Nonideal teleportation in coherent-state basis

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A coherent representation has been developed for entanglement and measurement, which is an elegant approach to continuous variable quantum teleportation. In the present paper this frame is used to deal with nonideal elements of the experimental situation, such as inefficiencies of the Bell measurement, loss, and thermal noise in the nonlinear crystal used for producing entangled pairs. A measure of fidelity is introduced for characterizing the quality of nonideal schemes.

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I. INTRODUCTION

Some of the most interesting features of quantum mechanics arise from entanglement and measurement. Two (or more) quantum systems are in a pure entangled state if their common state cannot be obtained by simply compounding pure states of the subsystems. Entangled states of light are relatively easy to produce by using proper laser sources and special crystals [1,2]. When the entangled light beams separate, the entanglement can be preserved at great distances: quantum correlations between light beams 10 km apart have already been shown [3].

The famous paradox of Einstein, Podolski, and Rosen (EPR) also concerns entanglement and measurement [4]: if two particles are in a special entangled state, then according to Neumann's principle, a measurement on one influences the results of possible measurements on the other (that is, the state of the other particle).

This is the essence of several quantum communication and state engineering methods such as quantum teleportation [5], entanglement swapping [6,7], distillation of entanglement [8], or quantum dense coding [9]. These methods are closely related to each other and form the base of quantum informatics. In this work we investigate some models of nonideal quantum teleportation.

Quantum teleportation is a communication protocol for transmitting the state of a quantum system from one place to another without passing the system itself. In the ideal case it is carried out by means of a maximally entangled (EPR) pair. A joint so-called Bell measurement is carried out on one member of the pair and the system carrying the state to be transmitted. This measurement changes the state of the other member of the pair in such a way that that and only that will contain information about the given quantum state thereafter.

The teleportation is nonideal if the pair is only partially entangled [10,11], if it is in a mixed state [12] or the Bell measurement is not perfect. As a consequence the communication distorts. From the point of view of teleportation this imperfection is a disadvantage, in quantum state engineering, however, it may be exploited.

Bennett's discrete teleportation scheme has been generalized by Vaidman [13] and Braunstein and Kimble [14] to teleport continuous variables. A coherent-state representation for entanglement has been developed, providing an especially elegant approach to continuous variable teleportation [15,16].

Coherent states are very important in quantum optics in several aspects. From the experimental point of view they are significant because the light of lasers can be approximately described by them. From the theoretical point of view the fact that they are eigenstates of annihilation operators is important. For the present work the most vital consequence of this property is that the transformation of coherent states on beam splitters is simple.

Coherent states, however, form an over-complete base, that is, a given state can be expanded by them in several ways. Nevertheless, using appropriate constraints for the weight function, the expansion can be made unique. According to Glauber, such a constraint is, for example, the prescription of the weight function to be analytic, but many such representations exist for one and two modes [17–20].

In this cadre it is quite convenient to deal with nonideal elements of the experimental situation [21]. Such elements are the inefficiencies of the Bell measurement of the sender (Alice), yielding an imperfect Bell measurement; and thermal noise and loss in the crystal during the creation of the entangled pair, resulting in a partially entangled pair being in mixed state. Our aim is to investigate the distorting effect of these elements.

This paper is organized as follows: in Sec. II we examine the ideal scheme of continuous teleportation and introduce a measure of fidelity. It will be shown that if the input state is a pure coherent state then fidelity is independent of the amplitude. Therefore the fidelity of the teleportation of coherent states may be used as a measure of the quality of the scheme. In Sec. III a model for nonideal Bell measurement is suggested and calculated. In Sec. IV we examine teleportation by means of a more realistic entangled pair obtained by solving the Langevin equations in a pumped crystal at finite temperature. During the investigation of each nonideal element all other elements are supposed to be ideal because we would like to concentrate on the distortion caused by the given nonideal element.

II. CONTINUOUS TELEPORTATION

In this section first we follow the discussion of Ref. [15]. The scenario is depicted in Fig. 1. Alice's state to be tele-

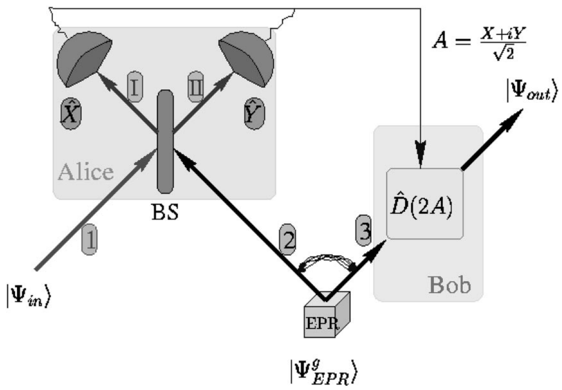

FIG. 1. Scheme for continuous teleportation of an unknown quantum state $|\Psi_{\mathrm{in}}\rangle$ from Alice to Bob by means of the shared entanglement, a joint measurement performed by Alice, and the classical information A sent from Alice to Bob.

ported is an arbitrary pure state of mode 1 written as

$$|\Psi_{\mathrm{in}}\rangle_1 = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha e^{-|\alpha|^2/2} f(\alpha^*)|\alpha\rangle_1 \tag{1}$$

in Glauber's coherent representation. In modes 2 and 3 we have the entangled pair

$$|\Psi^s_{\mathrm{EPR}}\rangle_{23} \propto \sum_n s^n |n\rangle_2 |n\rangle_3 . \tag{2}$$

