+(\hat{\mathbf{r}}) \\ u_1(r)\varphi^-(\hat{\mathbf{r}}) + u_2(r)\varphi^+(\hat{\mathbf{r}}), & iv_1(r)\varphi^A(\hat{\mathbf{r}}) + iv_2(r)\varphi^0(\hat{\mathbf{r}}) \end{bmatrix}, \tag{6.1}$$

for $-(-1)^J = (-1)^{J\pm1}$ parity eigenstates, and

$$F(\mathbf{r}) = \frac{1}{r}\begin{bmatrix} is_1(r)\varphi^-(\hat{\mathbf{r}}) + is_2(r)\varphi^+(\hat{\mathbf{r}}), & t_1(r)\varphi^A(\hat{\mathbf{r}}) + t_2(r)\varphi^0(\hat{\mathbf{r}}) \\ u_1(r)\varphi^A(\hat{\mathbf{r}}) + u_2(r)\varphi^0(\hat{\mathbf{r}}), & iv_1(r)\varphi^-(\hat{\mathbf{r}}) + iv_2(r)\varphi^+(\hat{\mathbf{r}}) \end{bmatrix} \tag{6.2}$$

for $-(-1)^{J\pm1} = (-1)^J$ parity eigenstates.

The 2×2 angular bispinor harmonics $\varphi^A(\hat{\mathbf{r}})$, $\varphi^0(\hat{\mathbf{r}})$, $\varphi^+(\hat{\mathbf{r}})$, $\varphi^-(\hat{\mathbf{r}})$, for given total angular-momentum quantum numbers J and $m_J \equiv M$, are

$$\varphi^A(\hat{\mathbf{r}}) = \frac{1}{\sqrt{2}} Y_J^M(\hat{\mathbf{r}})\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{6.3}$$

$$\varphi^0(\hat{\mathbf{r}}) = \frac{1}{\sqrt{2J(J+1)}}\begin{bmatrix} \sqrt{(J-M+1)(J+M)}\, Y_J^{M-1} & -MY_J^M \\ -MY_J^M & -\sqrt{(J+M+1)(J-M)}Y_J^{M+1} \end{bmatrix}, \tag{6.4}$$

$$\varphi^+(\hat{\mathbf{r}}) = \frac{1}{\sqrt{2J(2J-1)}}\begin{bmatrix} \sqrt{(J+M-1)(J+M)}Y_{J-1}^{M-1} & \sqrt{(J+M)(J-M)}Y_{J-1}^M \\ \sqrt{(J+M)(J-M)}Y_{J-1}^M & \sqrt{(J-M-1)(J-M)}Y_{J-1}^{M+1} \end{bmatrix}, \tag{6.5}$$

and

$$\varphi^-(\hat{\mathbf{r}}) = \frac{1}{\sqrt{2(J+1)(2J+3)}}\begin{bmatrix} \sqrt{(J-M+1)(J-M+2)}Y_{J+1}^{M-1} & -\sqrt{(J+M+1)(J-M+1)}Y_{J+1}^M \\ -\sqrt{(J+M+1)(J-M+1)}Y_{J+1}^M & \sqrt{(J+M+1)(J+M+2)}Y_{J+1}^{M+1} \end{bmatrix}. \tag{6.6}$$

We note that φ^A is antisymmetric and $\varphi^{0,\pm}$ are symmetric matrices. Furthermore, φ^0, φ^A, and $\varphi^\pm$ correspond to opposite parity because $Y_L^M(-\hat{\mathbf{r}}) = (-1)^L Y_L^M(\hat{\mathbf{r}})$ and φ^0, φ^A have $L=J$ whereas $\varphi^\pm$ have $L=J\pm1$. These four bispinor harmonics form an orthonormal set, in the sense that $\int d\hat{\mathbf{r}}\,\mathrm{Tr}(\varphi_i^\dagger \varphi_j) = \delta_{ij}$, where $i,j = A,0,+,-$ and the integrations are taken over the entire solid angle.

The eight radial functions in the bispinors (6.1) and (6.2) are solutions of the coupled radial equations that are obtained by substituting Eqs. (6.1) and (6.2) into Eq. (4.5) and equating the coefficients of the four independent bispinor harmonics.

We make use of the following identities in carrying out the radial reduction:

$$\boldsymbol{\sigma}\cdot\mathbf{p}f(r)\varphi(\hat{\mathbf{r}}) = -i\frac{df}{dr}\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\varphi(\hat{\mathbf{r}}) + \frac{i}{r}f(r)\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}\cdot\mathbf{l}\varphi(\hat{\mathbf{r}}), \tag{6.7}$$

where $\varphi(\hat{\mathbf{r}})$ is any 2×2 bispinor harmonic, $f(r)$ a radial function, $\hat{\mathbf{r}} = \mathbf{r}/r$, and $\mathbf{l} = \mathbf{r}\times\mathbf{p} = -i\mathbf{r}\times\nabla$. In addition, we note the following useful properties of the above bispinors harmonics:

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\,\varphi^A = A\varphi^- - B\varphi^+, \tag{6.8}$$

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\varphi^0 = B\varphi^- + A\varphi^+, \tag{6.9}$$

$$\boldsymbol{\sigma}\cdot\mathbf{l}\varphi^A = C\varphi^0, \tag{6.10}$$

$$\boldsymbol{\sigma}\cdot\mathbf{l}\varphi^0 = -\varphi^0 + C\varphi^A, \tag{6.11}$$

$$\boldsymbol{\sigma}\cdot\mathbf{l}\varphi^- = -(J+2)\varphi^-, \tag{6.12}$$

$$\boldsymbol{\sigma}\cdot\mathbf{l}\varphi^+ = (J-1)\varphi^+, \tag{6.13}$$

where

$$A = \sqrt{\frac{J+1}{2J+1}}, \quad B = \sqrt{\frac{J}{2J+1}}, \quad \text{and} \quad C = \sqrt{J(J+1)}. \tag{6.14}$$

It is evident from Eqs. (6.1) and (6.2) that, in general, eight coupled radial equations are obtained, for arbitrary $J>0$.

VII. RADIAL REDUCTION OF THE TWO-BODY EQUATIONS FOR $J=0$ STATES

For the $J=0$ states, namely, the $0^-(^1S_0)$ and $0^+(^3P_0)$ states, only two linearly independent bispinor harmonics arise, namely, φ^A and φ^- [Eqs. (6.3) and (6.6)], and so $s_2 = t_2 = u_2 = v_2 = 0$ in Eqs. (6.1) and (6.2). (Here, as elsewhere, we give in brackets the nonrelativistic limit designa-

tion, $^{2S+1}L_J$, corresponding to the J^P state in question.) Thus there is only one set of four coupled radial equations for each of $0^-(^1S_0)$ and $0^+(^3P_0)$ states:

$$[m_+ + V(r) - E]s(r) - t'(r) - \frac{K}{r}t(r) - u'(r) - \frac{K}{r}u(r)$$

$$+ \xi V(r)v(r) = 0, \tag{7.1}$$

$$[m_- + V(r) - E]t(r) + s'(r) - \frac{K}{r}s(r) + v'(r) - \frac{K}{r}v(r)$$

$$+ \eta V(r)u(r) = 0, \tag{7.2}$$

$$[-m_- + V(r) - E]u(r) + s'(r) - \frac{K}{r}s(r) + v'(r) - \frac{K}{r}v(r)$$

$$+ \eta V(r)t(r) = 0, \tag{7.3}$$

$$[-m_+ + V(r) - E]v(r) - t'(r) - \frac{K}{r}t(r) - u'(r) - \frac{K}{r}u(r)$$

$$+ \xi V(r)s(r) = 0, \tag{7.4}$$

where $m_\pm = m_1 \pm m_2$, $s' = ds/dr$, the potential $V(r) = -\alpha/r$ ($\alpha = |q_1 q_2|/4\pi$), and E is the eigenenergy (two-particle bound-state mass) to be determined, while $K = 1$ ($\xi = 2, \eta = 0$) for the $0^-(^1S_0)$ states and $K = -1$ ($\xi = 0, \eta = 2$) for the $0^+(^3P_0)$ states. As shown in Ref. [8], Eqs. (7.1)–(7.4) have the expected Schrödinger nonrelativistic limit and the Dirac one-body limit.

We note that the case with $\xi = \eta = 0$ corresponds to the simplified model without transverse-photon interactions, that is, the Coulomb-QED model of Ref. [8]. Similarly for $\xi = \eta = 0$, if the potential is $V(r) = -(q_1 q_2/4\pi)(e^{-\mu r}/r)$ and the sign of the potential is reversed in Eqs. (7.2) and (7.3), we recover the $0^\pm$ radial equation of the Yukawa model discussed in Ref. [30], for which the interfermion interaction is via a (massive or massless) scalar mediating field.

We should point out that Eqs. (7.1)–(7.4), like the Dirac equations, have both positive- and negative-energy solutions. Indeed, in this two-body case, there are solutions of four types: $E \simeq m_1 + m_2$, $E \simeq -m_1 + m_2$, $E \simeq m_1 - m_2$, and $E \simeq -m_1 - m_2$, as can be seen most easily from the $\alpha = 0$ case. Of these, two are positive-energy and two are negative-energy solutions.

