PHYSICAL REVIEW A **66**, 023804 (2002)

Quantum-entanglement production in a micromaser

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(Received 3 September 2001; published 6 August 2002)

We show that a micromaser can work as an effective source of highly correlated atoms. We consider a one-photon micromaser pumped by a Poissonian beam of atomic pairs. We show that the atoms forming the pairs leave the micromaser's cavity in entangled states and that they can violate the Bell inequality. We consider two aspects of the violation of the Bell inequality: we study the maximal value $\mathbf{B}_{\max}$ of an expression appearing in the Bell inequality and we evaluate a vector-dependent $\mathbf{B}(\vec{a},\vec{a}',\vec{b},\vec{b}')$ depending upon experimental setup. We calculate the entanglement of the formation of states of the atomic pairs. We show that the pairs of atoms, which fly out from the micromaser cavity, are entangled for almost all values of control parameters characterizing the considered two-atom micromaser. This happens even if they do not violate the Bell inequality.

DOI: 10.1103/PhysRevA.66.023804 PACS number(s): 42.50.Ct, 03.65.Ud, 03.67.−a

I. INTRODUCTION

Quantum information theory has been developed during recent years very rapidly. Recent progress in new experimental techniques is also significant. Quantum logic gates have been demonstrated in many cavity QED [1], ion trap [2], and NMR [3] experiments. Quantum teleportation has been realized in experiments using optical systems [4] and NMR [5]. It may be possible, in the near future, to store and process information encoded in microscopic quantum systems. Fast progress in high-speed photon detection, high-speed laser optoelectronics, wavelength, and time division multiplexing has occurred, making it possible for the first time to contemplate the design of high-speed quantum cryptography systems implemented in actual physical environments via either free-space or fiber-optic cable quantum channels [6]. Quantum correlations are the key feature of quantum systems, which allows us to perform many computational and communicational tasks with an efficiency unattainable using classic devices. We consider a micromaser, which is an experimental realization of the Jaynes-Cummings model of a single two-level atom interacting with a single mode of the electromagnetic field [7], as a promising source of such highly correlated quantum systems. The micromaser can also be used to investigate many interesting quantum effects, such as quantum revivals [8], trapping states [9], sub–Poissonian photon statistics [10], or to prepare pure photon number states [11]. Preparation of EPR states and testing of the Bell inequality in the micromaser have also been recently considered [12–14].

In this paper, we consider a two-atom micromaser pumped by a Poissonian beam of pairs of excited two-level atoms. We investigate how interaction of the atoms with the electromagnetic field in the micromaser causes creation of quantum correlations between the atoms. We show that the atoms leaving the cavity are in entangled states. We check the nonlocality of the states of the atoms in two ways: first, we analyze the violation of the Bell inequality, and second, we calculate an entanglement of formation of the atoms [15]. In Sec. II, we present the theory of the micromaser pumped by the Poissonian beam of atomic pairs. We work out a steady-state photon distribution, which will be used in the next section to calculate an atomic correlation function. In Sec. III, we demonstrate the violation of the Bell inequality. We consider a CHSH version of the Bell inequality [16]. The two-atom correlation function used in the Bell inequality depends upon some parameters, which in the original formulation related to an experiment with polarized photons have an interpretation of vectors characterizing a spatial orientation of polarizers. We analyze also the maximal value of the expression appearing in the Bell-CHSH inequality. The maximal value of this expression is the quantity, which is already independent of any additional parameters characterizing experimental setups, and it is very useful in the evaluation of the quantum correlations actually present in the atomic pairs. In Sec. IV, we calculate the entanglement of formation of the states of the atomic pairs. This quantity is much more sensitive to the nonlocal quantum correlations and it allows us to identify the entangled states, even if they do not violate the Bell inequality. We compare different ranges of the parameters used to a characterization of the considered model of the two-atom micromaser. We find these values of the parameters for which the Bell inequality is violated and these values for which the entanglement of formation is greater than zero. The comparison will allow us to judge how the micromaser can be efficient as a source of two- (or more) atom correlated systems. Finally, in Sec. V, we summarize the results and we present conclusions.

II. TWO-ATOM MICROMASER WITH POISSONIAN PUMPING

In the one-atom micromaser, the pumping rate is so low that at most only one atom at a time is present inside the cavity. This condition, easy to fulfill in the case of regular pumping by appropriate choosing of the intensity of the atomic pumping beam, is also fulfilled with very good accuracy in the case of the Poissonian pumping. Such a situation is particularly desirable in experiments, when confirmation of the possibility of the generation of an electromagnetic field in such a system, and nonclassical properties of the obtained field, are investigated. However, there are effects that should be more evident when the simultaneous presence of more than one atom in the cavity is allowed. Recently, it was shown that the atoms leaving the micromaser's cavity can violate the Bell inequality [12,14]. We think that the quantum correlations between the atoms flying out from the micromaser should be distinctly stronger when more atoms present in the same time in the cavity will interact with the electromagnetic field. We concentrate on the simplest case when two atoms fly simultaneously through the cavity [17,18].