We use this pair because the ideal down-conversion results in this two-mode squeezed vacuum state. For calculations it is convenient to rewrite this state in coherent-state representation

$$|\Psi^g_{\mathrm{EPR}}\rangle_{23} = \frac{\mathcal{N}(g)}{\pi}\int_{\mathbb{C}} d^2\alpha e^{-|\alpha|^2/g^2}|\alpha\rangle_2|\alpha^*\rangle_3 . \tag{3}$$

It is straightforward to see that Eqs. (2) and (3) are equivalent to

$$g = \sqrt{\frac{s}{1-s}} . \tag{4}$$

$\mathcal{N}(g)$ is a normalization factor that becomes important when we investigate the fidelity of the scenario as a function of g. This pair becomes maximally entangled as $g \to \infty$ $(s \to 1)$, but it is not normalizable in this limit. Later in Sec. IV we shall see that a nonideal entangled pair in an experiment is not as simple as Eq. (3), but at the moment this is the simplest way to deal with nonideal entangled pairs.

Alice carries out a joint measurement on modes 1 and 2. She makes them interfere on a symmetric beam splitter BS, then carries out $\hat{X}$ and $\hat{Y}$ quadrature measurements on the out-modes. This measurement projects the state of modes 1 and 2 onto

$$|\Psi_B(A)\rangle_{12} = \frac{1}{\pi}\int_{\mathbb{C}} d^2\lambda e^{\lambda^* A - \lambda A^*}|\lambda + A\rangle_1 |\lambda^* - A^*\rangle_2 , \tag{5}$$

with

$$A := \frac{X+iY}{\sqrt{2}} \tag{6}$$

reflecting the result of Alice's measurement.

After Alice's measurement the state of mode 3 is projected onto

$$|\Psi_f\rangle_3 \propto \langle \Psi_{\mathrm{Bell}}(A)|\Psi_i\rangle_{123} , \tag{7}$$

where the symbol $\propto$ indicates that we have an additional normalization factor arising from such a projection. This, applying successively Glauber's useful identity for analytic f functions

$$f(\beta^*) = \frac{1}{\pi}\int d^2\alpha f(\alpha^*)\, e^{-|\alpha|^2 + \alpha\beta^*} , \tag{8}$$

yields

$$|\Psi_f\rangle_3 \propto \frac{\mathcal{N}(g) e^{-2|A|^2}}{\pi}\int d^2\alpha f(\alpha^* + 2A^*) \times e^{-(1/g^2+1/2)|\alpha|^2 - 2\alpha^* A}|\alpha\rangle_3 . \tag{9}$$

Now, if Alice passes to Bob the result A of her measurement via a classical channel, he may perform a coherent displacement by $2A$ leading to the state

$$|\Psi_{\mathrm{out}}\rangle_3 = \hat{D}(2A)|\Psi_f\rangle_3 \propto \frac{\mathcal{N}(g)}{\pi}\int d^2\alpha f(\alpha^*) \times \exp\left[-\frac{|\alpha - 2A|^2}{g^2} - \frac{|\alpha|^2}{2}\right]|\alpha\rangle_3 . \tag{10}$$

We perceive that in the limit of $g \to \infty$ this state reproduces the input state. This is the case of maximal entanglement and the teleportation is without distortion.

In any other case, the transfer distorts: we have a Gaussian smoothing factor next to the function f. In the case of finite g, the state $|\Psi_{\mathrm{out}}\rangle$ is obtained by an obviously nonunitary transformation of $|\Psi_{\mathrm{in}}\rangle$, making the reconstruction of the original state really difficult.

In this scheme, if g is finite, it is interesting to ask how good a transfer provided by the teleportation is, in other words, how closely the output state resembles the input [22]. To answer this question let us calculate first the density operator of the "expectation state" of mode 3 after Alice's measurement (a probabilistic event) and Bob's displacement. This expectation state will be compared with the input.

It is straightforward to see that the expectation state is simply

$$\hat{\rho}^e_{\mathrm{out}} = \frac{1}{\pi}\int d^2A |\Psi_{\mathrm{out}}(A)\rangle\langle\Psi_{\mathrm{out}}(A)| , \tag{11}$$

because the probability of measuring A is just the inverse of the missing additional normalization factor appearing in Eq. (7). So the fidelity will be defined as

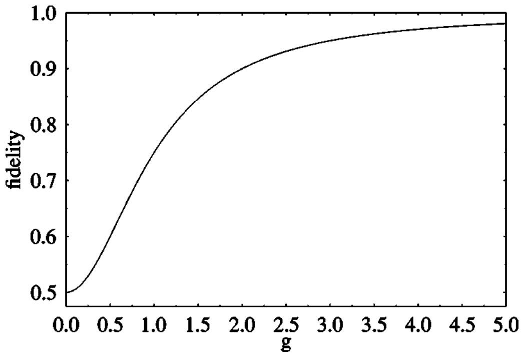

FIG. 2. Fidelity of teleportation in the ‘‘ideal’’ scheme as a function of the g parameter of the entangled pair. $g=0$ represents the case of classical teleportation with fidelity $\frac{1}{2}$ because in this case the shared state is the vacuum; while $g=\infty$ is the case of maximal shared entanglement, that is, perfect teleportation with fidelity 1.

$$\mathcal{F}=\mathrm{Tr}\{\hat{\rho}_{\mathrm{in}}\hat{\rho}_{\mathrm{out}}^{e}\}. \tag{12}$$

We will not calculate this for the general situation when the input state has the form (1), because in this case fidelity depends on the choice of the function f, thus it gives little information on the quality of the scheme itself.

