Theorem for the beam-splitter entangler

Wang Xiang-bin*
Imai Quantum Computation and Information Project, ERATO, Japan Science and Technology Corporation, Daini Hongo White Bulding 201, 5-28-3, Hongo, Bunkyo, Tokyo 113-0033, Japan
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It has been conjectured that the entangled output state from a beam splitter requires nonclassicality in the input state [M. S. Kim, W. Son, V. Buzek, and P. L. Knight, Phys. Rev. A **65**, 032323 (2002)]. Here we give a proof for this conjecture.

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The beam splitter is one of the few quantum devices that may act as an entangler. The entangler properties of a beam splitter have been studied in the past [2–5]. In particular, Kim *et al.* [5] studied the entangler properties with many different input states, such as a Fock number state, a coherent state, a squeezed state, and mixed states in Gaussian form. It was conjectured there that, to obtain an entangled output state, a necessary condition is that the input state should be nonclassical. Unfortunately, there was no proof for this conjecture in [5]. In this paper, we give a very simple proof of this conjecture.

Consider a lossless beam splitter (see Fig. 1). We can distinguish the field mode a and mode b by the different propagation directions. Most generally, the properties of a beam-splitter operator $\hat{B}$ in the Schrödinger picture can be summarized by the following equations (see, e.g., Ref. [1]):

$$\rho_{out}=\hat{B}\rho_{in}\hat{B}^{-1}, \tag{1}$$

$$\hat{B}^{\dagger}=\hat{B}^{-1}, \tag{2}$$

$$\hat{B}\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix}\hat{B}^{-1}=M_B\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix}, \tag{3}$$

$$M_B=\begin{pmatrix}\cos\theta e^{i\phi_0} & \sin\theta e^{i\phi_1}\\ -\sin\theta e^{-i\phi_1} & \cos\theta e^{-i\phi_0}\end{pmatrix}, \tag{4}$$

$$\hat{B}|00\rangle=|00\rangle. \tag{5}$$

Here ρ_{in} and ρ_{out} are the density operators for the input and output states, respectively. Both of them are two-mode states including mode a and mode b. The elements in the matrix M_B are determined by the beam splitter itself, $\hat{a},\hat{b}$ are the annihilation operators for mode a and mode b, respectively, and $|00\rangle$ is the vacuum state for both modes. Equation (5) is due to the simple fact of no input, no output. Without any loss of generality, we can express ρ_{in} in the P representation in the following form:

$$\rho_{in}=\int_{-\infty}^{\infty}P(\alpha_a,\alpha_b,\alpha_a^*,\alpha_b^*)|\alpha_a,\alpha_b\rangle\langle\alpha_a,\alpha_b|d^2\alpha_a d^2\alpha_b, \tag{6}$$

where $|\alpha_a,\alpha_b\rangle$ is a coherent state in two-mode Fock space, i.e.,

$$|\alpha_a,\alpha_b\rangle=\hat{D}_{ab}(\alpha_a,\alpha_b)|00\rangle \tag{7}$$

and

$$\hat{D}_{ab}(\alpha_a,\alpha_b)=e^{\hat{a}^{\dagger}\alpha_a-\hat{a}\alpha_a^*+\hat{b}^{\dagger}\alpha_b-\hat{b}\alpha_b^*}. \tag{8}$$

If ρ_{in} is a classical state, the distribution function $P(\alpha_a,\alpha_b,\alpha_a^*,\alpha_b^*)$ must be non-negative definite in the whole complex plane. In such a case, the ouput state is

$$\rho_{out}=\int_{-\infty}^{\infty}P(\alpha_a,\alpha_b,\alpha_a^*,\alpha_b^*)\times\hat{B}|\alpha_a,\alpha_b\rangle\langle\alpha_a,\alpha_b|\hat{B}^{-1}d^2\alpha_a d^2\alpha_b, \tag{9}$$

which is equivalent to

$$\rho_{out}=\int_{-\infty}^{\infty}P(\alpha_a,\alpha_b,\alpha_a^*,\alpha_b^*)\hat{B}\hat{D}_{ab}(\alpha_a,\alpha_b)\times\hat{B}^{-1}\hat{B}|00\rangle\langle00|\hat{B}^{-1}\hat{B}\hat{D}_{ab}\hat{B}^{\dagger}. \tag{10}$$

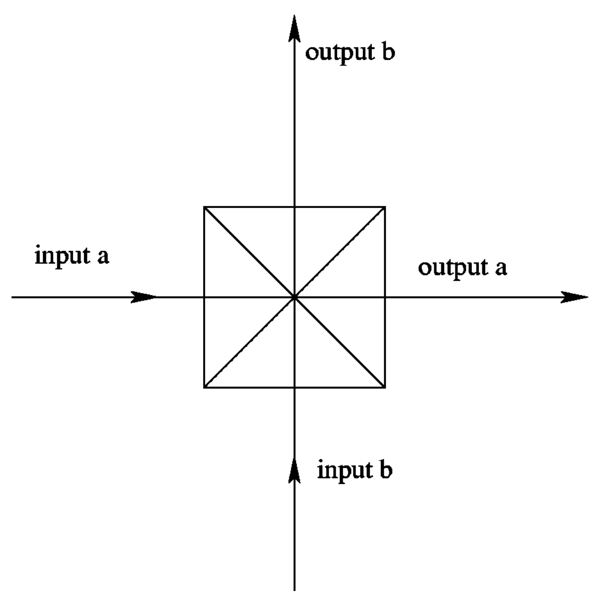

FIG. 1. A schematic diagram for the beam-splitter operation. Both the input and the output are two-mode states. The different modes are distinguished by the propagation direction of the field.