Since we do not have analytic solutions for the eigenenergies of the present QED case, it is useful to illustrate this phenomenon on the scalar Yukawa (or Wick-Cutkosky) model, in which scalar particles interact via a massive or massless mediating field. For such a scalar model, analytic expressions for the two-body bound-state energy eigenvalues are available in the massless-exchange case [3]:

$$E = \sqrt{m_1^2 + m_2^2 \pm 2m_1 m_2 \left[1 - \left(\frac{\alpha}{n}\right)^2\right]^{1/2}}, \tag{7.5}$$

where α is the effective dimensionless coupling constant, analogous to the fine-structure constant of QED, and n is the principal quantum number. The $\pm$ in Eq. (7.5) correspond to

two segments of a distorted semicircle. The upper branch of this distorted semicircle corresponds to the upper (positive) sign in Eq. (7.5). It begins from $E = m_1 + m_2$ at $\alpha = 0$ [indeed, $E = m_1 + m_2 - 1/2m_r(\alpha/n)^2 - 1/8m_r(1 + m_r/m_+)(\alpha/n)^4 + \cdots$ for $\alpha/n \ll 1$], and decreases to $E = \sqrt{m_1^2 + m_2^2}$ at the critical value of $\alpha = n$, beyond which E ceases to be real, and the wave functions cease to be normalizable. The lower branch, by contrast, begins from $E = |m_1 - m_2|$ at $\alpha = 0$ and rises monotonically to meet the upper branch at the same critical point $E(\alpha/n) = \sqrt{m_1^2 + m_2^2}$. These $|m_1 - m_2|$-type bound-state eigenenergies do not have the correct Balmer limit, since for this branch

$$E = |m_1 - m_2| + \frac{1}{2}\left(\frac{m_1 m_2}{|m_1 - m_2|}\right)\left(\frac{\alpha}{n}\right)^2$$

$$+ \frac{1}{8}\left(\frac{m_1 m_2}{|m_1 - m_2|}\right)\left(1 - \frac{m_1 m_2}{(m_1 - m_2)^2}\right)\left(\frac{\alpha}{n}\right)^4 + \cdots \tag{7.6}$$

for $m_1 \neq m_2$, but $E = m[(\alpha/n) + 1/8(\alpha/n)^3 + \cdots]$ for $m_1 = m_2 = m$. Thus, this "mixed energy" $E \simeq |m_1 - m_2|$ bound-state spectrum must be regarded as unphysical. There are also negative-energy solutions of the $E \simeq -m_1 - m_2$ and $E \simeq -|m_1 - m_2|$ type, but they are not bound states, since the potential effectively reverses sign for the negative-energy solutions (as happens also in the Dirac-Coulomb case). The same type of behavior of the energy spectrum is observed in another analytically solvable case, namely, a fermion and a scalar particle interacting via massless scalar quantum exchange [31]. Thus we expect that the energy eigenvalue spectrum of Eqs. (7.1)–(7.4) will be qualitatively similar to that of the scalar exchange models just described.

We have not been able to determine solutions to the coupled radial equations (7.1)–(7.4) in terms of common analytic functions. It is of interest, therefore, to consider the properties and general behavior of the solutions before commencing with numerical solutions.

In analogy with the scalar model just described, and with the Coulomb-QED case [8] we expect that, as α increases, the eigenenergy spectrum $E(\alpha)$ of Eqs. (7.1)–(7.4) will have a qualitative behavior similar to that of the Dirac spectrum, namely, that $E(\alpha)$ decreases monotonically from $E(\alpha = 0) = m_1 + m_2$ until α hits a critical value α_c, beyond which $E(\alpha)$ ceases to be real. It is possible to infer the value of α_c by considering the ultrarelativistic limit, $p \to \infty$, in which case we can neglect the masses m_1 and m_2, and seek solutions of Eqs. (7.1)–(7.4) with $E = m_1 = m_2 = 0$. (This approach, when applied to the one-body Dirac-Coulomb case, yields the correct critical values $\alpha_c = |\kappa| = |j + 1/2|$.) In this ultrarelativistic approximation, Eqs. (7.1)–(7.4) have the solutions $t = u$, $s = v$, $|t| = |s| = 1$ (i.e., $F \propto 1/r$) with $\alpha_c^2 = 4K^2/(1 + \xi)(1 + \eta)$, which gives $\alpha_c = 2/\sqrt{3} = 1.1547\ldots$ for all $0^\pm$ states. Note, however, that this result does not mean that the value of the two-fermion rest mass E at α_c is necessarily the same for the 0^- and 0^+ states (certainly, such is not the case in the one-body limit). Note also that the result $\alpha_c = 2/\sqrt{3}$ for $0^\pm$ states is independent of the masses,

that is, we expect it to be the same for all finite m_1/m_2. The value $\alpha_c = 2/\sqrt{3}$ is different, and somewhat larger, than the known one-body limit (Dirac-Coulomb) value of $\alpha_c = 1$ for $|\kappa| = 1$ states. Also, this value is much smaller that the value $\alpha_c = 2$, which is obtained for the two-fermion Coulomb-QED case (where $\xi = \eta = 0$) for $0^{\pm}$ states.

For the Coulomb potential $V = -\alpha/r$, where $\alpha = |q_1 q_2|/4\pi$, it is often convenient to rescale the radial variable, that is, to let $\rho = r/a$, where a is a suitable scale parameter. For example, the radial functions s, t, u, v have the large r (negligible V and K/r) behavior $s \sim e^{-\rho}$, etc. for positive-energy $J = 0$ bound states, where a is given by

$$\frac{1}{a^2} = \frac{[m_+^2 - E^2][E^2 - m_-^2]}{4E^2} \quad \text{or} \quad \frac{1}{a^2} = m^2 - \left(\frac{E}{2}\right)^2$$

$$\text{if } m_1 = m_2. \tag{7.7}$$

Equation (7.7) implies that a is positive only for $|m_1 - m_2| \leq E \leq m_1 + m_2$, which means that the bound-state spectrum must lie in this domain [cf. Eq. (7.5)]. From this, and in analogy with the scalar model results, we can infer that the critical value $E(\alpha_c = 2/\sqrt{3})$ lies between $E = m_+$ and $E = |m_-|$, and likely closer to the former rather than the latter.

With the rescaling $\rho = r/a$, Eqs. (7.1)–(7.4) become modified slightly, in that r is replaced by $a\rho$ in all of them. For purposes of numerical integration of the radial equations, the scale parameter a can be chosen to be anything that is convenient, be it that given in Eq. (7.7), or $a = 1$ or $a = 1/\mu\alpha$, or whatever.

For a power-series analysis of the radial equations it is useful to make the replacement $s = \bar{s}e^{-\rho}$, etc. Assuming solutions of the form

$$\bar{s} = \rho^{\gamma}[a_0 + a_1\rho + a_2\rho^2 + \cdots], \tag{7.8}$$

$$\bar{t} = \rho^{\gamma}[b_0 + b_1\rho + b_2\rho^2 + \cdots], \tag{7.9}$$

$$\bar{u} = \rho^{\gamma}[c_0 + c_1\rho + c_2\rho^2 + \cdots], \tag{7.10}$$

$$\bar{v} = \rho^{\gamma}[d_0 + d_1\rho + d_2\rho^2 + \cdots], \tag{7.11}$$

we find, upon substitution into the radial equations for $\bar{s}, \bar{t}, \bar{u}, \bar{v}$ and equating coefficients of powers of $\rho^{\gamma + \nu - 1}$, that the coefficients a_j, b_j, c_j, d_j must satisfy the following recursion relations:

$$a(m_+ - E)a_{\nu-1} - \alpha a_{\nu} - (\gamma + K + \nu)b_{\nu} + \delta b_{\nu-1}$$
$$- (\gamma + K + \nu)c_{\nu} + \delta c_{\nu-1} - \xi\alpha d_{\nu} = 0, \tag{7.12}$$

$$(\gamma - K + \nu)a_{\nu} - \delta a_{\nu-1} + a(m_- - E)b_{\nu-1} - \alpha b_{\nu}$$
$$+ (\gamma - K + \nu)d_{\nu} - \delta d_{\nu-1} - \eta\alpha c_{\nu} = 0, \tag{7.13}$$

$$(\gamma - K + \nu)a_{\nu} - \delta a_{\nu-1} + a(-m_- - E)c_{\nu-1} - \alpha c_{\nu}$$
$$+ (\gamma - K + \nu)d_{\nu} - \delta d_{\nu-1} - \eta\alpha b_{\nu} = 0, \tag{7.14}$$

$$(\gamma + K + \nu)b_{\nu} - \delta b_{\nu-1} + (\gamma + K + \nu)c_{\nu} - \delta c_{\nu-1}$$
$$+ a(m_+ + E)d_{\nu-1} + \alpha d_{\nu} + \xi\alpha a_{\nu} = 0, \tag{7.15}$$

where $\delta = 1$. If $\delta = 0$ then Eqs. (7.12)–(7.15) are the recursion relations for the power-series representations of the functions $s(r)$, etc., rather than for $\bar{s}(r)$, etc.

For $\nu = 0$, and bearing in mind that $a_{-1} = b_{-1} = c_{-1} = d_{-1} = 0$, Eqs. (7.12)–(7.15) yield four coupled homogeneous equations for the parameters a_0, b_0, c_0, d_0, which have nontrivial (and nonsingular) solutions only if

$$\gamma = \sqrt{K^2 - \frac{1}{4}(1+\xi)(1+\eta)\alpha^2} = \sqrt{1 - \frac{3\alpha^2}{4}} \tag{7.16}$$

for the $J = 0$ states, for any values of m_1, m_2, whereupon

$$\frac{d_0}{a_0} = 1, \quad \frac{b_0}{a_0} = \frac{c_0}{a_0} = -\frac{(1+\xi)\alpha}{2(\gamma+K)} = \frac{2(\gamma-K)}{(1+\eta)\alpha}. \tag{7.17}$$

The condition (7.16) implies that the radial equations have real bound-state solutions of the form of Eqs. (7.8)–(7.11) only for $\alpha \leq 2/\sqrt{3}$, for any values of m_1 and m_2. This, in turn, implies that $\alpha_c \leq 2/\sqrt{3}$ for the $0^{\mp}$ states for any (finite) values of m_1 and m_2, in agreement with the ultrarelativistic limit discussed above. This condition for bound-states, $\alpha \leq 2/\sqrt{3}$, is additional to the one that follows from Eq. (7.7), namely, that $|m_1 - m_2| \leq E \leq m_1 + m_2$.

The recursion relations (7.12)–(7.15), with Eqs. (7.16) and (7.17), can be used to generate the power-series form of the solutions of Eqs. (7.8)–(7.11). These series converge in the domain $r \lesssim \alpha/m_+$, as discussed below and in [8]. Such a series can be used, for example, as a starting procedure for the numerical integration of the radial equations (7.1)–(7.4).