We consider the one-photon micromaser pumped by the beam of pairs of excited two-level atoms. Velocities of the atoms in the pairs can differ, so times of flight of the atoms through the cavity can also be different, $\tau_1 \neq \tau_2$, but we assume that they are of the same order of magnitude. We assume that delays of the second atoms, dt, and the times of flight τ_1 and τ_2 are the same for all pairs. We assume also that the lifetime of atoms, t_{at}, is much larger than the times of the interaction of the atoms with the field in the cavity, $\tau_{1,2}$, and we neglect the spontaneous emission. We assume that the following conditions are fulfilled: $\tau_{1,2} \ll t_p \ll t_{cav}$, where t_p is a time distance between succeeding pairs, and $t_{cav} = 1/\kappa$ is a cavity damping time. The delay of the second atom can change from 0, when both atoms fly into the cavity at the same time, to τ_1, when the atoms interact with the field separately. We use the delay dt and the interaction times $\tau_{1,2}$ as control parameters of the model. The pairs enter the cavity according to the Poisson process with mean spacing $1/R$ between events, where R is the flux of the pairs. In $t_0 = 0$, let the first atom from the pair fly into the cavity. Next, at the moment t_1 the second atom arrives and both atoms fly through the resonator together. At the moment t_2 the first atom leaves the cavity and, at the end, in the moment t_3 the second atom flies out from the cavity. We neglect all kinds of direct interatomic interactions such as the dipole-dipole interaction or the van der Waals interaction. However, during the period $t_c = t_2 - t_1$, when both atoms are in the cavity, they interact with the same field and they entangle due to this interaction. We assume that the atoms entering the cavity are in the same excited state $\hbar\omega_0$. The Hamiltonian in the interaction picture has the following form:

$$H^I(t) = H^I_{\mathrm{JC},1}\eta_1(t) + H^I_{\mathrm{JC},2}\eta_2(t),$$

$$H^I_{\mathrm{JC},1} = \frac{1}{2}\Omega[(I\otimes\sigma^\dagger)a + (I\otimes\sigma)a^\dagger], \tag{1}$$

$$H^I_{\mathrm{JC},2} = \frac{1}{2}\Omega[(\sigma^\dagger\otimes I)a + (\sigma\otimes I)a^\dagger],$$

where $H^I_{\mathrm{JC},(1,2)}$ are the time-independent Jaynes-Cummings Hamiltonians for the first and the second atom, $\eta_1(t) = \theta(t) - \theta(t-t_2)$ and $\eta_2(t) = \theta(t-t_1) - \theta(t-t_3)$ are step functions equal to 1, when, respectively, the first or the second atom is present in the resonator, $a^\dagger$ and a are photon creation and annihilation operators, and $\sigma^\dagger$ and σ are raising and lowering atomic operators, respectively. We assume also that the atoms are in the resonance with the field, i.e., that $\Delta = \omega - \omega_0 = 0$. We expand the state vector of the whole system in the basis of states $|n,(\pm)_2(\pm)_1\rangle$,

$$\begin{aligned}|\Psi(t)\rangle = \sum_{n=0}^{\infty} [&c_{n,++}(t)|n,++\rangle + c_{n+1,-+}(t)|n+1,-+\rangle \\ &+ c_{n+1,+-}(t)|n+1,+-\rangle \\ &+ c_{n+2,--}(t)|n+2,--\rangle],\end{aligned} \tag{2}$$

and we solve the Schrödinger equation

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = H^I(t)|\Psi(t)\rangle. \tag{3}$$

In our notation, the right symbol at the vector $|.,.(\pm)_1\rangle$ corresponds to the first atom, which flies into the cavity, and the left symbol, $|.,(\pm)_2.\rangle$, corresponds to the second atom. The time dependence of the Hamiltonian $H^I(t)$ is trivial and we can solve the Schrödinger equation in each time interval $[t_0,t_1]$, $[t_1,t_2]$, and $[t_2,t_3]$ separately. For $t\in[0,t_1]$, we have the following set of equations for amplitudes of the basis states appearing in the expansion of the state vector $|\Psi(t)\rangle$:

$$\dot{c}_{n,++}(t) = -\frac{1}{2}i\Omega\sqrt{n+1}c_{n+1,+-}(t), \tag{4}$$

$$\dot{c}_{n+1,+-}(t) = -\frac{1}{2}i\Omega\sqrt{n+1}c_{n,++}(t),$$

$$\dot{c}_{n+1,-+}(t) = -\frac{1}{2}i\Omega\sqrt{n+2}c_{n+2,--}(t),$$

$$\dot{c}_{n+2,--}(t) = -\frac{1}{2}i\Omega\sqrt{n+2}c_{n+1,-+}(t),$$

for $n\in\mathbf{N}\cup\{0\}$ and

$$\dot{c}_{0,-+}(t) = 0, \quad \dot{c}_{0,+-}(t) = 0,$$

$$\dot{c}_{0,--}(t) = 0, \quad \dot{c}_{1,--}(t) = 0, \tag{5}$$

for the amplitudes of the remaining states, which do not participate in the interaction. Because we assume that the atoms fly into the cavity in the excited states, the initial condition has the following form: $c_{n,++}(0) = c_n(0), c_{n+1,-+}(0) = 0, c_{n+1,+-}(0) = 0$, $c_{n+2,--}(0) = 0$. The amplitudes $c_n(0) = \sqrt{p_n(0)}$ correspond to the initial probability distribution of the photons in the cavity. We set the amplitudes of the noninteracting states equal to zero, $c_{0,-+}(0) = c_{0,+-}(0) = c_{0,--}(0) = c_{1,--}(0) = 0$. At the moment t_1 the state vector is the following:

$$\begin{aligned}|\Psi(t_1)\rangle = \sum_{n=0}^{\infty} [&\cos(x_n t_1)|n,++\rangle \\ &- i\sin(x_n t_1)|n+1,+-\rangle],\end{aligned} \tag{6}$$

where $x_n = \frac{1}{2}\Omega\sqrt{n+1}$. We use the amplitudes $c_{n,\pm\pm}(t_1)$ as the initial condition in calculations of the evolution from t_1 to t_2. The equations for the amplitudes have a similar form to that in the previous case,

$$\dot{c}_{n,++}(t) = -\frac{1}{2}i\Omega\sqrt{n+1}[c_{n+1,+-}(t)+c_{n+1,-+}(t)], \tag{7}$$

$$\begin{aligned}\dot{c}_{n+1,+-}(t) &= -\frac{1}{2}i\Omega[\sqrt{n+1}c_{n,++}(t)\\ &\quad+\sqrt{n+2}c_{n+2,--}(t)],\\ \dot{c}_{n+1,-+}(t) &= -\frac{1}{2}i\Omega[\sqrt{n+1}c_{n,++}(t)\\ &\quad+\sqrt{n+2}\,c_{n+2,--}(t)],\\ \dot{c}_{n+2,--}(t) &= -\frac{1}{2}i\Omega\sqrt{n+2}[c_{n+1,+-}(t)+c_{n+1,-+}(t)].\end{aligned}$$

We solve them and obtain the amplitudes

$$\begin{aligned}c_{n,++}(t_2) &= \frac{(n+2)+(n+1)\cos[y_n(t_2-t_1)]}{2n+3}c_{n,++}(t_1)\\ &\quad-i\sqrt{\frac{n+1}{2(2n+3)}}\sin[y_n(t_2-t_1)]c_{n+1,+-}(t_1),\\ c_{n+1,+-}(t_2) &= -i\sqrt{\frac{n+1}{2(2n+3)}}\sin[y_n(t_2-t_1)]c_{n,++}(t_1)\\ &\quad+\cos^2\left[\frac{1}{2}y_n(t_2-t_1)\right]c_{n+1,+-}(t_1),\\ c_{n+1,-+}(t_2) &= -i\sqrt{\frac{n+1}{2(2n+3)}}\sin[y_n(t_2-t_1)]c_{n,++}(t_1)\\ &\quad-\sin^2\left[\frac{1}{2}y_n(t_2-t_1)\right]c_{n+1,+-}(t_1),\\ c_{n+2,--}(t_2) &= -\frac{2\sqrt{(n+1)(n+2)}}{2n+3}\\ &\quad\times\sin^2\left[\frac{1}{2}y_n(t_2-t_1)\right]c_{n,++}(t_1)\\ &\quad-i\sqrt{\frac{n+1}{2(2n+3)}}\sin[y_n(t_2-t_1)]c_{n+1,+-}(t_1),\end{aligned} \tag{8}$$

where $y_n = \Omega\sqrt{\frac{1}{2}(2n+3)}$. Again the amplitudes $c_{n,\pm\pm}(t_2)$ are used as the initial condition in calculations of the next stage of the evolution in the interval $[t_2,t_3]$. The state vector of the system, when the second atom leaves the cavity, is given by the expression

$$|\Psi(t_3)\rangle = \sum_{i=0}^{\infty}\sum_{\alpha,\beta=\pm,\pm} c_{n,\beta\alpha}(t_3)|n,\beta\alpha\rangle,$$

where the amplitudes are the following:

$$\begin{aligned}c_{n,++}(t_3) &= \cos[x_n(t_3-t_2)]c_{n,++}(t_2)\\ &\quad-i\sin[x_n(t_3-t_2)]c_{n+1,-+}(t_2),\\ c_{n+1,-+}(t_3) &= -i\sin[x_n(t_3-t_2)]c_{n,++}(t_2)\\ &\quad+\cos[x_n(t_3-t_2)]c_{n+1,-+}(t_2),\\ c_{n+1,+-}(t_3) &= \cos[x_{n+1}(t_3-t_2)]c_{n+1,+-}(t_2)\\ &\quad-i\sin[x_{n+1}(t_3-t_2)]c_{n+2,--}(t_2),\\ c_{n+2,--}(t_3) &= -i\sin[x_{n+1}(t_3-t_2)]c_{n+1,+-}(t_2)\\ &\quad+\cos[x_{n+1}(t_3-t_2)]c_{n+2,--}(t_2).\end{aligned} \tag{9}$$

We want to obtain the probability distribution of *n*-photon states. We need to take into account cavity losses. We calculate a reduced density operator $\hat{\rho}_f$ of the electromagnetic field. At the moment $t_{i+1}=t_p+t_3+t_i$, when the first atom of the next pair flies into the resonator, the field is given by the density matrix

$$\hat{\rho}_f(t_{i+1}) = e^{\hat{L}t_p}\hat{G}(t_3)\hat{\rho}_f(t_i),$$

where the operator $\hat{G}(t_3)$ describes the interaction of the atoms with the field, and its explicit form can be obtained from the calculations presented above:

$$\begin{aligned}\langle n|\hat{G}(t_3)\rho_f(t_i)|n\rangle &\equiv p_n(t_i+t_3)\\ &= \sum_{i,j=\pm\pm}|c_{n,ij}(t_i+t_3)|^2\\ &= |d_{n,++}(t_3)|^2 p_n(t_i)\\ &\quad+|d_{n,+-}(t_3)|^2 p_{n-1}(t_i)\\ &\quad+|d_{n,-+}(t_3)|^2 p_{n-1}(t_i)\\ &\quad+|d_{n,--}(t_3)|^2 p_{n-2}(t_i),\end{aligned} \tag{10}$$

where $|d_{n,\pm\pm}(t_3)|^2$ denotes transition probabilities between the appropriate states. The amplitudes $d_{.,..}(t_3)$ are obtained by the composition of the formulas describing the evolution of the system in three separately considered time intervals,

$$\begin{aligned}c_n(0) &\to c_{\{(n,++),(n+1,+-)\}}(t_1)\\ &= d_{\{(n,++),(n+1,+-)\}}(t_1)c_n(0)\to c_\gamma(t_2)\\ &= d_\gamma(t_2)c_n(0)\to c_\gamma(t_3)=d_\gamma(t_3)c_n(0),\end{aligned} \tag{11}$$

where the index γ takes the values $(n,++)$, $(n+1,-+)$, $(n+1,+-)$, and $(n+2,--)$. The Liouville superoperator $\hat{L}$ describes the damping of the field in the cavity,

$$\frac{d\hat{\rho}_f}{dt}=\hat{L}\hat{\rho}_f=\frac{\kappa}{2}(n_b+1)(2a\hat{\rho}_f a^\dagger - a^\dagger a\hat{\rho}_f - \hat{\rho}_f a^\dagger a) + \frac{\kappa}{2}n_b(2a^\dagger\hat{\rho}_f a - aa^\dagger\hat{\rho}_f - \hat{\rho}_f aa^\dagger). \quad (12)$$

We assume that the pairs of atoms arrive at the cavity in time intervals given by the Poissonian distribution with mean value $1/R$. We average the equation for $\hat{\rho}_f$ over the Poissonian distribution $P(t_p)=R\exp(-Rt_p)$ and obtain the following equation:

$$\hat{\bar{\rho}}_f(t_{i+1})=\left(1-\frac{\hat{L}}{R}\right)^{-1}\hat{G}(t_3)\hat{\bar{\rho}}_f(t_i). \quad (13)$$

In order to obtain the steady-state solution, we equate the density matrices describing the field in the cavity at the moments t_i and t_{i+1} $\hat{\bar{\rho}}_f(t_{i+1})=\hat{\bar{\rho}}_f(t_i)$. We have to solve the following equation: $[1-(\hat{L}/R)]\hat{\bar{\rho}}_{f,\mathrm{st}}=\hat{G}(t_3)\hat{\bar{\rho}}_{f,\mathrm{st}}$, which in the basis of the Fock states takes the form

$$\begin{aligned} p_n - \frac{1}{N_{\mathrm{ex}}}(n_b+1)[(n+1)p_{n+1}-np_n] \\ + n_b[np_{n-1}-(n+1)p_n] \\ = |d_{n,++}(t_3)|^2 p_n + [|d_{n,+-}(t_3)|^2 + |d_{n,-+}(t_3)|^2]p_{n-1} \\ + |d_{n,--}(t_3)|^2 p_{n-2}, \end{aligned} \quad (14)$$

