Wave functions of a time-dependent harmonic oscillator in a static magnetic field

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In this paper, we solve for the exact wave functions of a two-dimensional harmonic oscillator under the influence of a static magnetic field. This is done through the use of the so-called Ermakov invariants, similarity transformed so as to obtain a time-independent Schrödinger equation. In the same manner, time-dependent eigenvalues are also computed. It is shown that previous results for the invariants are particular cases for the ones found in this work.

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I. INTRODUCTION

The quantum-mechanical problem of finding exact wave functions for time-dependent harmonic oscillators (TDHO), as well as for oscillatorlike systems, when related to nonlinear, dynamical, coupled and time dependent Ermakov systems, is based on the determination of linear and Hermitian operators, called Ermakov invariants. These operators are constants of motion, subjected to quantum-mechanical superposition laws derived from the Ermakov pair solutions, which can be similarity transformed so as to establish a Hermitian operator, which plays the role of a time-independent Schrödinger's equation, whose eigenfunctions are exactly found. The Ermakov pairs are then, time-dependent, coupled differential equations, related through the invariants.

The same problem has been treated through path integrals, approximation methods, and transformation techniques [1–14], as well as through a different transformation technique called de Broglie–Bohm hydrodynamic interpretation [24]. A review of all these techniques can be found in Ref. [22] and others references therein. But as propagators and invariants better represent the dynamical evolution for these kinds of systems, their relationships have also been established. The case of a TDHO under the influence of an oscillating magnetic field has already been studied through path integrals [5,7], but apparently its description through Ermakov invariants has not been established. In this work, based on what has been established for the application of external fields on Ermakov systems [3,8,11,12], we first apply the invariant technique on a TDHO under the influence of a *static* magnetic field, thus finding its eigenfunctions and permitting additional studies to be realized towards oscillating magnetic and electromagnetic fields. The choice of a static magnetic field is due to the specification of a time-dependent auxiliary equation, which embodies in its definition a general frequency of oscillation that is derived from time-dependent coefficients, which involves time derivatives. Since the magnetic field is considered to be static, its time derivative vanishes and does not change the definition of the auxiliary equation. In the same manner, the relationship between propagators and invariants found in this work can be established further.

This work is divided in the following way. In Sec. II we define the time-dependent Hamiltonian, the equations of motion, the Ermakov-Pinney auxiliary equation, and the invariants to be used. By using Hartley and Ray techniques, in Sec. III the invariant operator in one dimension only is similarity transformed so as to get a time-independent Schrödinger's equation and compute its eigenfunctions. The establishment of the system energy is achieved altogether, where a matrix element is calculated. Finally, in Sec. IV we discuss particular cases derived from the formalism. We show that, when the external field is "switched off," previous cases studied in literature are readily recovered.

II. SCHRÖDINGER EQUATION, EQUATIONS OF MOTION, AND THE ERMAKOV SYSTEM

The Hamiltonian to be considered is the well-known one in which a charged particle of mass $M(t)$ is moving in an axially, symmetric, static magnetic field [5]

$$\mathbf{H}(t)=\frac{1}{2M(t)}\mathbf{P}^2+\frac{1}{2}\overline{\omega}_c L_z+\frac{1}{2}M(t)\Omega^2(t)(x^2+y^2), \tag{2.1}$$

where $\mathbf{P}$ is the linear momentum operator, $\overline{\omega}_c=eB_0/M(t)c$ is the *cyclotronic frequency* of oscillation, L_z is the angular-momentum operator in the axial direction, and $\Omega(t)$ is the general frequency of oscillation given by [5]

$$\Omega^2(t)=\tfrac{1}{4}\overline{\omega}_c^2(t)+\omega^2(t). \tag{2.2}$$

Equations (2.1) and (2.2) are obtained by a choice of a specific gauge $\mathbf{A}=(-yB_0/2,xB_0/2,0)$, which is a Coulomb gauge, since $\boldsymbol{\nabla}\cdot\mathbf{A}=0$.

The equations of motion are obtained through Heisenberg's equation

$$\dot{\mathbf{Q}}=\frac{1}{i\hbar}[\mathbf{Q},\mathbf{H}(t)]+\frac{\partial\mathbf{Q}}{\partial t}, \tag{2.3}$$

where $\mathbf{Q}$ is an operator and $\mathbf{H}$ is the Hamiltonian. For the two dimensions considered in this paper, these equations are [22]