Unlike in the Dirac case, the recursion relations (7.12)–(7.15) do not admit power-series solutions of the form (7.8)–(7.11), which terminate at the same power, say $\nu = n'$, so that $a_{n'+1} = b_{n'+1} = c_{n'+1} = d_{n'+1} = 0$. In particular, the ground-state solution is not of the simple form

$$\bar{s} = a_0\rho^{\gamma}, \quad \bar{t} = b_0\rho^{\gamma}, \quad \bar{u} = c_0\rho^{\gamma}, \quad \bar{v} = d_0\rho^{\gamma} \tag{7.18}$$

as it is for the two radial Dirac equations. This is perhaps to be expected, since in the Dirac case there are only two functions, say $\bar{s}$ and $\bar{t}$, and four unknowns to be determined, namely, b_0/a_0, γ, a, and E. Since the two coupled radial Dirac equations yield four equations (the coefficients of ρ^{γ} and $\rho^{\gamma-1}$), it is not surprising that a solution is obtained. In the present case, we have four coupled radial equations (7.1)–(7.4), which yield eight equations (the coefficients of ρ^{γ} and $\rho^{\gamma-1}$) to be satisfied by the six unknowns of the proposed solutions (7.18), namely, b_0, c_0, d_0, γ, a, and E. Thus the system is overdetermined and no solution of the form (7.18) is possible. This situation persists for any solution of the form (7.8)–(7.11), where the polynomials all terminate at the same degree. Therefore, we shall solve the radial equations (7.1)–(7.4) numerically.

Equations (7.1)–(7.4) are not independent. Indeed, elementary manipulations of these equations, namely, subtracting Eq. (7.4) from Eq. (7.1) and similarly Eq. (7.3) from Eq. (7.2) show that

$$v(r) = \frac{E - m_+ - (1-\xi)V(r)}{E + m_+ - (1-\xi)V(r)} s(r),$$

$$u(r) = \frac{E - m_- - (1-\eta)V(r)}{E + m_- - (1-\eta)V(r)} t(r). \tag{7.19}$$

Thus, the number of equations can be reduced from four to two.

We introduce the auxiliary functions $f(r) = s(r) + v(r)$ and $g(r) = t(r) + u(r)$. Then, adding Eq. (7.1) to Eq. (7.4) and Eq. (7.2) to Eq. (7.3), and using Eq. (7.19) yields the equations

$$f'(r) = \frac{K}{r} f(r) + W_g(r) g(r),$$

$$-g'(r) = \frac{K}{r} g(r) + W_f(r) f(r), \tag{7.20}$$

where

$$W_g(r) = \frac{1}{2} \left[E - V_\eta(r) - \frac{(m_1 - m_2)^2}{E - \bar{V}_\eta(r)} \right], \tag{7.21}$$

$$W_f(r) = \frac{1}{2} \left[E - V_\xi(r) - \frac{(m_1 + m_2)^2}{E - \bar{V}_\xi(r)} \right], \tag{7.22}$$

and where

$$V_\xi(r) = (1+\xi)V(r), \quad \bar{V}_\xi(r) = (1-\xi)V(r),$$

$$V_\eta(r) = (1+\eta)V(r), \quad \bar{V}_\eta(r) = (1-\eta)V(r). \tag{7.23}$$

For the present QED case in the Coulomb gauge, for the 0^- states (for which $\xi = 2, \eta = 0$), $V_\eta = \bar{V}_\eta = V$, while $V_\xi = 3V$ and $\bar{V}_\xi = -V$. In this case we see that $W_f(r)$ is singular at $r_1 = \alpha/E = \alpha^2(m_r/E)(1/m_r\alpha)$, where m_r is the reduced mass and $1/m_r\alpha$ is the reduced Bohr radius. This singular point is quite close to the origin (in units of the reduced Bohr radius) for small α. The appearance of this singularity may signal difficulties in the numerical determination of eigensolutions of the equations (7.20) by standard "shooting" methods. For the $0^+(^3P_0)$ states (for which $\xi = 0, \eta = 2$), the singularity at $r_1 = \alpha/E$ occurs in W_g, but only if $m_1 \neq m_2$. Thus for the equal-mass 0^+ states, Eqs. (7.20) have only the usual $1/r$ singularities at the origin, and are amenable to solution by standard methods, as discussed below.

VIII. PERTURBATIVE DETERMINATION OF THE RELATIVISTIC CORRECTION TO THE TWO-BODY EIGENENERGIES FOR $J=0$ STATES

Equations (7.20) can be written in the matrix form

$$H|\psi\rangle = \epsilon|\psi\rangle, \quad \text{where} \quad H = \begin{bmatrix} \epsilon - W_f & -\dfrac{d}{dr} - \dfrac{K}{r} \\[2mm] \dfrac{d}{dr} - \dfrac{K}{r} & \epsilon - W_g \end{bmatrix},$$

$$\psi = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{8.1}$$

and where $\epsilon = E - (m_1 + m_2)$. If W_f is replaced by $W_f^{\mathrm{nr}} = \epsilon_{\mathrm{nr}} - V$ and W_g by $W_g^{\mathrm{nr}} = 2\mu$, where $\mu = m_1 m_2/(m_1 + m_2)$, then Eq. (7.20), or Eq. (8.1), is equivalent to the radial Schrödinger equation. The first-order correction to the nonrelativistic energy $\epsilon_{\mathrm{nr}} = -1/2\mu\alpha^2(1/n^2)$ is then given by

$$\Delta\epsilon = \frac{\langle \psi_{\mathrm{nr}} | H - H_{\mathrm{nr}} | \psi_{\mathrm{nr}} \rangle}{\langle \psi_{\mathrm{nr}} | \psi_{\mathrm{nr}} \rangle} = \frac{\langle f_{\mathrm{nr}} | \epsilon - \epsilon_{\mathrm{nr}} + W_f^{\mathrm{nr}} - W_f | f_{\mathrm{nr}} \rangle + \langle g_{\mathrm{nr}} | \epsilon - \epsilon_{\mathrm{nr}} + W_g^{\mathrm{nr}} - W_g | g_{\mathrm{nr}} \rangle}{\langle f_{\mathrm{nr}} | f_{\mathrm{nr}} \rangle + \langle g_{\mathrm{nr}} | g_{\mathrm{nr}} \rangle}. \tag{8.2}$$

If we expand the coefficients W_f and W_g [Eqs. (7.10) and (7.11)] in powers of V/m_i, and keep only the lowest-order terms, we obtain

$$\epsilon - \epsilon_{\mathrm{nr}} + W_f^{\mathrm{nr}} - W_f \simeq \frac{1}{2m_+}(\epsilon_{\mathrm{nr}} - V_\xi)^2, \tag{8.3}$$

$$\epsilon - \epsilon_{\mathrm{nr}} + W_g^{\mathrm{nr}} - W_g \simeq -\left(1 - 2\frac{\mu}{m_+}\right)\epsilon_{\mathrm{nr}} + \frac{1}{2}V_\eta + \delta^2 \bar{V}_\eta, \tag{8.4}$$

where $\delta = m_-/m_+$. This leads to the following $O(\alpha^4)$ correction to the nonrelativistic energy for the $J=0$ states:

$$\Delta\epsilon = \frac{1}{2m_+}[\epsilon_{\mathrm{nr}}^2 - 2\epsilon_{\mathrm{nr}}(1-\xi)\langle V \rangle + (1-\xi)^2\langle V^2 \rangle]$$

$$- \left(1 - 2\frac{\mu}{m_+}\right)\epsilon_{\mathrm{nr}}\langle\langle 1 \rangle\rangle + \frac{1}{2}[(1+\eta) + (1-\eta)\delta^2]\langle\langle V \rangle\rangle. \tag{8.5}$$

We use the notation $\langle X \rangle = \langle f_{\mathrm{nr}} | X | f_{\mathrm{nr}} \rangle / \langle f_{\mathrm{nr}} | f_{\mathrm{nr}} \rangle$ but $\langle\langle X \rangle\rangle = \langle g_{\mathrm{nr}} | X | g_{\mathrm{nr}} \rangle / \langle f_{\mathrm{nr}} | f_{\mathrm{nr}} \rangle$.

For the $0^-(n^1S_0)$ states (for which $K=1$ and $\xi=2, \eta=0$) this formula gives, for the lowest-order relativistic correction, the result

$$\Delta \epsilon[0^-(n^1S_0)] = \mu \alpha^4 \left\{ \frac{1}{n^4} \left(\frac{3}{8} - \frac{1}{8} \frac{\mu}{m_+} \right) \right.$$
$$\left. + \frac{1}{n^3} \left(-\frac{1}{2} + 2\frac{\mu}{m_+} \right) \right\}, \qquad (8.6)$$

which becomes $\frac{11}{64} m\alpha^4$ in the equal-mass, $m_1 = m_2 = m$, case. This does not agree with the known positronium value of $-\frac{21}{64} m\alpha^4$ [18]. This is not surprising, since the Breit equation, without modification, is known to give the incorrect fine structure for hydrogen and positronium. Brown and Ravenhall [32] argue that the reason for this is the mixing of positive- and negative-energy one-particle states [which arises, in our formalism, because of our use of the empty vacuum (3.4)]. This difficulty of the unmodified Breit equation is discussed in various works (e.g., Refs. [32,33,18]). The modification that is needed to bring the result into agreement with the observed fine structure of H or Ps is to subtract off the expectation value of the operator [33,34],

$$H' = \frac{\alpha^2}{4 m_+ r^2} (3 - 2\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \sigma_{1r}\sigma_{2r}), \quad \sigma_{ir} = \boldsymbol{\sigma}_i \cdot \mathbf{r}/r,$$
$$(8.7)$$

where, in this equation, we use the notation of [18]. The expectation value of Eq. (8.7) (with respect to the nonrelativistic eigenfunctions) is

$$\langle H' \rangle_{\mathrm{nr}} = \mu \alpha^4 \frac{1-\delta^2}{n^3(2J+1)} f_S = \mu \alpha^4 \frac{4f_S}{n^3(2J+1)} \frac{\mu}{m_+},$$
$$(8.8)$$

where $f_S = 1$ for the singlet $S=0$ states, while $f_S = 1/4$ for the triplet $S=1$ states with $J>0$, but $f_S = 0$ for the triplet states with $J=0$ (see, also, Sec. X below). The expression (8.8) gives the value $\frac{1}{2} m\alpha^4$ for the equal-mass ground state, which, when subtracted from the Breit value of $\frac{11}{64} m\alpha^4$ gives the expected positronium result $-\frac{21}{64} m\alpha^4$. More generally for arbitrary masses, if we subtract Eq. (8.8) from the expression (8.5), we obtain the corrected result

$$\Delta \epsilon_c[n0^-(^1S_0)] = \mu \alpha^4 \left\{ \frac{1}{n^4} \left(\frac{3}{8} - \frac{1}{8} \frac{\mu}{m_+} \right) \right.$$
$$\left. + \frac{1}{n^3} \left(-\frac{1}{2} - 2\frac{\mu}{m_+} \right) \right\}. \qquad (8.9)$$

This same result (8.9) was obtained previously for the $n=2$ state by Darewych and Horbatsch [13(b)], who used a perturbative approximation on variationally derived equations.