Instead we examine the special case when the input state is a pure coherent state $|\alpha\rangle$. In this case, curiously enough, fidelity turns out to be independent of the amplitude α. Moreover, we find the same result in a more general situation than the scheme discussed above, if before the measurement the state of modes 2 and 3 is a general two-mode mixed state and we try to use this for teleportation. At the first look this may seem surprising because the entangled pair (3) with the Gaussian cutoff contains large amplitudes with everdiminishing weight. Thus it does not have enough energy to teleport large-amplitude coherent states faithfully.

The solution of this paradox lies in Bob’s displacement that compensates the energy loss in such a way that coherent states can be teleported with equal fidelity irrespective of the amplitude. After Alice’s measurement, one part of the information about the input state remains with Alice in the measured result, while the other part is transferred to Bob encoded in the state of mode 3. Since Alice communicates her result to Bob, he will already have enough information to reconstruct coherent states independently of the amplitude.

Since in the teleportation of coherent states the fidelity defined in Eq. (12) does not depend on the amplitude, we shall regard it as a characteristic of the teleportation scheme itself. The fidelity of the scheme represented in Fig. 1 turns out to be

$$\mathcal{F}(g)=\frac{1+2g^2}{2(1+g^2)}. \tag{13}$$

This function is shown in Fig. 2. It is interesting that even in the case of $g=0$ when the state (3) is vacuum, coherent states may be teleported with fidelity $\frac{1}{2}$ due to Bob’s displacement governed by Alice’s classical message. This process may be called ‘‘classical teleportation’’ because there is no quantum correlation here: the system is measured somewhere, the results are sent elsewhere and on the basis of the results of the measurement it is ‘‘assembled’’ again there. It can be shown that the fidelity of the classical teleportation of coherent states with a perfect measurement can be at most $\frac{1}{2}$, we find this result in Ref. [14]. As we have already mentioned above, with $g\to\infty$ we tend to the ideal case when the fidelity tends to 1.

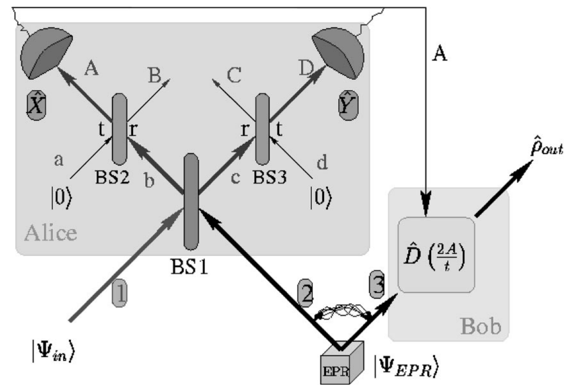

FIG. 3. Teleportation scheme for modeling inefficiencies of Alice’s measurement by the addition of two asymmetric beam splitters BS2 and BS3 between the modes to be measured and the detectors. These beam splitters scatter out some of the photons to be measured, resulting in energy loss.

III. INEFFICIENCIES OF THE BELL MEASUREMENT

In an experiment Alice’s Bell measurement is not perfect because the detector efficiency is finite and beam splitters are not ideal. In this section we examine the distorting effect of the inefficiencies of the measurement. Since our model consists of auxiliary beam splitters, here we shall see the real advantage of using a coherent-state representation.

The nonideal Bell measurement is modeled by introducing two additional asymmetric beam splitters between the out-modes of the original beam splitter and the detectors. These additional beam splitters scatter out some of the photons that we would like to measure. The resulting Bell measurement will be nonideal, and projects the state of modes 1 and 2 onto a mixed state. Naturally, Bob’s output state in this case will be mixed as well.

The proposed scheme is depicted in Fig. 3. BS1 is a symmetric beam splitter while BS2 and BS3 are asymmetric beam splitters with transmission coefficient t and reflection coefficient r; these are supposed to be real for the sake of simplicity (we have $r^2+t^2=1$). The limit of ideal measurement is the case $t=1,\ r=0$.

Here Alice wants to teleport the

$$\hat{\rho}_{\mathrm{in}}=\frac{1}{\pi^2}\int d^2\alpha \int d^2\beta R(\alpha^*,\beta)e^{-(|\alpha|^2+|\beta|^2)/2}|\alpha\rangle\langle\beta| \tag{14}$$

general state of mode 1, with $R(\alpha^*,\beta)=f(\alpha^*)f^*(\beta)$ if the state to be teleported is pure, but the calculation is similar for general mixed states. The state of modes 2 and 3 is an ideal EPR pair obtained from Eq. (3) by performing the limit $g \to \infty$,

$$|\Psi_{\mathrm{EPR}}\rangle_{23} = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha |\alpha\rangle_2 |\alpha^*\rangle_3, \tag{15}$$

which is a non-normalizable state.

