Rabi oscillations and macroscopic quantum superposition states

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A two-level atom interacting with a single radiation mode is considered, without the rotating-wave approximation, in the strong-coupling regime. It is shown that, in agreement with the recent results on Rabi oscillations in a Josephson junction [Y. Nakamura, Yu. A. Pashkin, and J. S. Tsai, Phys. Rev. Lett. **87**, 246601 (2001)], the Rabi frequency is indeed proportional to first kind integer order Bessel functions in the limit of a large number of photons and the dressed states are macroscopic quantum superposition states. To approach this problem, analytical use of the dual Dyson series and the rotating-wave approximation is made.

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A recent experimental finding on Josephson junctions [1] has shown as Rabi oscillations happen in strong electromagnetic fields for the two-level model. One of the main results of this experiment was the proportionality of the Rabi frequency to first kind integer order Bessel functions with the order given by the photon number involved in the transition, in agreement with a pioneering treatment by Cohen-Tannoudji *et al.* [2], with the contribution of the two-level model being systematically neglected.

The study of a two-level model in a cosine time-dependent perturbation in the strong-coupling regime [3] proved that in this case Rabi oscillations involve only odd order first kind Bessel functions and partially explains the results of Ref. [1]. On the same ground we approached the problem of a two-level atom interacting with a single radiation mode in the strong-coupling regime in Ref. [4]. In this paper we want to extend the results of Ref. [4] by discussing the case of the experiment of Ref. [1] with a large number of photons involved. We will show that theory and experiment indeed agree.

The Hamiltonian we start with has the simple form as also given in Ref. [1] (neglecting the tunneling contribution as not essential),

$$H=\omega a^\dagger a+\frac{\Delta}{2}\sigma_3+g\sigma_1(a^\dagger+a), \tag{1}$$

where ω is the frequency of the radiation mode, Δ is the separation between the two levels of the atom, g is the coupling between the radiation field and the atom, σ_1 and σ_3 are the Pauli matrices, and a and $a^\dagger$ are the annihilation and creation operators for the radiation field. With respect to the model used in Ref. [1], we have interchanged σ_1 and σ_3 but this is irrelevant as the two Hamiltonians are connected by the unitary transformation $\exp[i\sigma_2(\pi/4)]$. This apparently simple model is not exactly solvable unless the rotating-wave approximation is done, in this latter case the solution is exactly known and the model is then named the Jaynes-Cummings model [5]. The model (1) is able to describe the results of Ref. [1] when the coupling g is large with respect to the level separation.

To reach our goal we apply the duality principle in perturbation theory as given in Refs. [6,7]. The idea is to consider, contrarily to small perturbation theory as given, e.g., in Ref. [8], as unperturbed Hamiltonian the term

$$H_0=\omega a^\dagger a+g\sigma_1(a^\dagger+a). \tag{2}$$

The Schrödinger equation ($\hbar=1$),

$$H_0U_F(t)=i\frac{\partial}{\partial t}U_F(t) \tag{3}$$

has the solution [4]

$$U_F(t)=\sum_{n,\lambda} e^{-iE_nt}|[n;\alpha_\lambda]\rangle\langle[n;\alpha_\lambda]||\lambda\rangle\langle\lambda|, \tag{4}$$

where

$$E_n=n\omega-(g^2/\omega),\quad \alpha_\lambda=\lambda g^2/\omega,\quad \text{and}$$

$$|[n;\alpha_\lambda]\rangle=e^{(g/\omega)\lambda(a-a^\dagger)}|n\rangle \tag{5}$$

is a displaced number state [9], n an integer starting from zero is the eigenvalue of the operator $a^\dagger a$, and $\lambda=\pm1$ is the eigenvalues of σ_1. These states represent the dressed states for the system but our analysis complies with the one given in Ref. [7] by a dual Dyson series otherwise, no Rabi oscillations can be obtained theoretically.

At this point we are able to write down a dual Dyson series for the Hamiltonian (1) as

$$U(t)=U_F(t)T\exp\left[-i\int_0^t dt' H_F(t')\right], \tag{6}$$

having put

$$H_F(t)=U_F^\dagger(t)\frac{\Delta}{2}\sigma_3U_F(t). \tag{7}$$

It is important to note that in the dual Dyson series (6), as also happens in the small perturbation case, when there is a resonance between the two-level atom and the radiation field, perturbative terms appear that are unbounded in the limit $t\to\infty$ and the perturbation series is useless unless we are able to resum such terms, named ‘‘secularities’’ as in celestial mechanics, at all order. This can be done, e.g., by renormalization group methods [10] but here we limit the complexity of the mathematical analysis by simply doing the rotating-wave approximation and ignoring any correction to it.