Somewhat surprisingly, the unequal-mass $O(\alpha^4)$ corrections for arbitrary states seem to have been worked out fully only relatively recently. We refer to the work of Connell [35], who used a quasipotential formalism based on the work of Todorov [36], and of Hersbach [37], who used a formal-

TABLE I. Values of E/m for the $n=2$, $0^+(^3P_0)$ state ($m_1 = m_2 \equiv m$).

α	Perturbative [Eq. (9.1)]	Numeric [Eqs. (7.20)]
1/137	1.999 996 669 953 2	1.999 996 669 953 2
0.01	1.999 993 749 690 8	1.999 993 749 690 8
0.05	1.999 843 556 722 0	1.999 843 5564
0.1	1.999 371 907 552 1	1.999 371 886
0.5	1.982 442 220 052 1	1.982 028 02
0.7	1.961 950 032 552 1	1.957 997 74
0.9	1.929 085 449 218 8	1.902 4531
1.0	1.906 575 520 833 3	1.838 781 05
1.1	1.879 098 470 052 1	1.688 2317
1.15	1.863 256 642 659 5	1.436 9434
1.154	1.861 924 191 099 5	1.355 170 76
1.1547	1.861 689 995 054 9	1.301 3199
1.154 700 538 379 2	1.861 689 814 814 8	1.299 74

ism based on a relativistic generalization of the Lippmann-Schwinger equation due to De Groot and Ruijgok [38]. Our corrected expression (8.9) agrees with the results of these authors. [The $O(\alpha^4)$ corrections for hydrogen and muonium quoted in standard references are expansions in m_1/m_2 (e.g., [39,40]).]

For the $n0^+(^3P_0)$ states (for which $K=-1$ and $\xi=0, \eta=2$) Eq. (8.5) gives

$$\Delta \epsilon[n0^+(^3P_0)] = \mu \alpha^4 \left\{ \frac{1}{n^4} \left(\frac{3}{8} - \frac{1}{8} \frac{\mu}{m_+} \right) \right.$$
$$\left. + \frac{1}{n^3} \left(-\frac{1}{2} - \frac{2}{3} \frac{\mu}{m_+} \right) \right\}, \qquad (8.10)$$

which *does* agree, in the equal-mass case, with the Ps values for all the $n0^+(^3P_0)$ states, as well as with the unequal-mass expressions of Connell [35] and Hersbach [37] for these states. This agreement implies that the "correction" $\langle H' \rangle_{\mathrm{nr}}$ vanishes for the $^3P_0$ states, as indeed it does.

We might note, in passing, that formula (8.5) gives the correct $O(\alpha^4)$ results for the Coulomb-QED ($\xi = \eta = 0$) case [8], for which the W coefficients are nonsingular for $r>0$.

IX. NUMERICAL SOLUTIONS FOR SOME $J=0^+$ STATES

In the case of equal masses $m_1 = m_2 \equiv m$ the radial equations (7.20) for $J=0^+(^3P_0)$ states are free of singularities. Thus the boundary-value problem is well posed, and it can be solved by means of a standard numerical ODE-solving procedure. We solved it by the "shooting" method using the MAPLE Runge-Kutta program.

The corresponding perturbative spectrum [cf. Eq. (8.10)],

$$E/m = 2 - \frac{\alpha^2}{4n^2} + \alpha^4 \left\{ \frac{11}{64n^4} - \frac{1}{3n^3} \right\}, \qquad (9.1)$$

TABLE II. Values of E/m for the $n=3,4$, $0^+(^3P_0)$ states ($m_1 = m_2 \equiv m$).

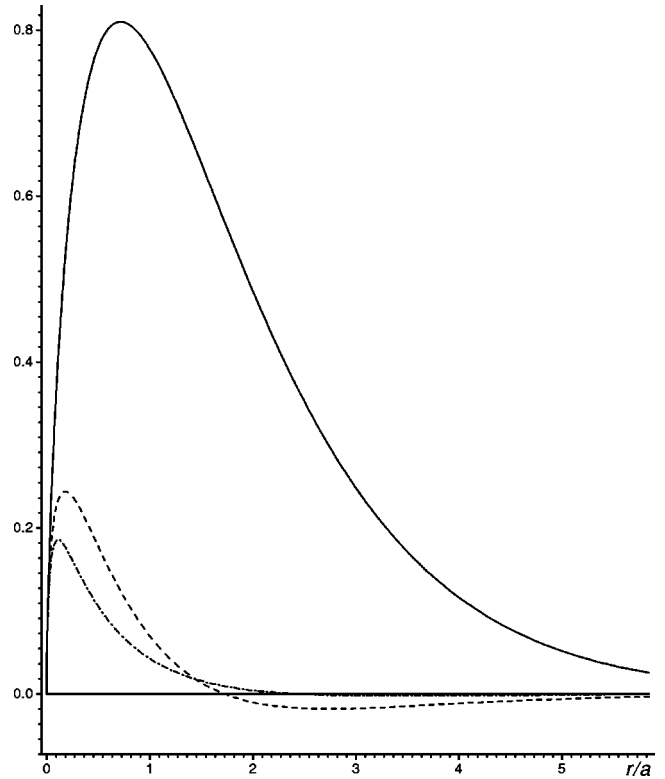| n	α	Perturbative [Eq. (9.1)]	Numeric [Eqs. (7.20)]
3	1/137	1.999 998 519 989 2	1.999 998 519 989 2
	0.01	1.999 997 222 120 0	1.999 997 222 120 0
	0.05	1.999 930 491 657 0	1.999 930 491 657 0
	0.1	1.999 721 199 845 7	1.999 721 193 728
	0.5	1.992 416 570 216 0	1.992 303 736
	0.7	1.983 934 162 808 6	1.982 908 59
	0.9	1.970 792 187 500 0	1.964 645 8
	1.0	1.961 998 456 790 1	1.948 028 09
	1.1	1.951 420 273 919 8	1.918 619 6
	1.15	1.945 382 459 249 6	1.884 435 8
	1.154	1.944 876 373 119 8	1.875 613
	1.1547	1.944 787 448 407 1	1.870 569 7
	1.154 700 538 379 2	1.944 787 379 972 6	1.870 423 4
4	1/137	1.999 999 167 497 4	1.999 999 167 497 4
	0.01	1.999 998 437 454 6	1.999 998 437 454 6
	0.05	1.999 960 909 144 1	1.999 960 909 144 1
	0.1	1.999 843 296 305 3	1.999 843 293 9
	0.5	1.995 810 190 836 6	1.995 766 294 7
	0.7	1.991 254 429 117 8	1.990 864 9
	0.9	1.984 367 059 326 2	1.982 144 1
	1.0	1.979 838 053 385 4	1.974 981 6
	1.1	1.974 451 206 461 6	1.963 685
	1.15	1.971 400 789 515 2	1.952 562
	1.154	1.971 145 810 108 7	1.950 122 5
	1.1547	1.971 101 018 265 9	1.948 751
	1.154 700 538 379 2	1.971 100 983 796 3	1.948 714

FIG. 1. Reduced radial wave functions for the lowest, $m_1 = m_2 \equiv m$, $n=2$, $0^+(^3P_0)$ state for $\alpha=1$, $E/m=1.838\,781$. $s(\rho)$, full curve; $t(\rho)=u(\rho)$, broken curve; $v(\rho)$, chain curve. $\rho=r/a$, where $a=2.542\,291(1/m)$. Units $\hbar=c=1$ are used.

agrees with the orthopositronium spectrum, since the contribution of the extra terms [cf. Eq. (8.7)] (caused by positive-energy–negative-energy mixing) vanishes in this case.

In Table I the numeric and perturbative results are presented for the lowest-energy 0^+ state (i.e., $J=0$, $\ell=1$, $n=2$) for different values of $\alpha \leqslant \alpha_c = 2/\sqrt{3} \approx 1.154\,700\,538\,379\,251\,5$.

There are, as we explained in Sec. VII, $E>0$ "mixed energy" solutions of the form $E/m = \alpha + O(\alpha^3)$, which are unphysical, because they do not have the Balmer nonrelativistic limit. We do not list such solutions here, though they can be calculated readily enough in the same way as those in the Table I. This unphysical branch of the $n=2$ 0^+ state rises uniformly from zero at $\alpha=0$ to join the physical branch of Table I smoothly at $E(\alpha_c)$. As mentioned previously, the two branches together resemble a distorted semicircle [cf. Eq. (7.5)].