First we determine the state the nonideal measurement projects the state of modes 1 and 2 onto. On mode A an $\hat{X}$ quadrature measurement is carried out so here we have $\hat{X}$ eigenstate, while the state of mode a is vacuum. The connection between the modes of the beam splitter is

$$\hat{a}_A = t\hat{a}_b + r\hat{a}_a, \quad \hat{a}_B = -r\hat{a}_b + t\hat{a}_a, \tag{16}$$

that is,

$$\hat{a}_b = \frac{\hat{a}_A + r\hat{a}_a}{t}, \quad \hat{a}_B = \frac{\hat{a}_a - r\hat{a}_A}{t}. \tag{17}$$

The state of modes A,a is

$$|\Psi\rangle_{Aa} = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} da\, e^{-iXa} |X+ia\rangle_A |0\rangle_a, \tag{18}$$

with X being the result of the quadrature measurement. Using this and Eq. (17) the state of modes b,B,

$$|\Psi\rangle_{bB} = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} da\, e^{-iXa} \left|\frac{X+ia}{t}\right\rangle_b \left| -\frac{r(X+ia)}{t}\right\rangle_B, \tag{19}$$

from which we form the common density operator of modes b and B,

$$\hat{\rho}_{bB} = \frac{1}{\pi}\int_{\mathbb{R}} da \int_{\mathbb{R}} db\, e^{iX(b-a)} \left|\frac{X+ia}{t}\right\rangle_b \left| -\frac{r(X+ia)}{t}\right\rangle_B \times \left\langle\frac{X+ib}{t}\right|_b \left\langle -\frac{r(X+ib)}{t}\right|_B, \tag{20}$$

which gives the state of mode b by calculating the partial trace

$$\hat{\rho}_b \propto \mathrm{Tr}_B\{\hat{\rho}_{bB}\} = \frac{1}{\pi}\int_{\mathbb{R}} da \int_{\mathbb{R}} db \exp\left[iX(b-a)\left(1-\frac{r^2}{t^2}\right) - \frac{r^2}{2t^2}(b-a)^2\right] \times \left|\frac{X+ia}{t}\right\rangle_b \left\langle\frac{X+ib}{t}\right|_b, \tag{21}$$

where the symbol $\propto$ means that a state obtained by calculating partial trace is not normalized. We see that the nonideal measurement projects the state of mode b onto a statistically mixed state. This is quite natural because modes b and B are entangled, and the subsystems of a system in a pure entangled state are in a mixed state when regarded separately.

Quite similarly we obtain the state of mode c,

$$\hat{\rho}_c \propto \frac{1}{\pi}\int_{\mathbb{R}} dc \int_{\mathbb{R}} dd \exp\left[iY(c-d)\left(1+\frac{r^2}{t^2}\right) - \frac{r^2}{2t^2}(c-d)^2\right] \times \left|\frac{c+iY}{t}\right\rangle_c \left\langle\frac{d+iY}{t}\right|_c, \tag{22}$$

with Y being the result of the $\hat{Y}$ quadrature measurement. So the common density operator of modes b and c is

$$\begin{aligned}\hat{\rho}_{bc} &= \hat{\rho}_b \otimes \hat{\rho}_c \\ &\propto \frac{1}{\pi^2}\int_{\mathbb{R}} da \int_{\mathbb{R}} db \int_{\mathbb{R}} dc \int_{\mathbb{R}} dd \exp\Bigg\{iY(c-d)\left(1+\frac{r^2}{t^2}\right) \\ &\quad + iX(b-a)\left(1-\frac{r^2}{t^2}\right) - \frac{r^2}{2t^2}[(b-a)^2+(c-d)^2]\Bigg\} \\ &\quad \times \left|\frac{X+ia}{t}\right\rangle_b \left\langle\frac{X+ib}{t}\right|_b \otimes \left|\frac{c+iY}{t}\right\rangle_c \left\langle\frac{d+iY}{t}\right|_c,\end{aligned} \tag{23}$$

from which, using Eqs. (16) in the special case of a symmetric beam splitter, we obtain the density operator of modes 1 and 2 after the Bell measurement,

$$\begin{aligned}\hat{\rho}_{12}^{\mathrm{Bell}} &\propto \frac{1}{\pi^2}\int_{\mathbb{R}} da \int_{\mathbb{R}} db \int_{\mathbb{R}} dc \int_{\mathbb{R}} dd \exp\Bigg\{iY(c-d)\left(1+\frac{r^2}{t^2}\right) \\ &\quad + iX(b-a)\left(1-\frac{r^2}{t^2}\right) - \frac{r^2}{2t^2}[(b-a)^2+(c-d)^2]\Bigg\} \\ &\quad \times \left|\frac{c+ia+X+iY}{t\sqrt{2}}\right\rangle_1 \left\langle\frac{d+ib+X+iY}{t\sqrt{2}}\right|_1 \\ &\quad \otimes \left|\frac{c-ia-X+iY}{t\sqrt{2}}\right\rangle_2 \left\langle\frac{d-ib-X+iY}{t\sqrt{2}}\right|_2.\end{aligned} \tag{24}$$

Using again

$$A := \frac{X+iY}{\sqrt{2}} \tag{25}$$

for representing the result of Alice's measurement, and turning to complex integration variables we have

$$\hat{\rho}_{12}^{\mathrm{Bell}} \propto \frac{\mathcal{N}(A,t)}{\pi^2} \int_{\mathbb{C}} d^2\gamma \int_{\mathbb{C}} d^2\delta \exp\Bigg[\gamma^* A - \gamma A^* + \delta A^* - \delta^* A - \frac{r^2}{t^2}(|\gamma|^2 + |\delta|^2 - \gamma A + \gamma^* A^* - \delta^* A^* + \delta A - \{\gamma^*\delta + \gamma\delta^*\}) \Bigg] \left| \frac{\gamma + A}{t} \right\rangle_1 \left\langle \frac{\delta + A}{t} \right|_1 \otimes \left| \frac{\gamma^* - A^*}{t} \right\rangle_2 \left\langle \frac{\delta^* - A^*}{t} \right|_2 . \tag{26}$$