Let us look at the Hamiltonian (7). It is easily realized that it can be rewritten in the form [4]

$$H_F = H_0' + H_1 \tag{8}$$

using the result of Ref. [9],

$$\langle l|e^{(g/\omega)\lambda(a-a^\dagger)}|n\rangle = \sqrt{\frac{n!}{l!}}\left(\lambda\frac{g}{\omega}\right)^{l-n} \times e^{-\lambda^2(g^2/2\omega^2)} L_n^{(l-n)}\left(\lambda^2\frac{g^2}{\omega^2}\right), \tag{9}$$

with $l \geqslant n$ and $L_n^{(l-n)}(x)$ the associated Laguerre polynomial [11]. So, one has

$$H_0' = \frac{\Delta}{2}\sum_n e^{-(2g^2/\omega^2)} L_n\left(\frac{4g^2}{\omega^2}\right) \times [|[n;\alpha_1]\rangle\langle[n;\alpha_{-1}]||1\rangle\langle -1| + |[n;\alpha_{-1}]\rangle\langle[n;\alpha_1]||-1\rangle\langle 1|], \tag{10}$$

where L_n is the nth Laguerre polynomial [11] and

$$H_1 = \frac{\Delta}{2}\sum_{m,n,m\neq n} e^{-i(n-m)\omega t}[\langle n|e^{-(2g/\omega)(a-a^\dagger)}|m\rangle|[n;\alpha_1]\rangle \times\langle[m;\alpha_{-1}]||1\rangle\langle -1| + \langle n|e^{(2g/\omega)(a-a^\dagger)}|m\rangle|[n;\alpha_{-1}]\rangle \times\langle[m;\alpha_1]||-1\rangle\langle 1|]. \tag{11}$$

The Hamiltonian H_0' can be immediately diagonalized by the eigenstates

$$|\psi_n;\sigma\rangle = \frac{1}{\sqrt{2}}[\sigma|[n;\alpha_1]\rangle|1\rangle + |[n;\alpha_{-1}]\rangle|-1\rangle] \tag{12}$$

with eigenvalues

$$E_{n,\sigma} = \sigma\frac{\Delta}{2}e^{-(2g^2/\omega^2)}L_n\left(\frac{4g^2}{\omega^2}\right), \tag{13}$$

where $\sigma = \pm 1$. Then we can see that each level of the atom develop a band with an infinite subset of levels numbered by the integer number n. The eigenstates can be seen as macroscopic quantum superposition states (sometimes named in the literature as Schrödinger cat states) [5]. We will prove that the two-level atom shows Rabi oscillations between these states. To prove this result we look for a solution of the Schrödinger equation with the Hamiltonian H_F by taking

$$|\psi_F(t)\rangle = \sum_{\sigma,n} e^{-iE_{n,\sigma}t}a_{n,\sigma}(t)|\psi_n;\sigma\rangle, \tag{14}$$

which gives the equations for the amplitudes [4]

$$i\dot{a}_{m,\sigma'}(t) = \frac{\Delta}{2}\sum_{n\neq m,\sigma} a_{n,\sigma}(t)e^{-i(E_{n,\sigma}-E_{m,\sigma'})t} \times e^{-i(m-n)\omega t}\left[\langle m|e^{-(2g/\omega)(a-a^\dagger)}|n\rangle\frac{\sigma'}{2} + \langle m|e^{(2g/\omega)(a-a^\dagger)}|n\rangle\frac{\sigma}{2}\right]. \tag{15}$$

At this stage we can apply the rotating-wave approximation. The resonance condition is given by

$$E_{n,\sigma} - E_{m,\sigma'} - (n-m)\omega = 0 \tag{16}$$

and two Rabi frequencies are obtained. For interband transitions ($\sigma \neq \sigma'$) one has

$$\mathcal{R} = \Delta\left|\left\langle n\left|\sinh\left[\frac{2g}{\omega}(a-a^\dagger)\right]\right|m\right\rangle\right|, \tag{17}$$

while for intraband transitions one has

$$\mathcal{R}' = \Delta\left|\left\langle n\left|\cosh\left[\frac{2g}{\omega}(a-a^\dagger)\right]\right|m\right\rangle\right|. \tag{18}$$