Analogous results for the $n=3$ and $n=4$ $0^+(^3P_0)$ equal-mass two-fermion energies are given in Table II. The qualitative behavior of $E(\alpha)/m$ for these states is similar to that for the lowest such state ($n=2$), except that the critical value of α increases with n, as it does in the case of the analytically solvable scalar model of Eq. (7.5). However, here we obtain $\alpha_c/n = 0.649\,87, 0.602\,474\,5, 0.487\,178\,5$ for the $n=2,3$, and 4 states, respectively, in contrast to the scalar model values $\alpha_c/n = 1$ for all n.
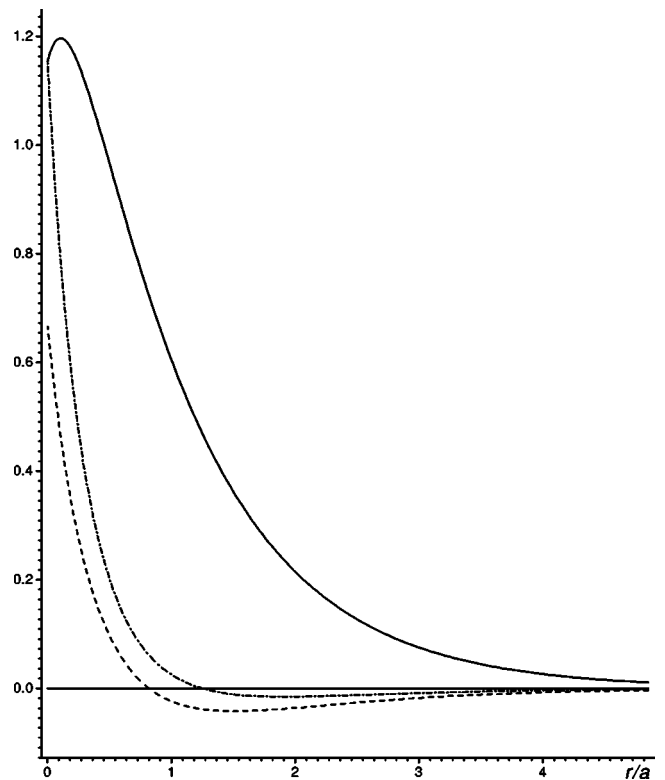

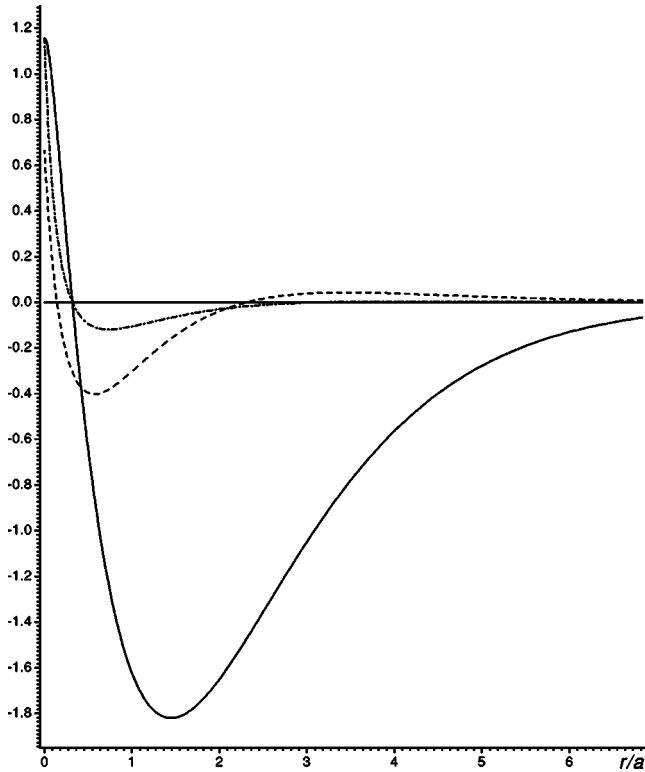FIG. 2. Same as Fig. 1 but $\alpha = \alpha_c = 2/\sqrt{3}$, $E/m = 1.299\,74$, and $a = 1.315\,711(1/m)$.

FIG. 3. Same as Fig. 1 but for the excited $n=3$, $0^+(^3P_0)$ state, with $\alpha=\alpha_c=2/\sqrt{3}$, $E/m=1.870\,423$, and $a=2.824\,148(1/m)$.

FIG. 4. Same as Fig. 1 but for the excited $n=4$, $0^+(^3P_0)$ state, with $\alpha=1$, $E/m=1.974\,982$, and $a=6.342\,094(1/m)$.

Note that the critical value of the two-body mass, $E(\alpha_c)/m$, increases with n, in contrast to the scalar model, for which $E(\alpha_c)/m=\sqrt{2}$ for all n.

Figure 1 is a plot of the unnormalized reduced radial wave functions $s(r)$, $t(r)=u(r)$, and $v(r)$ [see Eqs. (7.1)–(7.4)] in the case of equal masses, $m_1=m_2\equiv m$, for the lowest-energy $n=20^+(^3P_0)$ states, when $\alpha=1$. These wave functions are qualitatively similar to those obtained for these states in the Coulomb-QED case [8]. The large component $s(r)$ is nodeless while the small component $t(r)=u(r)$ and the doubly small one $v(r)$ have one node. The node at the origin, $r=0$, is a consequence of our use of reduced radial wave functions $s(r)$, etc., rather than the actual $s(r)/r$, etc. Indeed, the wave functions behave at small r as follows:

$$s(r)\approx v(r)\approx \text{const}\times \alpha r^\gamma,$$

$$t(r)=u(r)\approx \text{const}\times (2/3)(\gamma+1)r^\gamma, \qquad (9.2)$$

where $0<\gamma<1$ [see Eq. (7.18)]. Thus the matrix wave function $F(\mathbf{r})$ is singular, $F(\mathbf{r})\sim r^{\gamma-1}$, as happens also in the one-body Dirac equation with a Coulomb potential. Nevertheless, $F(\mathbf{r})$ is normalizable for all α up to and including $\alpha=\alpha_c$, at which point $t(0)=u(0)=(2/3)s(0)=(2/3)v(0)\neq 0$, as can be seen in Figs. 2 and 3.

Figure 3 represents the excited $n=30^+(^3P_0)$ state for the critical coupling strength $\alpha_c=2/\sqrt{3}$. In this case, the wave function $s(r)$ has one node, while $t(r)$ and $v(r)$ have two nodes. This behavior differs from that found in CQED [8] for the same case (where the number of nodes was two, one, and
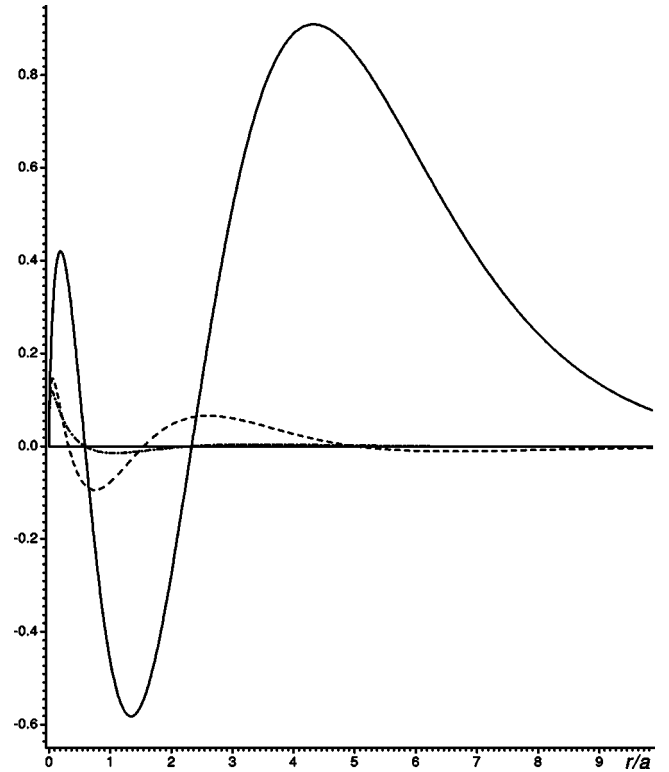
three, respectively). In the $n=40^+(^3P_0)$ case (Fig. 4) each of the functions $s(r)$, $t(r)$, and $v(r)$ gets one more node. This tendency likely continues for higher values of the quantum number n.

X. RADIAL REDUCTION FOR $J>0$ STATES

For states with $J>0$, the eigenstate problem reduces to a set of eight first-order differential equations for the functions $s_1(r)\cdots v_2(r)$ and the energy E [cf. Eqs. (6.1) and (6.2)]. It is convenient to present this set in the following matrix form. Let us introduce the eight-dimensional vector function

$$\mathsf{X}(r)=\begin{bmatrix} s_1(r) \\ s_2(r) \\ t_1(r) \\ \vdots \\ v_2(r) \end{bmatrix}. \qquad (10.1)$$

Then the set of radial equations reads

$$\mathcal{H}\mathsf{X}(r)\equiv\left\{\mathsf{H}\frac{d}{dr}+\mathsf{U}(r)\right\}\mathsf{X}(r)=E\mathsf{X}(r), \qquad (10.2)$$

where the 8×8 matrix $\mathsf{U}(r)$ has the following structure:

$$\mathsf{U}(r)=\mathsf{M}+[\mathsf{G}-\alpha(\mathsf{I}+\mathsf{S})]/r. \qquad (10.3)$$

Here I is the unit matrix, M is diagonal,

$$
\mathsf{M}=\begin{bmatrix}
m_+ & & & & & & & \\
& m_+ & & & & & & \\
& & m_- & & & & 0 & \\
& & & m_- & & & & \\
& & & & -m_- & & & \\
& & & & & -m_- & & \\
& 0 & & & & & -m_+ & \\
& & & & & & & -m_+
\end{bmatrix},
\tag{10.4}
$$