where $N_{\mathrm{ex}}=R/\kappa$ is an average number of the pairs that traverse the cavity during the lifetime of the field and n_b is the steady-state temperature-dependent mean photon number. We collect coefficients at the probabilities of the same number of photons, p_n, and we obtain the equation

$$a_n p_n + b_{n-1}p_{n-1} + c_{n+1}p_{n+1} + d_{n-2}p_{n-2}=0, \quad (15)$$

where

$$a_n=|d_{n,++}(t_3)|^2 - 1 - \frac{1}{N_{\mathrm{ex}}}[(n_b+1)n+n_b(n+1)], \quad (16)$$

$$b_{n-1}=|d_{n,+-}(t_3)|^2+|d_{n,-+}(t_3)|^2+\frac{n_b n}{N_{\mathrm{ex}}}, \quad (17)$$

$$c_{n+1}=\frac{(n_b+1)(n+1)}{N_{\mathrm{ex}}}, \quad (18)$$

$$d_{n-2}=|d_{n,--}(t_3)|^2. \quad (19)$$

Due to the relations

$$|c_{n,++}(t)|^2+|c_{n+1,-+}(t)|^2+|c_{n+1,+-}(t)|^2 + |c_{n+2,--}(t)|^2=p_n(t_i) \quad (20)$$

and

$$|d_{n,++}(t)|^2+|d_{n+1,-+}(t)|^2+|d_{n+1,+-}(t)|^2 + |d_{n+2,--}(t)|^2=1 \quad (21)$$

for $t\in[t_i,t_{i+1}]$, expressing the normalization of the probability distribution, Eq. (15) has a solution in the form of a chain fraction [19]:

$$p_n=p_0\prod_{k=1}^{n}\frac{1}{c_k}\left[b_{k-1}+d_{k-1}+\cfrac{d_{k-2}c_{k-1}}{b_{k-2}+d_{k-2}+\cfrac{d_{k-3}c_{k-2}}{\ddots \over b_1+d_1+\cfrac{d_0+c_1}{b_0+d_0}}}\right]. \quad (22)$$

The steady-state solution is fully determined by values of the following parameters: the vacuum Rabi frequency Ω, the pumping rate N_{ex}, the interaction times of the first and the second atom in the pair τ_1,τ_2, and the delay of the second atom dt. The times t_1, t_2, and t_3 are connected with the parameters τ_i and dt by the relations $t_1=dt$, $t_2=\tau_1$, and $t_3=\tau_2+dt$. In order to systematize and simplify analysis of the model, we relate the time parameters to the time interaction of the first atom $\tau\equiv\tau_1$. The time parameters are expressed by τ and the ratio $r=\tau_2/\tau_1$ in the following way: $\tau_2=r\tau$, $t_1=dt$, $t_2=\tau$, and $t_3=r\tau+dt$. We see that the amplitudes d_γ characterizing the changes of the state of the system depend upon the Rabi frequency Ω just by the product of the frequency Ω and appropriate interaction times. We get rid of the direct dependence upon Ω and we introduce a family of dimensionless time parameters defined by the function $\Theta(t)=\frac{1}{2}\Omega\sqrt{N_{\mathrm{ex}}}t$, where t can be any of the considered time parameters.[1] In this parametrization, the considered model is fully characterized by the pumping rate N_{ex}, the ratio of the interaction times r, the dimensionless delay of the second atom $d\Theta=\Theta(dt)$, and the dimensionless interaction time $\Theta=\Theta(\tau)$ of the first atom. All results of numerical computations presented in the next sections are parametrized by these quantities.

[1]The parameters x_n and y_n have in the Θ parametrization the following form: $x_n=\sqrt{n+1}/\sqrt{N_{ex}}$ and $y_n=[2\sqrt{(2n+3)/2}]/\sqrt{N_{\mathrm{ex}}}$.

III. VIOLATION OF THE BELL INEQUALITY

The Bell inequality was primarily introduced in order to test local hidden variable theories considered as alternative to the quantum mechanics [15]. Many modified versions of the inequality proposed originally by Bell, adapted to different experimental proposals, were studied [16]. A violation of the Bell inequality was demonstrated in many experiments [20], and now the old question, namely whether the Bell inequality is violated, can be replaced by a new question: how strong is the violation of the Bell inequality when a given system is investigated? A degree of the violation of the Bell inequality is also one of the few quantitative measures of the quantum nonlocality.