By using Eq. (9), it is easy to show that

$$\mathcal{R} = \frac{\Delta}{2}\sqrt{\frac{n!}{m!}}\left(\frac{2g}{\omega}\right)^{m-n} e^{-(2g^2/\omega^2)}\left|L_n^{(m-n)}\left(\frac{4g^2}{\omega^2}\right)\right| \times[1-(-1)^{m-n}] \tag{19}$$

and

$$\mathcal{R}' = \frac{\Delta}{2}\sqrt{\frac{n!}{m!}}\left(\frac{2g}{\omega}\right)^{m-n} e^{-(2g^2/\omega^2)}\left|L_n^{(m-n)}\left(\frac{4g^2}{\omega^2}\right)\right| \times[1+(-1)^{m-n}], \tag{20}$$

and then, for interband transitions one can have Rabi oscillations only between states differing by an odd number and we write $m-n = 2N+1$, while interband Rabi oscillations can happen only for states differing by an even number and we write in this case $m-n = 2N$. So, finally

$$\mathcal{R} = \Delta\sqrt{\frac{n!}{(n+2N+1)!}}\left(\frac{2g}{\omega}\right)^{2N+1} e^{-(2g^2/\omega^2)}\left|L_n^{(2N+1)}\left(\frac{4g^2}{\omega^2}\right)\right| \tag{21}$$

and

$$\mathcal{R}' = \Delta\sqrt{\frac{n!}{(n+2N)!}}\left(\frac{2g}{\omega}\right)^{2N} e^{-(2g^2/\omega^2)}\left|L_n^{(2N)}\left(\frac{4g^2}{\omega^2}\right)\right|. \tag{22}$$

We can interpret this Rabi oscillations as involving an effective number of photons $2N+1$ and $2N$, respectively, in the transitions. So, we can take n to be very large and N small or zero and this is in agreement with the experiment described in Ref. [1]. This in turn means, in agreement with the experimental results,

$$\mathcal{R}\approx\Delta\left|J_{2N+1}\left(\frac{4\sqrt{n}g}{\omega}\right)\right| \tag{23}$$

and

$$\mathcal{R}'\approx\Delta\left|J_{2N}\left(\frac{4\sqrt{n}g}{\omega}\right)\right|, \tag{24}$$

where Stirling approximation has been used for the factorial $n!\approx e^{-n}n^n\sqrt{2\pi n}$ at large n and the equation [11]

$$J_\alpha(2\sqrt{n}x)=e^{-x/2}\left(\frac{x}{n}\right)^{\alpha/2}L_n^\alpha(x) \tag{25}$$

holds in the limit of n going to infinity, both for $\mathcal{R}$ and $\mathcal{R}'$.

To complete this paper, we want to show how Rabi oscillations emerge from Eq. (15) in the limit of a large number of photons involved, when we start taking as initial state, e.g., $|0\rangle|g\rangle$, where $a|0\rangle=0$ and $\sigma_3|g\rangle=-|g\rangle$. Indeed, one has

$$|0\rangle|g\rangle=\sum_{n,\sigma}a_{n,\sigma}(0)|\psi_n;\sigma\rangle, \tag{26}$$

where

$$a_{n,\sigma}(0)=e^{-(g^2/2\omega^2)}\left(\frac{g}{\omega}\right)^n\frac{1}{2\sqrt{n!}}[\sigma+(-1)^n]. \tag{27}$$

For interband resonance, with $n-m=2N+1$, we get for $E_{n,(\sigma=1)}-E_{m,(\sigma'=-1)}=(2N+1)\omega$ with even n and odd m and $n>m$,

$$a_{m,-1}(t)=a_{m,-1}(0)\cos\left(\frac{\mathcal{R}}{2}t\right)-ia_{n,1}(0)\sin\left(\frac{\mathcal{R}}{2}t\right), \tag{28}$$

$$a_{n,1}(t)=a_{n,1}(0)\cos\left(\frac{\mathcal{R}}{2}t\right)-ia_{m,-1}(0)\sin\left(\frac{\mathcal{R}}{2}t\right), \tag{29}$$

which can be put in explicit form by the coefficients (27) giving

$$a_{m,-1}(t)=-e^{-(g^2/2\omega^2)}\left(\frac{g}{\omega}\right)^m\frac{1}{\sqrt{m!}}\cos\left(\frac{\mathcal{R}}{2}t\right)-ie^{-(g^2/2\omega^2)}\left(\frac{g}{\omega}\right)^n\frac{1}{\sqrt{n!}}\sin\left(\frac{\mathcal{R}}{2}t\right), \tag{30}$$

$$a_{n,1}(t)=e^{-(g^2/2\omega^2)}\left(\frac{g}{\omega}\right)^n\frac{1}{\sqrt{n!}}\cos\left(\frac{\mathcal{R}}{2}t\right)+ie^{-(g^2/2\omega^2)}\left(\frac{g}{\omega}\right)^m\frac{1}{\sqrt{m!}}\sin\left(\frac{\mathcal{R}}{2}t\right), \tag{31}$$