$m_\pm = m_1 \pm m_2$, and the form of 8×8 matrices H, G, and S depends on the parity P:

$$
\mathsf{H}=\begin{bmatrix}
0 & 0 & -A & B & -A & B & 0 & 0 \\
0 & 0 & B & A & -B & -A & 0 & 0 \\
A & -B & 0 & 0 & 0 & 0 & A & B \\
-B & -A & 0 & 0 & 0 & 0 & -B & A \\
A & B & 0 & 0 & 0 & 0 & A & -B \\
-B & A & 0 & 0 & 0 & 0 & -B & -A \\
0 & 0 & -A & B & -A & B & 0 & 0 \\
0 & 0 & -B & -A & B & A & 0 & 0
\end{bmatrix},
\tag{10.5}
$$

$$
\mathsf{G}=\begin{bmatrix}
0 & 0 & -(J+1)A & -JB & -(J+1)A & -JB & 0 & 0 \\
0 & 0 & (J+1)B & -JA & -(J+1)B & JA & 0 & 0 \\
-(J+1)A & (J+1)B & 0 & 0 & 0 & 0 & -(J+1)A & -(J+1)B \\
-JB & -JA & 0 & 0 & 0 & 0 & -JB & JA \\
-(J+1)A & -(J+1)B & 0 & 0 & 0 & 0 & -(J+1)A & (J+1)B \\
-JB & JA & 0 & 0 & 0 & 0 & -JB & -JA \\
0 & 0 & -(J+1)A & -JB & -(J+1)A & -JB & 0 & 0 \\
0 & 0 & -(J+1)B & JA & (J+1)B & -JA & 0 & 0
\end{bmatrix},
\tag{10.6}
$$

$$
\mathsf{S}=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -B^2 & -AB & 0 & 0 \\
0 & 0 & 0 & 0 & -AB & -A^2 & 0 & 0 \\
0 & 0 & -B^2 & -AB & 0 & 0 & 0 & 0 \\
0 & 0 & -AB & -A^2 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\tag{10.7}
$$

$$
\mathsf{H}=\begin{bmatrix}
0 & 0 & -A & B & -A & -B & 0 & 0 \\
0 & 0 & B & A & B & -A & 0 & 0 \\
A & -B & 0 & 0 & 0 & 0 & A & -B \\
-B & -A & 0 & 0 & 0 & 0 & B & A \\
A & -B & 0 & 0 & 0 & 0 & A & -B \\
B & A & 0 & 0 & 0 & 0 & -B & -A \\
0 & 0 & -A & -B & -A & B & 0 & 0 \\
0 & 0 & B & -A & B & A & 0 & 0
\end{bmatrix},
\tag{10.8}
$$

for $P = (-)^{J \pm 1}$, and

$$G=\begin{bmatrix} 0 & 0 & (J+1)A & -(J+1)B & (J+1)A & (J+1)B & 0 & 0 \\ 0 & 0 & JB & JA & JB & -JA & 0 & 0 \\ (J+1)A & JB & 0 & 0 & 0 & 0 & (J+1)A & JB \\ -(J+1)B & JA & 0 & 0 & 0 & 0 & (J+1)B & -JA \\ (J+1)A & JB & 0 & 0 & 0 & 0 & (J+1)A & JB \\ (J+1)B & -JA & 0 & 0 & 0 & 0 & -(J+1)B & JA \\ 0 & 0 & (J+1)A & (J+1)B & (J+1)A & -(J+1)B & 0 & 0 \\ 0 & 0 & JB & -JA & JB & JA & 0 & 0 \end{bmatrix}, \tag{10.9}$$

$$S=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -B^2 & -AB \\ 0 & 0 & 0 & 0 & 0 & 0 & -AB & -A^2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -B^2 & -AB & 0 & 0 & 0 & 0 & 0 & 0 \\ -AB & -A^2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{10.10}$$

for $P=(-)^J$, where A, B, and C are defined in Eq. (6.14). Due to the properties

$$\mathsf{H}^T=-\mathsf{H}, \quad \mathsf{U}^T=\mathsf{U}, \tag{10.11}$$

the radial Hamiltonian $\mathcal{H}$ is a Hermitian operator with respect to the inner product

$$\langle \mathsf{Y}|\mathsf{X}\rangle_8=\int_0^\infty dr\,\mathsf{Y}^\dagger(r)\mathsf{X}(r), \tag{10.12}$$

where the subscript 8 denotes the dimensions of the vector functions X,Y.

In the subsequent reduction of the set (10.2) one can use the fact that rank$(\mathsf{H})=4$ (for either parity). Thus one can reduce the number of differential equations from eight to four. We perform this reduction in a way that ensures, as far as possible, the Hamiltonian structure of the equations.

First of all we perform the orthogonal transformation

$$\tilde{\mathsf{X}}(r)=\mathsf{E}\mathsf{X}(r), \quad \tilde{\mathcal{H}}=\mathsf{E}\mathcal{H}\mathsf{E}^{-1}, \tag{10.13}$$

where

$$\mathsf{E}=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & A & -B & A & -B & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & B & A & -B & -A & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & B & A & B & A & 0 & 0 \\ 0 & 0 & A & -B & -A & B & 0 & 0 \end{bmatrix}, \tag{}$$

$$P=(-)^{J\pm1}, \tag{10.14}$$

$$\mathsf{E}=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -A & B & 0 & 0 & 0 & 0 & -A & B \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ -B & -A & 0 & 0 & 0 & 0 & B & A \\ B & A & 0 & 0 & 0 & 0 & B & A \\ A & -B & 0 & 0 & 0 & 0 & -A & B \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \tag{}$$

$$P=(-)^J. \tag{10.15}$$

It preserves the inner product (10.12) and reduces Eqs. (10.2) to the form

$$\tilde{\mathcal{H}}\tilde{\mathsf{X}}(r)\equiv\left\{\tilde{\mathsf{H}}\frac{d}{dr}+\tilde{\mathsf{U}}(r)\right\}\tilde{\mathsf{X}}(r)=E\tilde{\mathsf{X}}(r), \quad \tilde{\mathsf{U}}=\mathsf{E}\mathsf{U}\mathsf{E}^{-1}. \tag{10.16}$$

It is convenient at this stage to express the eight-dimensional vectors and matrices in terms of four-dimensional blocks:

$$\tilde{\mathsf{X}}=\begin{bmatrix}\tilde{\mathsf{X}}_1 \\ \tilde{\mathsf{X}}_2\end{bmatrix}, \quad \tilde{\mathsf{V}}(r)\equiv\tilde{\mathsf{U}}(r)-E\mathsf{I}=\begin{bmatrix}\tilde{\mathsf{V}}_{11}(r) & \tilde{\mathsf{V}}_{12}(r) \\ \tilde{\mathsf{V}}_{21}(r) & \tilde{\mathsf{V}}_{22}(r)\end{bmatrix}, \quad \text{etc.} \tag{10.17}$$

Then the matrix $\tilde{\mathsf{H}}$ takes the form

$$\tilde{\mathsf{H}}=\begin{bmatrix}\tilde{\mathsf{H}}_{11} & 0 \\ 0 & 0\end{bmatrix}, \quad \tilde{\mathsf{H}}_{11}=2\begin{bmatrix}\mathsf{J} & 0 \\ 0 & \mathsf{J}\end{bmatrix}, \quad \mathsf{J}=\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}, \tag{10.18}$$

and the set of Eqs. (10.16) becomes

$$\tilde{\mathsf{H}}_{11}\tilde{\mathsf{X}}_1'(r)+\tilde{\mathsf{V}}_{11}(r)\tilde{\mathsf{X}}_1(r)+\tilde{\mathsf{V}}_{12}(r)\tilde{\mathsf{X}}_2(r)=0, \tag{10.19}$$

$$\tilde{V}_{21}(r)\tilde{X}_1(r)+\tilde{V}_{22}(r)\tilde{X}_2(r)=0. \qquad (10.20)$$

The set (10.20) is purely algebraic. It permits us to express $\tilde{X}_2(r)$ in the terms of $\tilde{X}_1(r)$:

$$\tilde{X}_2(r)=-\tilde{V}_{22}^{-1}(r)\tilde{V}_{21}(r)\tilde{X}_1(r). \qquad (10.21)$$

Substitution of Eq. (10.21) into Eq. (10.19) yields the closed set of four first-order differential equations:

$$\mathcal{L}(E)\tilde{X}_1(r)\equiv\left\{\frac{1}{2}\tilde{H}_{11}\frac{d}{dr}+\tilde{W}(E)\right\}\tilde{X}_1(r)=0, \qquad (10.22)$$

where

$$\tilde{W}(E)\equiv(\tilde{V}_{11}-\tilde{V}_{12}\tilde{V}_{22}^{-1}\tilde{V}_{21})/2=\frac{1}{2}
\begin{bmatrix}
-(E+3\alpha/r)+\dfrac{m_\pm^2}{E-\alpha/r}+\dfrac{4C^2/r^2}{E} & -2/r & 0 & -\dfrac{2Cm_\mp/r}{E} \\[2em]
-2/r & -(E+\alpha/r)+\dfrac{m_\mp^2}{E+\alpha/r} & -\dfrac{2Cm_\mp/r}{E+\alpha/r} & 0 \\[2em]
0 & -\dfrac{2Cm_\mp/r}{E+\alpha/r} & -(E+2\alpha/r)+\dfrac{m_\pm^2}{E}+\dfrac{4C^2/r^2}{E+\alpha/r} & 0 \\[2em]
-\dfrac{2Cm_\mp/r}{E} & 0 & 0 & -(E+2\alpha/r)+\dfrac{m_\mp^2}{E}
\end{bmatrix}. \qquad (10.23)$$

Here the upper sign corresponds to $P=(-)^{J+1}$ and the lower sign corresponds to $P=(-)^J$.

The operator $\mathcal{L}(E)$ in Eq. (10.22) is formally Hermitian, i.e., given E, it is Hermitian with respect to the inner product $\langle\cdots|\cdots\rangle_4$. But any two solutions of Eq. (10.22), X_1 and Y_1, corresponding to different values of the energy, E and E', are not orthogonal. This is due to the nonlinear dependence of $\mathcal{L}(E)$ on E. Orthogonality can be instated by using the following definition of the inner product:

$$\langle\langle Y_1|X_1\rangle\rangle_4=\left\langle Y_1\left|\frac{\mathcal{L}(E')-\mathcal{L}(E)}{E'-E}\right|X_1\right\rangle_4. \qquad (10.24)$$

This inner product follows directly by substitution of Eq. (10.21) into Eq. (10.12).