The violation of the Bell inequality by the atoms interacting with the photons in the micromaser has recently been studied theoretically [12,14]. The standard model of the one-atom micromaser with the very weak pumping beam was used in these considerations. We extend the earlier obtained results to the case of the two-atom micromaser. We expect that in the case in which two atoms simultaneously interact with the field in the micromaser's cavity, the quantum correlations should be much more distinct. We are more interested in the opportunity to use the micromaser as a source of nonlocal multiatomic systems than in testing the local realistic theories, and we want to use the degree of violation of the Bell inequality to quantitatively evaluate the quantum correlations between the atoms [12].

We suppose that at the beginning the field in the micromaser is in a steady state described by probability distribution p_n [Eq. (22)] and a probe pair of atoms arrives into the cavity. The initial state of the atom-field system, at the moment $t=0$, when the first atom enters the resonator, is described by the state vector

$$|\Psi(0)\rangle=\sum_{n=0}^{\infty} c_n(0)|n,++\rangle, \tag{23}$$

where the amplitudes $c_n(0)=\sqrt{p_n}$. Atoms fly through the cavity, and at the moment t_3, when the second atom leaves the cavity, the system is in the state

$$\begin{aligned}|\Psi(t_3)\rangle=\sum_{n=0}^{\infty} [&c_{n,++}(t_3)|n,++\rangle+c_{n+1,+-}(t_3)|n+1,+-\rangle\\ &+c_{n+1,-+}(t_3)|n+1,-+\rangle\\ &+c_{n+2,--}(t_3)|n+2,--\rangle].\end{aligned} \tag{24}$$

Amplitudes $c_{n,\pm\pm}(t_3)$ are given by Eqs. (9). We calculate a correlation function

$$\begin{aligned}E(\vec{a},\vec{b})&=\langle\vec{a}\cdot\hat{\vec{\sigma}}^1\vec{b}\cdot\hat{\vec{\sigma}}^2\rangle\\ &=\sum_{i,j=0}^{3} a_i b_j\langle\Psi(t_3)|\hat{\sigma}_i^1\hat{\sigma}_j^2|\Psi(t_3)\rangle,\end{aligned} \tag{25}$$

where $\vec{a}$ and $\vec{b}$ are unit vectors and $\hat{\sigma}_i$ are the Pauli matrices. In experiments with spin-$\frac{1}{2}$ particles or photons, the vectors $\vec{a}$ and $\vec{b}$ have the usual meaning of polarization vectors of a Stern-Gerlach apparatus or light polarizers. Now the vectors have another interpretation because entanglement between internal degrees of freedom of the different atoms is considered. Instead of a projection on the "polarization" vectors, an auxiliary interaction of the atoms with an electromagnetic field is necessary in order to transform the states of the atoms in a manner assigned by the vectors. Then a standard measurement of the atoms in their upper or lower states is enough to obtain a value of the correlation function $E(\vec{a},\vec{b})$. If the atoms are in the state in which the quantum correlations are present, then the following inequality is violated:

$$\mathbf{B}:=|E(\vec{a},\vec{b})-E(\vec{a},\vec{b}\,')|+|E(\vec{a}\,',\vec{b})+E(\vec{a}\,',\vec{b}\,')|\leq 2. \tag{26}$$

The quantity **B** appearing in the Bell inequality is a function of the state of the quantum system and the vectors $\vec{a}$, $\vec{b}$, $\vec{a}\,'$, $\vec{b}\,'$. It is known that **B** takes a maximal value equal to $2\sqrt{2}$ for the singlet state $|\Psi^-\rangle=1/\sqrt{2}(|01\rangle-|10\rangle)$, when vectors $\vec{a}$, $\vec{b}$, $\vec{a}\,'$, and $\vec{b}\,'$ lie in one plane and angles between two consecutive vectors are the same, and they are equal to 45°. An optimal configuration of the vectors, for which **B** is maximal, depends upon the state and should be fitted for different states separately. This requirement makes the quantity **B** appearing in the Bell inequality slightly inconvenient to analyze. It is rather impractical and actually computationally almost impossible to look for a new optimal configuration of the vectors every time the state of the system changes. Besides **B** depending upon the set of vectors, we consider also the maximal value of **B**, which does not already depend upon any additional parameters except the state of the system. A compact expression for the maximal value of **B** was recently obtained by Horodecki *et al.* [21]. $\mathbf{B}_{\max}$ is given directly by the formula $\mathbf{B}_{\max}=2\sqrt{m(\hat{\rho})}$, where $m(\hat{\rho})=\max_{i<j}(u_i+u_j)$, and $u_{i=1,2,3}$ are eigenvalues of the matrix $U(\hat{\rho})=T(\hat{\rho})^\dagger T(\hat{\rho})$, where $T_{i,j}(\hat{\rho})=Tr[\hat{\rho}\hat{\sigma}_i^{(1)}\otimes\hat{\sigma}_j^{(2)}]$. Using this expression, we investigate the quantum correlations of the atoms leaving the micromaser's cavity. The state of the atoms leaving the cavity depends upon the interaction times and the delay. We check how $\mathbf{B}_{\max}$ changes with the change of the parameters r, $d\Theta$, and Θ. We neglect a destructive influence of the background field and we assume that the number of thermal photons $n_b=0$. This assumption is not too restrictive because for typical temperatures attainable in real experiments n_b is really very close to zero. We decide to neglect the thermal field present in the micromaser's cavity also due to another reason. We are interested in an estimation of an upper limit of the quantum correlations between the atoms leaving the cavity of the micromaser. It is known that in experiments there are a lot of sources of various kinds of noises, which can reduce subtle quantum effects and which should be taken into account in the analysis of experimental results. However, we want to check whether it is in general possible to obtain entangled atoms in the micromaser. If the answer is no, in this slightly idealized situation, then all ad-