Here in the argument of the exponential function we have denoted with braces the two terms, $\{\gamma^*\delta + \gamma\delta^*\}$, which make the state statistically mixed. This state is non-normalizable. It will become convenient, however, to separate the dependence of the normalization factor on A and t in order to be able to calculate fidelity and investigate its dependence on t,

$$\mathcal{N}(A,t) = \frac{2-t^2}{t^2} \exp\left(\frac{(1-t^2)^2}{t^2(2-t^2)} (A + A^*)^2 \right). \tag{27}$$

Now let us examine the output of teleportation by means of the nonideal Bell measurement introduced above. After the measurement, the state of Bob's mode 3 becomes

$$\hat{\rho}_f \propto \mathrm{Tr}_{12}\{ \hat{\rho}^{\mathrm{Bell}} (|\Psi_{\mathrm{in}}\rangle\langle\Psi_{\mathrm{in}}| \otimes |\Psi_{\mathrm{EPR}}\rangle\langle\Psi_{\mathrm{EPR}}|)\}. \tag{28}$$

This, applying four times Glauber's identity (8), yields

$$\hat{\rho}_f \propto \frac{1}{\pi^2} \int d^2\alpha \int d^2\beta R\left(\alpha^* + \frac{2A^*}{t}, \beta + \frac{2A}{t} \right) \times \exp\Bigg[-r^2(|\alpha|^2 + |\beta|^2) + \frac{r^2}{t}(\alpha^* A - \alpha A^* - \alpha A + \alpha^* A^* - \beta^* A + \beta A^* + \beta A - \beta^* A^*) + r^2\{\alpha\beta^* + \alpha^*\beta\} \Bigg] \times \exp\left[-\frac{|\alpha|^2 + |\beta|^2}{2} - \frac{2}{t}(\alpha^* A + \beta A^*) \right] |\alpha\rangle\langle\beta|. \tag{29}$$

Further investigations show that Bob should now perform his displacement by $2A/t$ to obtain the best output. To this end, Alice should send Bob the result of her measurement and Bob needs to know the losses of Alice's measurement. The state obtained by the displacement is the output of the scheme,

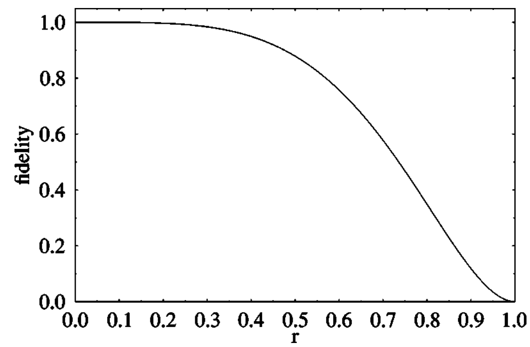

FIG. 4. Fidelity of the scheme with energy losses at the Bell measurement as a function of the reflectance r. $r=0$ represents measurement without energy loss, that is, teleportation with fidelity 1; while $r=1$ is just as if Alice did not perform a measurement at all, making classical teleportation fail, resulting in fidelity 0.

$$\hat{\rho}_{\mathrm{out}} = \hat{D}\left(\frac{2A}{t}\right) \hat{\rho}_f \hat{D}^\dagger\left(\frac{2A}{t}\right) \propto \frac{1}{\pi^2} \int d^2\alpha \int d^2\beta R(\alpha^*, \beta) \times \exp\Bigg[-r^2(|\alpha|^2 + |\beta|^2) + \frac{r^2}{t}(\alpha^* A - \alpha A^* - \alpha A + \alpha^* A^* - \beta^* A + \beta A^* + \beta A - \beta^* A^*) + r^2\{\alpha\beta^* + \alpha^*\beta\} \Bigg] \exp\left[-\frac{|\alpha|^2 + |\beta|^2}{2} \right] |\alpha\rangle\langle\beta|. \tag{30}$$

In the second line we can see the exponential smoothing factor we did not have in the ideal case, and inside of this again two terms are noted with braces, $\{\alpha\beta^* + \alpha^*\beta\}$, which make the output mixed even if the input were a pure state.

In this scheme it is very useful again to calculate fidelity defined by Eq. (12) in the special case when the input state is a pure coherent state of mode 1. Here we have an ideal EPR pair but Alice's measurement is not perfect. However, it turns out again that the fidelity is independent of the amplitude of the coherent state, so it is again a good measure of the quality of the teleportation as a function of the energy loss. The calculation of the fidelity as a function of r yields

$$\mathcal{F}(r) = (1 - r^4)^2. \tag{31}$$

This dependence is shown in Fig. 4. The case $r=0$ reflects the ideal case. For small r, that is, for a small loss, Bob's displacement corrects the defects of Alice's measurement quite efficiently: even at $r=0.5$, for example, we still have $\mathcal{F}=0.88$. The other limiting case is $r=1$, as if Alice did not perform a measurement at all, so classical teleportation also fails, yielding zero fidelity.

IV. ENTANGLEMENT WITH NOISE AND LOSSES

In this section we deal in detail with how entangled pairs are produced for teleportation in the experiment. This is carried out by means of a second-order nondegenerate process of a nonlinear crystal, most frequently a BBO

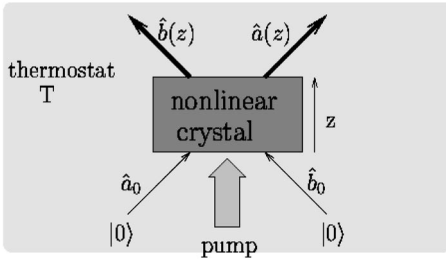

FIG. 5. Scheme for the creation of entanglement between modes a and b by pumping a nonlinear crystal at temperature T.