The set of first-order equations (10.22) can also be expressed as a second-order equation. For this purpose it is convenient to permute the elements of X_1 by means of the matrix L, where

$$L=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{for} \quad P=(-)^{J\pm1} \qquad (10.25)$$

and

$$L=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{for} \quad P=(-)^J. \qquad (10.26)$$

Then, in terms of the two-dimensional blocks

$$\bar{X}_1=L\tilde{X}_1\equiv\begin{bmatrix}\Psi_1\\\Psi_2\end{bmatrix}, \qquad (10.27)$$

the equations take the form

$$-\Psi_2'+\bar{W}_{11}\Psi_1+\bar{W}_{12}\Psi_2=0, \qquad (10.28)$$

$$\Psi'_1 + \bar{W}_{21}\Psi_1 + \bar{W}_{22}\Psi_2 = 0, \tag{10.29}$$

where $\bar{W} = L\tilde{W}L^{-1}$. Elimination of Ψ_2 leads to the following equation for the 2×1 vector function Ψ_1:

$$\mathcal{F}(E) \equiv \left[\left\{ \frac{d}{dr} - \bar{W}_{12} \right\} \bar{W}_{22}^{-1} \left\{ \frac{d}{dr} + \bar{W}_{21} \right\} + \bar{W}_{11} \right] \Psi_1(r) = 0. \tag{10.30}$$

Eigenstates Ψ_1, Φ_1 corresponding to different values of the energy are orthogonal with respect to the inner product:

$$\langle\langle \Phi_1 | \Psi_1 \rangle\rangle_2 = \left\langle \Phi_1 \left| \frac{\mathcal{F}(E') - \mathcal{F}(E)}{E' - E} \right| \Psi_1 \right\rangle_2, \tag{10.31}$$

which also follows from the reduction procedure.

XI. PERTURBATIVE SOLUTIONS FOR $J>0$ STATES

The form of Eqs. (10.30) is convenient for examining the energy spectrum of $J>0$ bound-states perturbatively to $O(\alpha^4)$. For this purpose we introduce the dimensionless quantities

$$\rho = \mu\alpha r, \quad \lambda = \frac{E - m_+}{\mu\alpha^2}, \quad \text{and} \quad \delta = \frac{m_-}{m_+}, \tag{11.1}$$

where $\mu = m_1 m_2/m_+$ is the reduced mass. We now perform a perturbative expansion in α of Eq. (10.30). To order α^2, Eq. (10.30) takes the Hamiltonian form (hereafter we omit the subscript 1 of the wave function Ψ_1)

$$\mathcal{H}\Psi(\rho) \approx \{\mathcal{H}^{(0)} + \alpha^2 \mathcal{H}^{(1)}(\lambda)\}\Psi(\rho) = \lambda\Psi(\rho), \tag{11.2}$$

where

$$\Psi(\rho) = \begin{bmatrix} \psi_1(\rho) \\ \psi_2(\rho) \end{bmatrix} \tag{11.3}$$

is a two-component wave function, $\mathcal{H}$ is the Hamiltonian divided by $\mu\alpha^2$ and expressed in terms of dimensionless quantities (11.1),

$$\mathcal{H}^{(0)} = -\frac{1}{2}\left\{ \frac{d}{d\rho^2} - \frac{1}{\rho^2}\mathcal{J} \right\} - \frac{1}{\rho} \tag{11.4}$$

is the unperturbed (i.e., zeroth-order) Hamiltonian, and

$$\mathcal{H}^{(1)}(\lambda) = \frac{d}{d\rho}\mathcal{K}(\rho,\lambda)\frac{d}{d\rho} + \mathcal{M}(\rho,\lambda) \tag{11.5}$$

is the perturbative correction to Eq. (11.4). The form of the symmetric 2×2 matrices $\mathcal{J}$, $\mathcal{K}(\rho,\lambda)$, $\mathcal{M}(\rho,\lambda)$ depends on the parity. In order to obtain the energy spectrum to $O(\alpha^4)$, it is sufficient to calculate the eigenvalues of λ to $O(\alpha^2)$, i.e., $\lambda \approx \lambda^{(0)} + \alpha^2\lambda^{(1)}$, where $\lambda^{(0)}$ will be calculated exactly, while for $\lambda^{(1)}$, first-order perturbation theory in α^2 is sufficient. Hence, the dependence of $\mathcal{H}^{(1)}(\lambda)$ on λ is not crucial: to the accuracy required, it can be replaced by $\lambda^{(0)}$. In addition, the kernel of the inner product (10.31) can be set to unity.

In the case $P = (-)^{J\pm 1}$ we have $\mathcal{J} = C^2 I$, so that $\mathcal{H}^{(0)}$ is the (dimensionless) radial Coulomb Hamiltonian H_J with the angular momentum $\ell = J$, repeated twice:

$$\mathcal{H}^{(0)} = \begin{bmatrix} H_J & 0 \\ 0 & H_J \end{bmatrix}, \quad H_J = -\frac{1}{2}\left\{ \frac{d}{d\rho^2} - \frac{J(J+1)}{\rho^2} \right\} - \frac{1}{\rho}. \tag{11.6}$$

The matrices $\mathcal{K}(\rho,\lambda)$ and $\mathcal{M}(\rho,\lambda)$ are

$$\mathcal{K}(\rho,\lambda) = \frac{1}{8} \begin{bmatrix} (1+\delta^2)\left(\lambda + \frac{1}{\rho}\right) & 0 \\ 0 & (1+\delta^2)\lambda + \frac{2}{\rho} \end{bmatrix}, \tag{11.7}$$

$$\mathcal{M}(\rho,\lambda) = \frac{1}{8} \begin{bmatrix} (1-\delta^2)\lambda\left(\lambda - \frac{2}{\rho}\right) + \frac{1-\delta^2 - C^2\lambda(1+\delta^2)}{\rho^2} + \frac{1+\delta^2(1-2C^2)}{\rho^3} & \frac{2C\delta}{\rho^3} \\ \frac{2C\delta}{\rho^3} & (1-\delta^2)\lambda^2 - \frac{C^2(1+\delta^2)}{\rho^2}\left(\lambda + \frac{1}{\rho}\right) \end{bmatrix}. \tag{11.8}$$

The eigenvalues of the zeroth-order Hamiltonian (11.6), namely,

$$\lambda^{(0)} = -1/(2n^2), \quad n = 1, 2, \dots \tag{11.9}$$

are twofold degenerate, each with the two eigenstates

$$\Psi_{(1)}^{(0)} = \begin{bmatrix} |n,J\rangle \\ 0 \end{bmatrix} \quad \text{and} \quad \Psi_{(2)}^{(0)} = \begin{bmatrix} 0 \\ |n,J\rangle \end{bmatrix}, \tag{11.10}$$

where $|n,J\rangle$ is a solution of the Coulomb problem $H_J|n,J\rangle = \lambda^{(0)}|n,J\rangle$. Thus, the correction $\lambda^{(1)}$ must be calculated appropriately for the degenerate situation:

$$\lambda_{(1,2)}^{(1)} = \frac{1}{2}[\Lambda_{11} + \Lambda_{22} \pm \sqrt{(\Lambda_{11} - \Lambda_{22})^2 + 4\Lambda_{12}^2}], \tag{11.11}$$

where the matrix Λ is defined as follows:

$$\Lambda = [\langle \Psi_{(i)}^{(0)} | \mathcal{H}^{(1)}(\lambda^{(0)}) | \Psi_{(j)}^{(0)} \rangle]$$

$$= \begin{bmatrix} \dfrac{11+\delta^2}{32n^4} - \dfrac{\delta^2}{(2J+1)n^3} & \dfrac{\delta}{2C(2J+1)n^3} \\[3mm] \dfrac{\delta}{2C(2J+1)n^3} & \dfrac{11+\delta^2}{32n^4} - \dfrac{(3+\delta^2)C^2+2}{4C^2(2J+1)n^3} \end{bmatrix}.$$

(11.12)

The mass spectrum

$$E_{(1,2)} = m_+ + \mu\alpha^2\lambda^{(0)} + \mu\alpha^2\lambda_{(1,2)}^{(1)} \qquad (11.13)$$

obtained with the use of Eqs. (11.9), (11.11), and (11.12) coincides neither with the muonium spectrum found in [35,37] nor (if $m_1 = m_2$) with the spectrum of parapositronium (see [18] and references therein). The reason, as for the $J=0$ states, lies in the use of the empty vacuum (3.4), which leads to single-particle states of positive and negative energies, and subsequently to the Breit equation with its spurious term H', Eq. (8.7), in the Hamiltonian [33,34,18], as discussed in connection with the $J=0$ states. We are going to show that the elimination of the contribution of H' from the spectrum leads to the correct result.

First of all we transform H' into a radial representation. For this purpose we note that $\psi_1 = s_1 + v_1$ ($\approx s_1$ in the non-relativistic limit), i.e., ψ_1 contains only those components of $F(\mathbf{r})$, Eq. (6.1), which are coefficients of bispinor harmonics $\phi^A(\mathbf{r})$. Similarly, $\psi_2 = -s_2 + v_2$ ($\approx -s_2$ in the nonrelativistic limit), i.e., ψ_1 contains only coefficients of the bispinor harmonics $\phi^0(\mathbf{r})$. Thus, for $P = (-)^{J\pm1}$ parity states the spurious term (divided by $\mu\alpha^4$) takes the following radial form:

$$\mathcal{H}' = \frac{1}{\mu\alpha^4}\left[\int d\hat{r}\,\mathrm{Tr}(\varphi_i^\dagger H'\varphi_j)\right] = \frac{1-\delta^2}{8\rho^2}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix},$$

(11.14)

where $i,j = A,0$, and the corresponding matrix elements are

$$\Lambda' = [\langle \Psi_{(i)}^{(0)} | \mathcal{H}' | \Psi_{(j)}^{(0)} \rangle] = \frac{1-\delta^2}{4(2J+1)n^3}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

(11.15)

If we now use $\bar{\Lambda} = \Lambda - \Lambda'$, instead of Λ, in Eqs. (11.11) and (11.13), we obtain the spectrum

$$E_{(1,2)} = m_+ - \frac{\mu\alpha^2}{2n^2} + \frac{\mu\alpha^4}{4n^3}\left\{\frac{11+\delta^2}{8n} - \frac{2J+1}{C^2}\right.$$

$$\left.\pm\frac{\sqrt{1+4C^2\delta^2}}{(2J+1)C^2}\right\},$$

(11.16)

which coincides with the results of Connell [35] and Hersbach [37] for the parity $(-1)^{J\pm1}$ states. Thus, correcting for the spurious terms in the Breit Hamiltonian, we obtain the expected $O(\alpha^4)$ results.