$$\ddot{x}+\Sigma(t)\dot{x}+\omega^2x=-\overline{\omega}_c\dot{y}, \tag{2.4a}$$

$$\ddot{y}+\Sigma(t)\dot{y}+\omega^2y=\overline{\omega}_c\dot{x}, \tag{2.4b}$$

where $\Sigma(t)=\dot{M}/M$ is a time-dependent dissipationlike parameter. Although Eqs. (2.4) do not take in consideration the general frequency $\Omega(t)$, they represent the equations of motion of a nonlinear, coupled, damped, and two-dimensional harmonic oscillator, whose damping factor is $\Sigma(t)$. Together with the Ermakov-Pinney equation [15–18]

$$\ddot{\sigma}+\Sigma(t)\dot{\sigma}+\sigma\Omega^2=\frac{k}{\sigma^3M^2}, \tag{2.5}$$

where k is a constant and σ is a c number, the whole system is coupled through the operators (Ermakov invariants) [22]

$$\mathbf{I}(x,t)=\frac{1}{2}\left\{k\left(\frac{x}{\sigma}\right)^2+(\sigma\mathbf{\Pi}_x-M\dot{\sigma}x)^2-\frac{1}{4}\sigma^2M^2\overline{\omega}_c^2x^2\right\}, \tag{2.6a}$$

$$\mathbf{I}(y,t)=\frac{1}{2}\left\{k\left(\frac{y}{\sigma}\right)^2+(\sigma\mathbf{\Pi}_y-M\dot{\sigma}y)^2-\frac{1}{4}\sigma^2M^2\overline{\omega}_c^2y^2\right\}, \tag{2.6b}$$

where $\mathbf{\Pi}_x$ and $\mathbf{\Pi}_y$ are canonical conjugate momenta along the x and y directions.

III. HARTLEY AND RAY TRANSFORMATION TECHNIQUE

Since Eqs. (2.6) satisfy Eq. (2.3), with $\dot{\mathbf{I}}=\mathbf{0}$, giving rise to a solution of the Schrödinger's equation [2], they have constant eigenvalues and satisfy altogether an eigenvalue equation

$$\mathbf{I}\phi_{n,m}(x,y,t)=\lambda_{n,m}\phi_{n,m}(x,y,t), \tag{3.1}$$

where n and m are quantum numbers representing, respectively, the principal and angular momentum of the particle. Applying an unitary transformation in Eq. (3.1), such that

$$\phi'_{n,m}(x,y,t)=\mathbf{U}\phi_{n,m}(x,y,t), \tag{3.2}$$

where the unitary operator is given by

$$\mathbf{U}^\dagger=\exp\left[\frac{iM(t)}{2\hbar\rho}\dot{\rho}(x^2+y^2)\right], \tag{3.3}$$

Eq. (3.1) is similarity transformed to

$$\mathbf{I}'\phi'_{n,m}(x,y,t)=\lambda_{n,m}\phi'_{n,m}(x,y,t). \tag{3.4}$$

In Eqs. (2.6a) and (2.6b) and in Eq. (3.4) $\mathbf{I}'$ is split into

$$\mathbf{I}'=\mathbf{U}[\mathbf{I}(x,t)+\mathbf{I}(y,t)]\mathbf{U}^+=\mathbf{I}'(x)+\mathbf{I}'(y) \tag{3.5}$$

in such a way that we may use separately Eqs. (2.6a) and (2.6b), now setting $\sigma=k^{1/4}\rho$. After the transformations have been worked out, operator $\mathbf{I}'$ becomes

$$\mathbf{I}'=-\frac{\hbar^2}{2}\rho^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\rho^2\frac{eB_0}{2c}L_z+\frac{1}{2}\left[\left(\frac{x}{\rho}\right)^2+\left(\frac{y}{\rho}\right)^2\right]. \tag{3.6}$$

Defining new variables $\eta=x/\rho$ and $\xi=y/\rho$, the eigenvalue equation finally becomes

$$\left\{-\frac{\hbar^2}{2}\left(\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\xi^2}\right)+\frac{1}{2}(\eta^2+\xi^2)+\rho^2\frac{eB_0}{2c}L_z\right\}\langle\eta,\xi|\varphi_{n,m}\rangle =\lambda_{n,m}\langle\eta,\xi|\varphi_{n,m}\rangle, \tag{3.7}$$

where we have introduced

$$\langle x,y,t|\phi'_{n,m}\rangle=\frac{1}{\rho}\langle\eta,\xi|\varphi_{n,m}\rangle, \tag{3.8}$$

and where we have made use of the Dirac notation. The factor $1/\rho$ gives that the normalization condition

$$\begin{aligned}&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dx\,dy\langle\phi'_{n,m}|x,y,t\rangle\langle x,y,t|\phi'_{n,m}\rangle\\&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\frac{\partial(x,y)}{\partial(\eta,\xi)}\right|d\eta d\xi\frac{1}{\rho^2}\langle\varphi_{n,m}|\eta,\xi\rangle\langle\eta,\xi|\varphi_{n,m}\rangle\\&=1\end{aligned} \tag{3.9}$$

holds. Here $|\partial(x,y)/\partial(\eta,\xi)|$ is the Jacobian.