(β-barium-borate) crystal. The scenario is shown in Fig. 5. During the propagation in the crystal, photons of the pump mode decay into two identical photons propagating in two different modes. The photon numbers of the two modes are correlated because the photons are created in pairs.

The pump mode is considered as a classical field and we neglect the attenuation of its intensity during propagation in the crystal. Thus, we have two quantum fields, a and b, and beside their parametric amplification we also take into account their attenuation in the crystal and their interaction with a thermostat of temperature T.

For the calculations we use the Heisenberg picture: the time evolution of ladder operators is investigated. However, time will be transformed to the traveled distance in the crystal, z, with the speed of light in the crystal. The Langevin equations of motion are

$$\frac{d\hat{a}}{dz}=x\hat{b}^\dagger-\gamma\hat{a}+\hat{Q}_a, \tag{32}$$

$$\frac{d\hat{b}}{dz}=x\hat{a}^\dagger-\gamma\hat{b}+\hat{Q}_b. \tag{33}$$

In these equations, the first term on the right side describes the pure entanglement in which all attributes of the pump are included in the x parameter taken to be real for the sake of simplicity. γ is also a positive real number characterizing energy loss during the propagation in the crystal. The third term on the right side describes the interaction with the thermostat. Assuming a δ correlation of the noise operators we can immediately write

$$\langle \hat{Q}_i^\dagger(z)\hat{Q}_j(z')\rangle_T=2\gamma n(T)\,\delta(z-z')\,\delta_{i,j}, \tag{34}$$

$$\langle \hat{Q}_i(z)\hat{Q}_j^\dagger(z')\rangle_T=2\gamma(n(T)+1)\,\delta(z-z')\,\delta_{i,j}, \tag{35}$$

$$\langle \hat{Q}(z)\hat{Q}(z')\rangle_T=0, \tag{36}$$

$$\langle \hat{Q}^\dagger(z)\hat{Q}^\dagger(z')\rangle_T=0, \tag{37}$$

where we have used

$$[\hat{Q}_i(z),\hat{Q}_j^\dagger(z')]=2\gamma\delta(z-z')\,\delta_{i,j} \tag{38}$$

for Eq. (35). In the above equations $n(T)$ denotes the expectation value of the number of thermal photons,

$$n(T)=(e^{\hbar\omega/k_BT}-1)^{-1}. \tag{39}$$

Now let us solve the Langevin equations with the given initial condition $\hat{a}_0,\hat{b}_0$,

$$\hat{a}(z)=C(z)\hat{a}_0+S(z)\hat{b}_0^\dagger+\hat{F}_a(z), \tag{40}$$

$$\hat{b}(z)=C(z)\hat{b}_0+S(z)\hat{a}_0^\dagger+\hat{F}_b(z), \tag{41}$$

with

$$C(z):=e^{-\gamma z}\cosh(xz), \tag{42}$$

$$S(z):=e^{-\gamma z}\sinh(xz), \tag{43}$$

and

$$\hat{F}_a(z)=\int_0^z dz'[C(z-z')\hat{Q}_a(z')+S(z-z')\hat{Q}_b^\dagger(z')], \tag{44}$$

$$\hat{F}_b(z)=\int_0^z dz'[C(z-z')\hat{Q}_b(z')+S(z-z')\hat{Q}_a^\dagger(z')]. \tag{45}$$

The calculation of the density operator of the entangled pair obtained by the evolution described by the Langevin equations above is quite straightforward in the characteristic function formalism. First, the normally ordered characteristic function of the two modes is calculated. The normally ordered characteristic operator

$$\hat{K}(\eta,\xi)=e^{\eta\hat{a}^\dagger(z)}e^{-\eta^*\hat{a}(z)}e^{\xi\hat{b}^\dagger(z)}e^{-\xi^*\hat{b}(z)} \tag{46}$$

will be expressed by the ladder operators of modes 0 so that it will be easy to apply the initial condition. Performing the commutations we obtain the following normally ordered form:

$$\begin{aligned}\hat{K}(\eta,\xi)=&\,e^{-S^2(z)(|\xi|^2+|\eta|^2)+C(z)S(z)(\eta^*\xi^*+\eta\xi)}e^{[\eta C(z)-\xi^*S(z)]\hat{a}_0^\dagger}e^{-[\eta^*C(z)-\xi S(z)]\hat{a}_0}e^{-[\eta^*S(z)-\xi C(z)]\hat{b}_0^\dagger}e^{[\eta S(z)-\xi^*C(z)]\hat{b}_0}\\&\times\exp\left\{\int_0^z dz'[\eta C(z-z')\hat{Q}_a^\dagger(z')+\xi S(z-z')\hat{Q}_a(z')]\right\}\exp\left\{-\int_0^z dz'[\eta^*C(z-z')\hat{Q}_a(z')+\xi^*S(z-z')\hat{Q}_a^\dagger(z')]\right\}\\&\times\exp\left\{\int_0^z dz'[\xi C(z-z')\hat{Q}_b^\dagger(z')+\eta S(z-z')\hat{Q}_b(z')]\right\}\exp\left\{-\int_0^z dz'[\xi^*C(z-z')\hat{Q}_b(z')+\eta^*S(z-z')\hat{Q}_b^\dagger(z')]\right\}.\end{aligned} \tag{47}$$