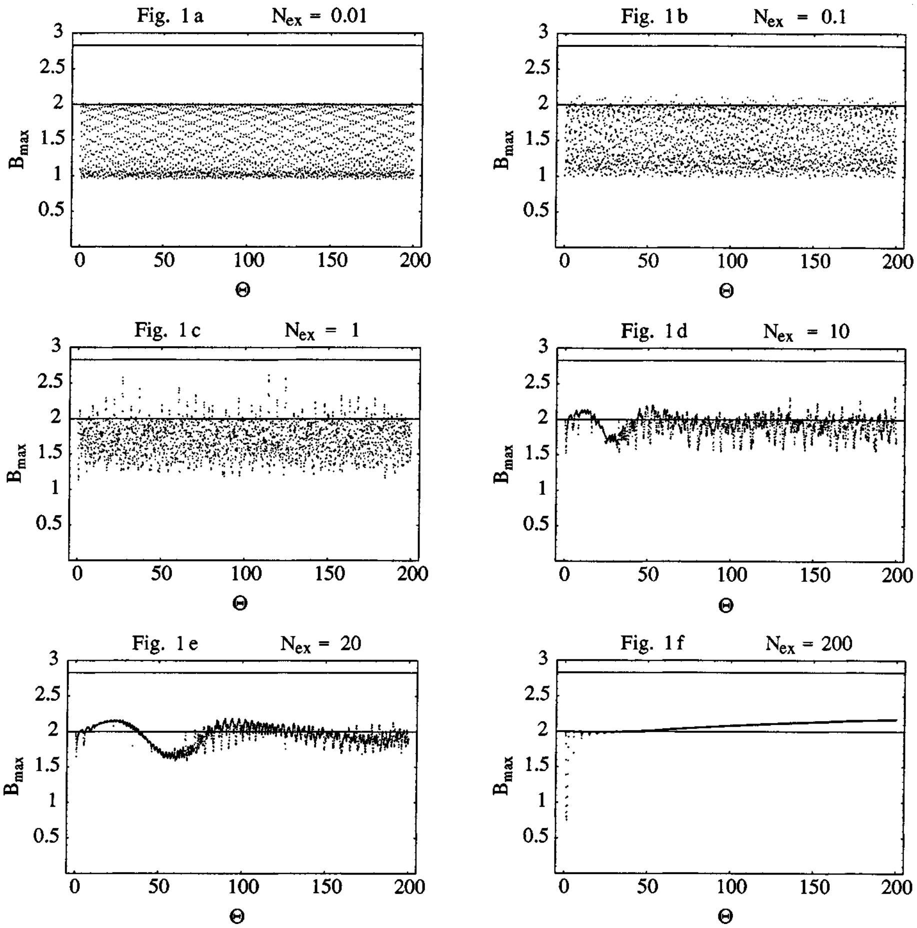

FIG. 1. The values of $\mathbf{B}_{\rm max}$ calculated for $r=1$ and $d\Theta=0$. Horizontal lines indicate the critical values: $\mathbf{B}_{\rm max}=2$, which separates the classical and quantum regions; and $\mathbf{B}_{\rm max}=2\sqrt{2}$, which is the upper limit of $\mathbf{B}_{\rm max}$.

ditional factors decreasing the quantum correlations, such as thermal fluctuations of the electromagnetic field, become unimportant.

First we calculate $\mathbf{B}_{\rm max}$ for $r=1$ and $d\Theta=0$, which means that both atoms enter and leave the cavity together. Such a situation may seem to be very similar to the case in which only one atom is inside the cavity, however it is not so trivial as one might suppose. Although the atoms do not interact directly, they interact with the same mode of the electromagnetic field in the cavity and their states are entangled due to this interaction. We will refer the results of the next computations to the results obtained in this case. In Fig. 1, we show values of $\mathbf{B}_{\rm max}$ calculated as a function of the interaction time Θ for pumping parameter $N_{\rm ex}$ equal to 0.01, 0.1, 1, 10, 20, and 200.