Now Eq. (3.8) represents the Schrödinger equation for a time-*independent*, two-dimensional, harmonic oscillator under the influence of a static magnetic field, whose solutions are known. By using cylindrical coordinates, the solution is given by [22]

$$\varphi_{n,m}(r,\phi)=\exp\left(-\frac{r^2}{2\hbar}\right)\left(\frac{1}{\sqrt{\hbar}}\right)r^{|m|}{}_1F_1\times\left(\frac{|m|+1}{2}-\gamma;\left|m\right|+1;\frac{r^2}{\hbar}\right)\exp(im\phi), \tag{3.10}$$

where r is the radius, γ is a parameter comprising the energies due to the angular momentum and the harmonic motion on the plane xy, and ${}_1F_1$ is Kummer's hypergeometric function of first kind. With the aid of Eqs. (3.8) and (3.2), the formal solution of the problem is [19,22]

$$\phi_{n,m}(r,\phi)=\frac{1}{\rho}\frac{r^{|m|}}{\sqrt{\hbar^{|m|}}}\exp\left[\frac{iM(t)}{2\hbar}\left(\rho\dot{\rho}+\frac{i}{M(t)}\right)r^2\right]\times\exp(im\phi)\,{}_1F_1\left(\frac{|m|+1}{2}-\gamma;|m|+1;\frac{r^2}{\hbar}\right), \tag{3.11}$$

where

$$\lambda_{n,m}=(2n+|m|+1)\hbar+\rho^2\frac{e\hbar}{2c}B_0m \tag{3.12}$$

are the eigenvalues. Since the general solution of the Schrödinger equation is given by [2]

$$\psi(x,y,t)=\sum_{n,m} C_{n,m}e^{i\alpha_{n,m}(t)}\phi_{n,m}(x,y,t), \qquad (3.13)$$

where $C_{n,m}$ are time-independent coefficients, the rest of the calculation is to compute the time-dependent phase $\alpha_{n,m}(t)$, which with the aid of Eq. (3.11) will give the exact evolution in time of the general wave function [21].

The phase is calculated through [1,2]

$$\dot{\alpha}_{n,m}(t)=\frac{1}{\hbar}\left\langle \phi_{n,m}\left| i\hbar\frac{\partial}{\partial t}-H(t)\right|\phi_{n,m}\right\rangle . \qquad (3.14)$$

Introducing $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dx\,dy|x,y\rangle\langle x,y|=1$ and using $\langle x,y,t|\phi_{n,m}\rangle=\mathbf{U}^{+}\langle x,y,t|\phi'_{n,m}\rangle$, Eq. (3.14) becomes

$$\hbar\dot{\alpha}_{n,m}(t)=\int_{-\infty}^{+\infty}dx\,dy\langle\phi'_{n,m}|x,y,t\rangle \times\left(i\hbar\mathbf{U}\frac{\partial}{\partial t}\mathbf{U}^{+}-\mathbf{U}H(t)\mathbf{U}^{+}\right)\langle x,y,t|\phi'_{n,m}\rangle . \qquad (3.15)$$

At this point a change of variables must be made, and the introduction of a suitable scale transformation [22] will suppress the time dependence of the mass in Eq. (2.5) and modify the unitary operator in Eq. (3.3). But since in a practical meaning this will not change the picture, the recovery of original variables will lead to

$$\hbar\dot{\alpha}_{n,m}(t)=\int_{-\infty}^{+\infty}dx\,dy\langle\phi'_{n,m}|x,y,t\rangle\left\{i\hbar\frac{\partial}{\partial t}+\frac{i\hbar}{\rho}\dot{\rho} +\frac{i\hbar}{\rho}\dot{\rho}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)-\frac{\mathbf{I}'}{M\rho^{2}}\right\}\langle x,y,t|\phi'_{n,m}\rangle . \qquad (3.16)$$

A new change of variables, by introducing $\langle x,y,t|\phi'_{n,m}\rangle =1/\rho\langle\eta,\xi|\varphi_{n,m}\rangle$ leads to

$$\hbar\dot{\alpha}_{n,m}(t)=\int_{-\infty}^{+\infty}d\eta\,d\xi\rho^{2}\frac{1}{\rho}\langle\varphi_{n,m}|\eta,\xi\rangle\left\{i\hbar\frac{\partial}{\partial t}+\frac{i\hbar}{\rho}\dot{\rho} +\frac{i\hbar}{\rho}\dot{\rho}\left(\eta\frac{\partial}{\partial\eta}+\xi\frac{\partial}{\partial\xi}\right)-\frac{\mathbf{I}'}{M\rho^{2}}\right\}\frac{1}{\rho}\langle\eta,\xi|\varphi_{n,m}\rangle . \qquad (3.17)$$

