Dense coding in entangled states

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We consider the dense coding of entangled qubits shared between two parties, Alice and Bob. The efficiency of classical information gain through quantum entangled qubits is also considered for the case of pairwise entangled qubits and maximally entangled qubits. We conclude that using the pairwise entangled qubits can be more efficient when two parties communicate whereas using the maximally entangled qubits can be more efficient when the N parties communicate.

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The quantum entanglement state [1,2] is an important tool distinguishing the quantum mechanics from the classical physics. In quantum information processing, there have been some examples utilizing quantum entanglement features such as quantum dense coding [3], quantum teleportation [4], and the compression of quantum information [5]. Here we will focus on quantum dense coding. Classically, the capacity of a single bit through the classical channel cannot exceed one bit. However, if we use a pair of entangled qubits between Alice and Bob, then we can send two bits of classical information from Alice to Bob [3]. For this protocol, Alice and Bob share a pair of entangled qubits in the Bell state. Alice performs one of the four 1-qubit unitary operations given by the identity I or the Pauli matrices $(\sigma_x, i\sigma_y, \sigma_z)$ on her qubit. Each of four unitary operations maps the Bell state to a different member of the four Bell states. Then Alice sends her qubit to Bob. Bob can obtain the two bits of classical information from the joint measurement on his qubit and her qubit. In this Brief Report we will extend quantum dense coding to the multiqubit entanglement with N pairwise qubits entangled and with maximally entangled N qubits shared between Alice and Bob.

First, we consider the $2N$ qubits that are entangled pairwise and shared between Alice and Bob. The qubits from the first to the Nth belong to Alice and next N qubits to Bob. The first and the $(N+1)$th qubits are made up Bell states, the second and $(N+2)$th qubits are made up Bell states, and so on. The total state is then written by

$$|\psi\rangle = |\psi^+\rangle_{1,N+1} \otimes |\psi^+\rangle_{2,N+2} \otimes \cdots \otimes |\psi^+\rangle_{N,2N}, \qquad (1)$$

where $|\psi^+\rangle_{i,j} = 1/\sqrt{2}(|00\rangle_{i,j} + |11\rangle_{i,j})$ is a Bell state and the subindex (i,j) denotes that the Bell state is made of the ith qubit and the jth qubit. For dense coding Alice performs unitary operation on her qubits. Each qubit has four possible unitary operations including the identity or Pauli operators and the total number of possible unitary operations is 4^N. Since these operations map the state (1) to the orthogonal states composed by the tensor products of Bell bases, the operations are independent. Then Alice can encode the classical 2^N bits in her unitary operations. Alice sends N qubits to Bob and Bob reads out the classical 2^N bits information after performing a Bell measurement on each joint state of the two qubits.

Next, suppose Alice and Bob are sharing a maximally entangled three qubits state (GHZ state):

$$|\psi^1\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \qquad (2)$$

The first and the second qubit are with Alice and the third with Bob. Alice applies one of the possible unitary operations on her qubits. The unitary operations are the tensor products of identity and Pauli's operators. Here 16 unitary operations are possible. These operators are to be applied on the first or the second qubits of the GHZ states:

$$\begin{aligned}
|\psi^1\rangle &= \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),\\
|\psi^2\rangle &= \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle),\\
|\psi^3\rangle &= \frac{1}{\sqrt{2}}(|100\rangle + |011\rangle),\\
|\psi^4\rangle &= \frac{1}{\sqrt{2}}(|100\rangle - |011\rangle), \qquad (3)\\
|\psi^5\rangle &= \frac{1}{\sqrt{2}}(|010\rangle + |101\rangle),\\
|\psi^6\rangle &= \frac{1}{\sqrt{2}}(|010\rangle - |101\rangle),\\
|\psi^7\rangle &= \frac{1}{\sqrt{2}}(|110\rangle + |001\rangle),
\end{aligned}$$

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$$|\psi^8\rangle=\frac{1}{\sqrt{2}}(|110\rangle-|001\rangle).$$

We need only eight unitary operators among 16 operators: tensor product operators of four operators (I, σ_x, $i\sigma_y$, σ_z) in the first qubit and two operators (σ_x, σ_z) in the second qubit as

$$\begin{aligned} U_0&=\sigma_z\otimes\sigma_z, \quad U_1=I\otimes\sigma_z,\\ U_2&=i\sigma_y\otimes\sigma_z, \quad U_3=\sigma_x\otimes\sigma_z,\\ U_4&=I\otimes\sigma_x, \quad U_5=\sigma_z\otimes\sigma_x,\\ U_6&=\sigma_x\otimes\sigma_x, \quad U_7=i\sigma_y\otimes\sigma_x. \end{aligned} \tag{4}$$

Then Alice has the eight unitary operators to map the state (2) to the different GHZ states (3). Alice applies U_i to her qubits and sends these two qubits to Bob. Bob works a joint GHZ measurement distinguishing the eight GHZ states because of each orthogonal state. Then Bob extracts the three bits of classical information after that measurement processing. This fact show that the GHZ states shared between two parties give rise to the quantum dense coding.

Now let us extend to maximally entangled $N+1$ qubits:

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|00\cdots0\rangle+|11\cdots1\rangle). \tag{5}$$

Alice has N qubits and Bob one qubit among the maximally entangled $N+1$ qubits. Alice intends to use this state to communicate classical information by performing unitary operations on the first N qubits. It is possible to apply four possible unitary operations chosen from the identity or the Pauli operators on the first qubit. On the other hand, one can apply only two possible unitary operations either σ_x or σ_z on the next qubits because the identity and $i\sigma_y$ cannot produce the distinguished states. Then the number of the possible unitary operations is $4\times2\times2\times\cdots\times2=2^{N+1}$ and the number of classical bits is $N+1$.

Now let us compare the efficiency of the maximally entangled case with the pairwise entangled case. First we consider the resource of the both cases. If we prepare the initial states, the pairwise entangled case requires N Hadamard gates and N CNOT gates to make up the entangled states because each single Bell state needs one Hadamard gate and one CNOT gate. If the operation times for the Hadamard gate and the CNOT gate are t_h and t_c, respectively, the rate of information gain in the pairwise entangled case is

$$r_p=\frac{2^N}{N(t_h+t_c)} \tag{6}$$

bits per unit time. The case of the maximally entangled state requires a Hadamard gate and $N+1$ CNOT gates from the initial state. The rate of information gain is

$$r_m=\frac{N+1}{t_h+Nt_c}. \tag{7}$$

If we assume that both gates have the same time scale of operation, that is $t_h=t_c$, then $r_p=2^N/2Nt_c$ and $r_m=N+1/(N+1)t_c=1/t_c$. Thus we can deduce that the pairwise entangled case is more efficient. If we define efficiency as the rate per the number of qubit used, then the pairwise entangled case is more efficient in large N although the maximally entangled case requires $N+1$ qubits less than the pairwise entangled case with $2N$ qubits.

This result is different from that of [6]. They considered that $N+1$ users are sharing maximally entangled qubits, possessed one qubit by each user including Bob and N pairwise entangled states, and possessed one particle by each user except Bob with N qubits. They concluded that the maximally entangled state is definitely more efficient. However, our case shows that the pairwise entangled case is more efficient in the case of large N. This describes that using the pairwise entangled qubits can be more efficient when two parties communicate whereas using the maximally entangled qubits can be more efficient when the N parties make communicate.

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