The initial condition is that in modes 0 there is a vacuum state

$$\hat{\rho}_0 = |0\rangle_a\langle 0|_a \otimes |0\rangle_b\langle 0|_b \otimes \hat{\rho}_T, \tag{48}$$

where $\hat{\rho}_T$ is the density operator of the thermostat. So the characteristic function of the two modes is

$$\begin{aligned}\chi(\eta,\xi) &= \mathrm{Tr}\{\hat{\rho}_0 \hat{K}(\eta,\xi)\}\\ &= e^{-S^2(z)(|\xi|^2+|\eta|^2)+C(z)S(z)(\eta^*\xi^*+\eta\xi)}\mathrm{Tr}\left(\hat{\rho}_T \exp\left\{\int_0^z dz'[\eta C(z-z')\hat{Q}_a^\dagger(z') + \xi S(z-z')\hat{Q}_a(z')]\right\}\right.\\ &\quad\times\exp\left\{-\int_0^z dz'[\eta^* C(z-z')\hat{Q}_a(z') + \xi^* S(z-z')\hat{Q}_a^\dagger(z')]\right\}\exp\left\{\int_0^z dz'[\xi C(z-z')\hat{Q}_b^\dagger(z') + \eta S(z-z')\hat{Q}_b(z')]\right\}\\ &\quad\left.\times\exp\left\{-\int_0^z dz'[\xi^* C(z-z')\hat{Q}_b(z') + \eta^* S(z-z')\hat{Q}_b^\dagger(z')]\right\}\right).\end{aligned} \tag{49}$$

The trace is calculated using the expression

$$\mathrm{Tr}\{\hat{\rho}_T e^{\hat{F}^\dagger} e^{-\hat{F}}\} = e^{-\langle \hat{F}^\dagger \hat{F}\rangle_T + 1/2\langle \hat{F}^\dagger \hat{F}^\dagger\rangle_T + 1/2\langle \hat{F}\hat{F}\rangle_T}, \tag{50}$$

where expectation values of the operator products are calculated using Eqs. (34)–(37). The computation yields

$$\chi(\eta,\xi) = e^{-m(|\xi|^2+|\eta|^2)+\sigma\eta\xi+\sigma\eta^*\xi^*}, \tag{51}$$

where we have introduced

$$\begin{aligned}m &:= S^2(z) + 2\gamma\left(n(T)\int_0^z dz' C^2(z')\right.\\ &\quad\left.+ (n(T)+1)\int_0^z dz' S^2(z')\right),\end{aligned} \tag{52}$$

$$\sigma := C(z)S(z) + 2\gamma[2n(T)+1]\int_0^z dz' C(z')S(z'). \tag{53}$$

Using the formula

$$\langle (\hat{a}^\dagger)^n \hat{a}^m\rangle = (-1)^m \frac{\partial^n}{\partial \eta^n}\frac{\partial^m}{\partial \eta^{*m}}\chi(\eta)|_{\eta=0} \tag{54}$$

for calculating momenta of ladder operators it is easy to see that the photon numbers of each mode are just m, while σ is connected to entanglement.

Now we are turning our attention to the density operator of our entangled pair. This will be calculated by means of the R function,

$$\hat{\rho} = \frac{1}{\pi^2}\int d^2\alpha \int d^2\beta R(\alpha^*,\beta) e^{-(|\alpha|^2+|\beta|^2)/2}|\alpha\rangle\langle\beta|. \tag{55}$$

The connection between the characteristic function and the R function is [23]

$$R(\alpha^*,\beta) = \frac{e^{\alpha^*\beta}}{\pi}\int d^2\eta\,\chi(\eta) e^{-|\eta|^2 - \eta\alpha^* + \eta^*\beta}. \tag{56}$$

Generalization of these formulas for two modes is straightforward [24], in this case R is a function of $\alpha^*,\beta,\gamma^*,\delta$. Performing the transformation (56) for the characteristic function (51) we obtain

$$R(\alpha^*,\beta,\gamma^*,\delta) = e^{\Omega(\alpha^*\beta+\gamma^*\delta)+\Xi\beta\delta+\Xi\gamma^*\alpha^*}, \tag{57}$$

with

$$\Omega := \frac{m^2+m-\sigma^2}{(m+1)^2-\sigma^2}, \tag{58}$$

$$\Xi := \frac{\sigma}{(m+1)^2-\sigma^2}. \tag{59}$$

So the normalized density operator of the imperfect entangled pair reads

$$\begin{aligned}\hat{\rho}_{\mathrm{EPR}} &= \frac{(1-\Omega)^2-\Xi^2}{\pi^2}\int d^2\gamma \int d^2\delta\\ &\quad\times e^{-1/2(1-\Omega^2-\Xi^2)(|\gamma|^2+|\delta|^2)+\Omega\Xi\gamma\delta^*+\Omega\Xi\gamma^*\delta}\\ &\quad\times|\gamma\rangle\langle\Omega\gamma+\Xi\delta|\otimes|\Omega\delta^*+\Xi\gamma^*\rangle\langle\delta^*|.\end{aligned} \tag{60}$$

Comparison with Eq. (15) yields that this density operator describes an ideal EPR pair in the limit of $\Omega=0$, $\Xi=1$. If $\Omega=0$ and $\Xi\neq 1$ the state remains pure and we obtain the state we have already seen in Eqs. (2) and (3) with

$$\Xi = s = \frac{g^2}{1+g^2}. \tag{61}$$

In any other case, that is, with finite Ω, the entangled pair is in a mixed state.