Now we are free to use angular variables in Eq. (3.17). The final result is

$$\hbar\dot{\alpha}_{n,m}(t)=\frac{1}{2\pi}\int_{0}^{+\infty}\int_{0}^{2\pi}r\,dr\,d\phi\langle\varphi_{n,m}|r,\phi\rangle\left\{i\hbar\frac{\partial}{\partial t}+\frac{i\hbar}{\rho}\dot{\rho} +\frac{i\hbar}{\rho}\dot{\rho}\left(r\frac{\partial}{\partial r}\right)-\frac{\mathbf{I}'}{M\rho^{2}}\right\}\langle r,\phi|\varphi_{n,m}\rangle . \qquad (3.18)$$

Again in Eq. (3.18) we have introduced another Jacobian and a factor of $1/\sqrt{2\pi}$, so that the normalization condition holds.

Now Eq. (3.18) has the same one-dimensional form studied by Hartley and Ray [3], Pedrosa [16], and Pedrosa, Serra, and Guedes [17], whose final result, after integration with respect to time, is

$$\alpha_{n,m}(t)=-(2n+|m|+1)\int_{0}^{t}\frac{dt'}{M(t')\rho^{2}(t')} -\frac{eB_{0}}{2c}m\int_{0}^{t}\frac{dt'}{M(t')}, \qquad (3.19)$$

where $\rho(t)$ is the solution of the Eq. (2.5) with $\sigma=k^{1/4}\rho$, $n=0,1,2,\ldots$ and $m=0,\pm1,\pm2,\pm3,\ldots$.

IV. SPECIAL CASES

As a final analytical extension, we now discuss the special cases derived from the considerations introduced above. We take in consideration three special cases.

No magnetic field. When the magnetic field is zero—or, if it is suddenly "switched off"—we have $\mathbf{B}_0=\mathbf{0}$ and $\mathbf{A}=\mathbf{0}$, in such a way that the equations of motion become the system of equations (2.4a), (2.4b) without the coupling term. The only change in Eq. (2.5) is due to the frequency. The invariants (2.6a), (2.6b) become a two-dimensional form, each one similar to the one-dimensional form found by Pedrosa [16] and Pedrosa, Serra, and Guedes [17]. Equation (3.7) is reduced to a form without the angular-momentum operator, whose solution is given by Eq. (3.11), and that can be also obtained by a suitable choice of another unitary operator [23], besides the one given by Eq. (3.3). In this case, the effect of suppressing the application of the magnetic field is the same as changing the reference frame. The eigenvalues are the same as in Eq. (3.12), without the second term.

One-dimensional case. For the one-dimensional case when there is no field, we can consider only the *x* dimension of the problem. The Ermakov system becomes the one-dimensional case considered by Pedrosa [16] and Pedrosa, Serra, and Guedes [17]. A common physical application of this case is the calculation of coherent states [16,17].

Constant mass, no magnetic field. In one dimension, the Ermakov system is given by the case studied by Reid and Ray [11] and by Lutzky [12]. The case of a constant mass and a static magnetic field does not make sense, due to gauge considered, since irrespective of using Cartesian, cyllindrical, or spherical coordinates, there will always be crossed terms involving coordinates and momenta operators, thus establishing a two-dimensional character only. A physical application of this case can be seen in Refs. [14] and [20], where the quantum motion of a particle in the Paul trap is studied.

V. CONCLUSIONS AND SUGGESTIONS

A two-dimensional, time-dependent, harmonic oscillator under the influence of a static, axially oriented, magnetic field was studied using an Ermakov system, obtained through standard techniques, where we have solved for the wave functions. Here the time-dependent parameters are the mass and the natural frequency of oscillation, thus considering previous works by Pedrosa [16] and Pedrosa, Serra, and Guedes [17]. In this case, the linear momentum of the particle is canonical conjugate, and the vector potential **A** is chosen in such a way that the magnetic field points out in an axial direction.

As suggestions, we firmly believe that a time-dependent magnetic field, namely, $\mathbf{B}=\mathbf{B}(x,t)$, can be applied to the present case, and the Ermakov invariants, as well as its eigenfunctions, can be established, thus giving rise to a final relationship between the invariants and its respective propagators. This relationship can be established through the approach of Khandekar and Lawande [4], and the results may be compared to the ones found by Nassar [5] and Bassalo *et al.* [7].

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