In the $P = (-1)^J$ case, the matrix $\mathcal{J}$ is not diagonal:

$$\mathcal{J} = \begin{bmatrix} C^2+2 & -2C \\ -2C & C^2 \end{bmatrix}.$$

(11.17)

It can be diagonalized by means of the orthogonal transformation, using the matrix:

$$\mathcal{R} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}.$$

(11.18)

Then

$$\tilde{\mathcal{J}} = \mathcal{R}\mathcal{J}\mathcal{R}^{-1} = \begin{bmatrix} (J+1)(J+2) & 0 \\ 0 & (J-1)J \end{bmatrix}, \quad (11.19)$$

so that the zeroth-order Hamiltonian becomes

$$\tilde{\mathcal{H}}^{(0)} = \mathcal{R}\mathcal{H}^{(0)}\mathcal{R}^{-1} = \begin{bmatrix} H_{J+1} & 0 \\ 0 & H_{J-1} \end{bmatrix}. \quad (11.20)$$

It possesses the doubly degenerate eigenvalues (11.9) with eigenstates

$$\Psi_{(1)}^{(0)} = \begin{bmatrix} |n,J+1\rangle \\ 0 \end{bmatrix} \quad \text{and} \quad \Psi_{(2)}^{(0)} = \begin{bmatrix} 0 \\ |n,J-1\rangle \end{bmatrix}. \quad (11.21)$$

The first-order correction $\tilde{\mathcal{H}}^{(1)}$, Eq. (11.5), with the matrices

$$\tilde{\mathcal{K}}(\rho,\lambda) = \frac{1}{8}\begin{bmatrix} (1+\delta^2)\lambda + \dfrac{(3-\delta^2)A^2+2B^2}{\rho} - \dfrac{C^2(1-\delta^2)}{\rho^2} & \dfrac{AB(1-\delta^2)}{\rho} \\[4mm] \dfrac{AB(1-\delta^2)}{\rho} & (1+\delta^2)\lambda + \dfrac{2A^2+(3-\delta^2)B^2}{\rho} - \dfrac{C^2(1-\delta^2)}{\rho^2} \end{bmatrix},$$

(11.22)

$$\tilde{\mathcal{M}}(\rho,\lambda)=\frac{1}{8}\begin{bmatrix} \begin{aligned} &(1-\delta^2)\lambda(\lambda+2A^2/\rho)\\ &-A^2[\{\lambda(6J^2+11J+2)-1\}\\ &\quad-\delta^2\{\lambda(2J^2+J-2)-1\}]/\rho^2\\ &-A^2[7J^2+22J+9-\delta^2(3J^2+8J+3)]/\rho^3\\ &+(1-\delta^2)J(J+1)^2(J+4)/\rho^4 \end{aligned} & \dfrac{AB(1-\delta^2)}{\rho}\left(2\lambda+\dfrac{1+2\lambda}{\rho}-\dfrac{C^2}{\rho^2}\right)\\[2em] \dfrac{AB(1-\delta^2)}{\rho}\left(2\lambda+\dfrac{1+2\lambda}{\rho}-\dfrac{C^2}{\rho^2}\right) & \begin{aligned} &(1-\delta^2)\lambda(\lambda+2B^2/\rho)\\ &-B^2[\{\lambda(6J^2+J-3)-1\}-\delta^2\{\lambda(2J^2+3J-1)-1\}]/\rho^2\\ &-B^2[7J^2-8J-6-\delta^2(3J^2-2J-2)]/\rho^3\\ &+(1-\delta^2)J(J+1)^2(J-3)/\rho^4 \end{aligned} \end{bmatrix}$$

(11.23)

generates diagonal matrix elements only:

$$\Lambda=[\langle\Psi_{(i)}^{(0)}|\tilde{\mathcal{H}}^{(1)}(\lambda^{(0)})|\Psi_{(j)}^{(0)}\rangle]=\begin{bmatrix} \dfrac{11+\delta^2}{32n^4}-\dfrac{(7+\delta^2)J^2+(17-\delta^2)J+8-2\delta^2}{4(2J+1)(J+1)(2J+3)n^3} & 0\\[2em] 0 & \dfrac{11+\delta^2}{32n^4}-\dfrac{(7+\delta^2)J^2-3(1-\delta^2)J-2}{4(2J-1)J(2J+1)n^3} \end{bmatrix}.$$

(11.24)

The energy spectrum (11.13) with $\lambda_{(1)}^{(1)}=\Lambda_{11}$, $\lambda_{(2)}^{(1)}=\Lambda_{22}$ contains the contribution of the spurious term (8.7). Again, we present this term in the radial form

$$\mathcal{H}'=\frac{1}{\mu\alpha^4}\left[\int d\hat{r}\,\mathrm{Tr}(\varphi_i^\dagger H'\varphi_j)\right]=\frac{1-\delta^2}{8\rho^2}\begin{bmatrix} B^2 & AB\\ AB & A^2 \end{bmatrix},$$

(11.25)

where, for the present $P=(-)^{J\pm1}$ parity case, $i,j=-,+$. The corresponding matrix elements are

$$\Lambda'=[\langle\Psi_{(i)}^{(0)}|\mathcal{H}'|\Psi_{(j)}^{(0)}\rangle]=\frac{1-\delta^2}{4n^3}\begin{bmatrix} \dfrac{B^2}{2J+3} & 0\\[1em] 0 & \dfrac{A^2}{2J-1} \end{bmatrix}.$$

(11.26)

The substitution of $\lambda_{(1)}^{(0)}=\Lambda_{11}-\Lambda'_{11}$ and $\lambda_{(2)}^{(0)}=\Lambda_{22}-\Lambda'_{22}$ into Eq. (11.13) yields the spectrum

$$E_{(1,2)}=m_+-\frac{\mu\alpha^2}{2n^2}+\mu\alpha^4\frac{11+\delta^2}{32n^4}-\frac{\mu\alpha^4}{2n^3}$$

$$\times\begin{cases} \dfrac{1}{J+1}+\dfrac{1-\delta^2}{(2J+3)(2J+1)}\\[1.5em] \dfrac{1}{J}-\dfrac{1-\delta^2}{(2J+1)(2J-1)}, \end{cases}$$

(11.27)

which coincides with the results of Connell [35] and Hersbach [37] for the parity $(-1)^J$ states.

XII. CONCLUDING REMARKS

We have studied a reformulation of QED, in which the coupled Dirac-Maxwell field equations are partially decoupled by expressing the mediating photon field in terms of the Dirac-particle field, using covariant Green functions. This allows us to reformulate the Hamiltonian of the theory so that the photon propagator appears directly in a quartic, nonlocal interaction term. We then consider a truncated model, in which there are no free (physical) photons. For such a model, each N-particle segment of the Fock space of the quantized, equal-time Hamiltonian is an invariant space, that is, there is no coupling among the various N-fermion segments. This is achieved by introducing an unconventional "empty" vacuum state. As a consequence, there exist exact few-particle eigenstates of the truncated Hamiltonian, which lead to Dirac-like two- and three-fermion wave equations. We show, in particular, that the two-fermion wave equation, in the Coulomb gauge, is just the Breit equation.

For specific J^P states, the Breit equation reduces to the radial form, and then to Dirac-like equations for $J=0$ states, and to a coupled pair of Schrödinger-like equations for $J>0$ states. The perturbative solution of these equations yields α^4 corrections to the nonrelativistic Rydberg spectrum, which do not reproduce the muonium spectrum as calculated by Connell [35] and Hersbach [37] (nor the positronium spectrum in the $m_1=m_2$ case). The apparent reason for this disagreement is the mixing of positive- and negative-energy states, which is characteristic of the Breit equation [32,33,18]. However, agreement is achieved if we subtract the contribution of the spurious operator (8.7), which appears in the Breit equation [cf. Eqs. (8.9), (8.10), (11.16), (11.27)]. We have not been able to obtain analytic solutions of the

radial equations. These radial wave equations have, in general, a singular point at $r_1 \sim \alpha/E > 0$, where E is the two-fermion bound-state energy (rest mass). The existence of such an "interior" singularity makes it difficult to obtain numerical solutions of the radial boundary-value problem by standard methods. The only exception is the case $J = 0^+$, $m_1 = m_2$, for which the radial equations are regular, and which we have therefore studied numerically. [It is noteworthy that this is the case where the contribution of the operator (8.7) is zero.]

Our numerical results for the equal-mass $J = 0^+$ states show that the dependence of the energy E on the coupling constant α is qualitatively similar to that obtained earlier for the Coulomb-QED model (for which transverse-photon interactions are ignored) [8]. For low α, the numerically obtained eigenenergies are in agreement with the result derived perturbatively. Thereafter, $E(\alpha)$ decreases monotonically to $E(\alpha_c) > 0$ as α approaches a critical value α_c. We find that $\alpha_c = 2/\sqrt{3}$ (in contrast to the CQED value of $\alpha_c = 2$).

The present approach has limitations, as has been pointed out below Eq. (4.5), particularly that no radiative (loop) effects are included. Nevertheless the approach allows one to determine relativistic two-fermion eigenstates (energies and wave functions), albeit of a truncated QED Hamiltonian, in a semianalytic fashion for arbitrary states. That is, the problem is shown to reduce to the solution of at most four radial equations of Dirac or Schrödinger type. In addition, as we have shown that the method is straightforwardly generalizable to systems of three particles.

We end with a remark on gauge invariance. In conventional QED, the use of the Lorentz gauge, rather than the Coulomb gauge, requires special consideration (cf. Gupta-Bleuler quantization) and the use of second-order perturbation theory to deduce the Breit equation [18]. In our approach it is straightforward to ensure, at the classical level, that different gauge conditions lead to the same results at the quantum level.

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