*Email address: wang@qci.jst.go.jp

From Eq. (5) we know that $\hat{B}|00\rangle\langle 00|\hat{B}^{-1}=|00\rangle\langle 00|$. By Eq. (3) we can see that

$$\hat{B}\hat{D}_{ab}(\alpha_a,\alpha_b)\hat{B}^{-1}=\hat{D}_{ab}(\alpha_a',\alpha_b') \tag{11}$$

and

$$(\alpha_a',\alpha_b')=(\alpha_a,\alpha_b)M_B. \tag{12}$$

In short, the following equation can easily be obtained from Eqs. (3)–(5):

$$\hat{B}|\alpha_a,\alpha_b\rangle\langle\alpha_a,\alpha_b|\hat{B}^{-1}=|\alpha_a',\alpha_b'\rangle\langle\alpha_a',\alpha_b'|. \tag{13}$$

Since $\det M_B=1$, we have the following formula for the output state:

$$\rho_{out}=\int_{-\infty}^{\infty}P(\alpha_a,\alpha_b,\alpha_a^*,\alpha_b^*)|\alpha_a',\alpha_b'\rangle\langle\alpha_a',\alpha_b'|d^2\alpha_a'd^2\alpha_b'. \tag{14}$$

This is equivalent to

$$\rho_{out}=\int_{-\infty}^{\infty}P'(\alpha_a,\alpha_b,\alpha_a^*,\alpha_b^*)|\alpha_a,\alpha_b\rangle\langle\alpha_a,\alpha_b|d^2\alpha_a d^2\alpha_b, \tag{15}$$

$$P'(\alpha_a,\alpha_b,\alpha_a^*,\alpha_b^*)=P(\alpha_a'',\alpha_b'',\alpha_a''^*,\alpha_b''^*), \tag{16}$$

and

$$(\alpha_a'',\alpha_b'')=(\alpha_a'',\alpha_b'')M_B^{-1}. \tag{17}$$

Since $P(\alpha_a,\alpha_b,\alpha_a^*,\alpha_b^*)$ is non-negative, the functional $P'(\alpha_a,\alpha_b,\alpha_a^*,\alpha_b^*)$ must also be non-negative. By the definition of separability, the state ρ_{out} defined by Eq. (15) must be separable. Therefore we have the following theorem.

Theorem. If the input state is a classical state, the output state of a beam splitter must be a separable state.

This is equivalent to saying that, in order to obtain an entangled output state, the nonclassicality of the input state is a necessary condition. This theorem can be extended to a more general situation in multimode Fock space. Let us consider the rotation operator $\hat{R}$ in n-mode Fock space. We have

$$\hat{R}\Lambda\hat{R}^{-1}=M_R\Lambda, \tag{18}$$

where $\Lambda=(\hat{c}_1,\hat{c}_2,\ldots,\hat{c}_n)^T$, $\hat{c}_i$ are the annihilation operators of the ith mode, and M_R is a n-dimensional unitary matrix. By using the BCH formula

$$e^{\mu}\nu e^{-\mu}=\nu+[\mu,\nu]+\frac{1}{2!}[\mu,[\mu,\nu]]+\cdots, \tag{19}$$

we have the following explicit formula for the operator $\hat{R}$:

$$\hat{R}=\exp(-\Lambda^{\dagger}\ln M_R\Lambda). \tag{20}$$

Therefore we know that

$$\hat{R}|00\cdots 0\rangle=|00\cdots 0\rangle. \tag{21}$$

Any classical multimode state in Fock space can be written in the following probabilistic distribution:

$$\rho=\int_{-\infty}^{\infty}\mathbf{P}(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|d^2\boldsymbol{\alpha}, \tag{22}$$

where $|\boldsymbol{\alpha}\rangle=|\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2\cdots\boldsymbol{\alpha}_n\rangle$ and $\mathbf{P}(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)$ is a non-negative functional provided that ρ is a classical state. Similarly to the two-mode case, we can show that

$$\rho=\int_{-\infty}^{\infty}\mathbf{P}'(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|d^2\boldsymbol{\alpha}, \tag{23}$$

$$\mathbf{P}'(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)=\mathbf{P}(\boldsymbol{\alpha}'',\boldsymbol{\alpha}''^*) \tag{24}$$

and

$$(\boldsymbol{\alpha}'')=(\boldsymbol{\alpha})M_R^{-1}. \tag{25}$$

Obviously, the funtional $\mathbf{P}'(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)$ is non-negative. Thus we draw the following conclusion in the multimode Fock space: A classical density operator in multimode Fock space is separable under arbitrary rotation.

Although nonclassicality in the input state is a necessary condition, it is obviously not a sufficient condition for entanglement in the output state of a beam splitter. Since a beam-splitter operater is unitary, it is reversible. It was shown in Ref. [5] that a nonclassical separable input state can be changed to an entangled state in the output. The inverse of such a process gives examples where, even though the input state is nonclassical, the output could still be separable. Some specific examples are given in [6,7].

Note added in proof. Recently we became aware of a related work [8] where the nonlocality and the nonclassicality property of light with the beam splitter is investigated.

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