Now let us turn our attention to teleportation by means of the more realistic entangled pair discussed above. Here we would like to concentrate on the distorting effects of the nonideal entangled pair, so Alice's measurement is supposed

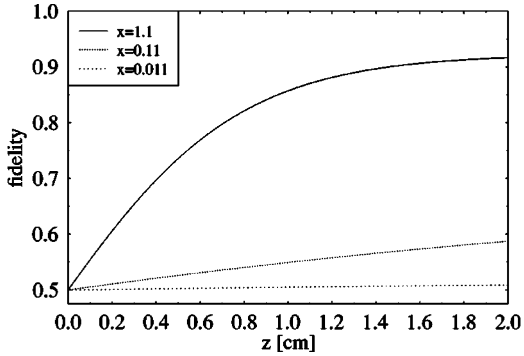

FIG. 6. Entanglement of the shared pair, measured by the fidelity of the teleportation of coherent states, as a function of the propagation distance z in the nonlinear crystal at three different values of the pump parameter x. Since the attenuation of the pump is neglected, entanglement is monotonic.

to be ideal. Alice's state to be teleported is again a general state of mode 1 characterized by the analytic function $f(\alpha^*)$ if the state is pure, or $R(\alpha^*,\beta)$ if it is mixed. Detailed investigations show that in this case Bob should perform his displacement again by $2A$. The calculation is quite similar to the one discussed in Sec. II. The state of Bob's mode 3 after Alice's measurement and Bob's displacement reads

$$\hat{\rho}_{\text{out}} \propto \frac{1}{\pi^4}\int d^2\alpha \int d^2\beta \int d^2\mu \int d^2\nu R(\alpha^*+2A^*,\beta+2A) \times \exp\Bigg[-|\alpha|^2-|\beta|^2-\frac{|\mu|^2+|\nu|^2}{2}+2(1-\Omega)(\mu^*A + \nu A^*)-2\alpha^*A-2\beta A^*-2\Xi(\alpha A^*+\beta^*A) + \Xi(\alpha\mu^*+\beta^*\nu)+\Omega(\alpha\beta^*+\mu^*\nu)\Bigg]|\mu\rangle\langle\nu|. \tag{62}$$

Using Glauber's formulas, we could evaluate two further integrals of the four, which leads to a difficult asymmetrical form.

For characterizing the quality of transmission by means of an entangled pair created in a realistic crystal we shall calculate again the fidelity of the teleportation of coherent states. A straightforward calculation yields

$$\mathcal{F}=\frac{1-\Omega+\Xi}{2}, \tag{63}$$

which, as we have already seen, does not depend on the amplitude of the coherent state.

We will investigate this expression with realistic parameters for BBO crystals. For typical wavelengths of about 700 nm at room temperature (300 K), $n(T)$ is, of course, totally negligible. At these wavelengths the typical value of the linear absorption coefficient is $\gamma=0.1\ \text{cm}^{-1}$. If the parameters of the crystal are fixed, x is simply proportional to the pump intensity. Let us suppose the length of the crystal to be 2 cm. First we investigate entanglement during the propagation in the crystal as a function of z, which we will measure by the fidelity of teleporting coherent states. In

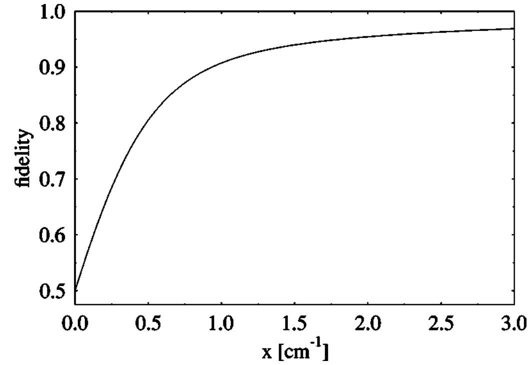

FIG. 7. Entanglement of the shared pair, after propagating 2 cm in a BBO crystal, as a function of the parameter x which is proportional to the pump intensity.

Fig. 6 we see the entanglement process in the crystal with three different pump intensities leading to $x=1.1, 0.11, 0.011\ \text{cm}^{-1}$. Since the attenuation of the pump is neglected, the entanglement is monotonic. The higher x we have, that is, the more intense pump we use, the better fidelity is obtained. Figure 7 depicts this dependence at the end of the crystal.

V. CONCLUSIONS

In this paper we have investigated nonideal continuous teleportation. To this end we have used a coherent representation that has proved to be a convenient tool because the transformation law of the coherent states on beam splitters is simple. First, by the example of the ideal scheme we have introduced a measure of fidelity and shown that in the teleportation of coherent states due to Bob's appropriate coherent displacement, fidelity does not depend on the amplitude of the signal even in such an extreme situation when an arbitrary two mode mixed state is shared between Alice and Bob and used for teleportation. So this quantity can be used as a measure for characterizing the quality of the scheme.

We have modeled inefficiencies of Alice's Bell measurement by introducing two auxiliary beam splitters into the original scheme. It has been shown that by this measurement Bob's state is projected onto a mixed state. We have found that in this case Bob should perform his unitary transformation differently from that in the ideal case. To this end he must know about Alice's losses. In this way the teleportation with small energy losses remains practically ideal.

Furthermore, we have investigated the production of the entangled pair, which is the other most important element of teleportation. We have taken into account distorting effects, such as energy loss and finite temperature in the nonlinear crystal. The Langevin equations of motion can be resolved exactly using a characteristic function formalism to obtain the density operator of the entangled pair. The dependence on temperature is through the number of thermal photons $n(T)$, which is negligible for typical wavelengths and temperatures. We have determined fidelity with several given pump parameters.

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