and finally

$$a_{m,-1}(t)=-e^{-(g^2/2\omega^2)}\left(\frac{g}{\omega}\right)^m\frac{1}{\sqrt{m!}}\left[\cos\left(\frac{\mathcal{R}}{2}t\right)+i\left(\frac{g}{\omega}\right)^{2N+1}\sqrt{\frac{m!}{(m+2N+1)!}}\sin\left(\frac{\mathcal{R}}{2}t\right)\right], \tag{32}$$

$$a_{m+2N+1,1}(t)=e^{-(g^2/2\omega^2)}\left(\frac{g}{\omega}\right)^m\frac{1}{\sqrt{m!}}\left[\left(\frac{g}{\omega}\right)^{2N+1}\times\sqrt{\frac{m!}{(m+2N+1)!}}\cos\left(\frac{\mathcal{R}}{2}t\right)+i\sin\left(\frac{\mathcal{R}}{2}t\right)\right]. \tag{33}$$

In the limit of a very large number of photons m, we easily realize that the Rabi frequency is

$$\mathcal{R}\approx\Delta\left|J_{2N+1}\left(\frac{4\sqrt{m}g}{\omega}\right)\right| \tag{34}$$

and we have the oscillating amplitudes

$$a_{m,-1}(t)=-e^{-(g^2/2\omega^2)}\left(\frac{g}{\omega}\right)^m\frac{1}{\sqrt{m!}}\cos\left(\frac{\mathcal{R}}{2}t\right), \tag{35}$$

$$a_{m+2N+1,1}(t)=e^{-(g^2/2\omega^2)}\left(\frac{g}{\omega}\right)^m\frac{1}{\sqrt{m!}}i\sin\left(\frac{\mathcal{R}}{2}t\right), \tag{36}$$

in agreement with the experimental results of Ref. [1]. It is interesting to note that the probability to find the atom in one of the two levels is in any case proportional to a Poisson distribution. Similar expressions can be obtained for resonant intraband transitions ($\sigma=\sigma'$) and so we can have Rabi oscillations with Rabi frequencies being proportional to odd or even order first kind Bessel functions. The situation is quite different if instead of a second quantized radiation field we use a classical cosine field [10]. In this case we can have Rabi frequency just with odd first kind Bessel functions. This is due to the disappearance of the band structure for a "classical" field and then to the disappearance of the intraband transitions.

The above computation gives a strong theoretical support to the experimental findings of Ref. [1]. On a different ground, we can state that the description through dressed states as generalized in Ref. [7] is sound. The experiment realized by Nakamura *et al.*, besides to be a first realization of a strongly perturbed two-level system by a radiation field, can be seen as the realization of oscillations between macroscopic quantum superposition states. However, it is important to point out that decoherence is observed whose source is to be identified in view of practical use of Josephson junction as gates for quantum computation. But, it is a fundamental result to have proven experimentally the existence of a two-level system in a strong-coupling regime.

In conclusion, we have shown how the experimental findings in Ref. [1] can be explained theoretically by the dual

Dyson series and the generalized understanding of dressed states described in Ref. [7]. A generalization of the approach described in this paper has also been recently provided by Fujii [12].

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[1] Y. Nakamura, Yu.A. Pashkin, and J.S. Tsai, Phys. Rev. Lett. **87**, 246601 (2001).
[2] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions* (Wiley, New York, 1992), pp. 485–489.
[3] M. Frasca, J. Opt. B: Quantum Semiclassical Opt. **3**, S15 (2001).
[4] M. Frasca, Z. Naturforsch., A: Phys. Sci. **56**, 197 (2001); in *Proceedings of the 3rd Workshop on Mysteries, Puzzles and Paradoxes In Quantum Mechanics*, edited by Rodolfo Bonifacio, David Vitali, and Berthold-Georg Englert (Verlag der Zeitschrift für Naturforschung, Gargnano, 2000). Unfortunately, the example given in this contribution of Rabi oscillations has the Rabi frequency being zero for the considered transition. Anyhow, the analysis of the two-level atom is given correctly.
[5] W.P. Schleich, *Quantum Optics in Phase Space* (Wiley-VCH, Berlin, 2001).
[6] M. Frasca, Phys. Rev. A **58**, 3439 (1998); **45**, 43 (1992); **47**, 2374 (1993).
[7] M. Frasca, Phys. Rev. A **60**, 573 (1999).
[8] A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1986), Vol. 2.
[9] F.A.M. de Oliveira, M.S. Kim, P.L. Knight, and V. Bužek, Phys. Rev. A **41**, 2645 (1990).
[10] M. Frasca, Phys. Rev. A **56**, 1548 (1997); **58**, 771 (1998).
[11] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1980).
[12] K. Fujii, e-print quant-ph/0203135.