We see that the Bell inequality is fulfilled for very weak pumping when $N_{\rm ex}=0.01$. $\mathbf{B}_{\rm max}$ does not exceed 2 for almost all values of Θ. In the few remaining cases in which the Bell inequality is violated, $\mathbf{B}_{\rm max}$ exceeds 2 only imperceptibly and the quantum correlations are actually negligible. When the pumping parameter increases to $N_{\rm ex}=0.1$, $\mathbf{B}_{\rm max}$ becomes, for some values of the interaction time Θ, distinctly greater than the critical value, and for $N_{\rm ex}=1$ the values of $\mathbf{B}_{\rm max}$ approach the maximal value equal to $2\sqrt{2}$. The violation of the Bell inequality, rather rare for the very weak pumping, becomes frequent for $N_{ex}=10$, but $\mathbf{B}_{\rm max}$ do not achieve as large values as for $N_{\rm ex}=1$. We interpret changes of the $\mathbf{B}_{\rm max}$ in the following way. The quantum correlations come into being in the case of the two-atom micromaser when both atoms interact with the field together. The field is weak for the low pumping and the connection between the atoms is small in this case. The quantum correlations do not appear and the Bell inequality is fulfilled because the atoms do not feel themselves enough strongly. The intensity of the field increases for larger $N_{\rm ex}$ and the atoms correlate so strongly that the Bell inequality can be violated. The connection between the atoms decreases for larger pumping because the interaction of one atom with photons cannot change the field strongly enough to have a significant influence on the state of the other atom, and $\mathbf{B}_{\rm max}$ is in this case not as large as for smaller values of $N_{\rm ex}$. The large field is too "inert" to create very strong quantum correlations between the atoms, and the degree of the violation of the Bell inequality is smaller in such cases. We notice that $\mathbf{B}_{\rm max}$ decreases slightly when the second atom enters the cavity with some delay $d\Theta$. This effect is common for all values of the pumping parameter $N_{\rm ex}$.

The properties of the micromaser and obviously the states of the atoms leaving the cavity depend upon the times of the interactions. Up until now, we have investigated the violation of the Bell inequality in the case in which the times of the interactions of both atoms were equal. Now we are going to analyze the situation when the second atom flies through the cavity longer than the first one. We put $r=1.5$ and $d\Theta=0$. The first atom resides in the resonator one and half times shorter than the second atom. In Fig. 2, we show the values

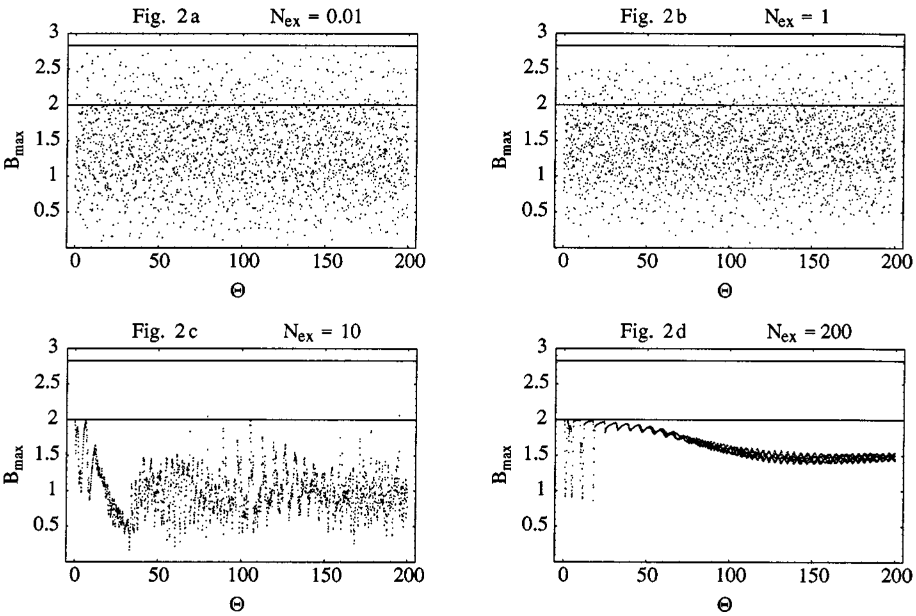

FIG. 2. The values of $\mathbf{B}_{max}$ calculated for $r=1.5$ and $d\Theta=0$. Horizontal lines indicate the critical values: $\mathbf{B}_{max}=2$, which separates the classical and quantum regions; and $\mathbf{B}_{max}=2\sqrt{2}$, which is the upper limit of $\mathbf{B}_{max}$.

of $\mathbf{B}_{max}$ calculated as a function of the interaction time Θ for pumping parameter N_{ex} equal to 0.01, 1, 10, and 200.

In this case the violation of the Bell inequality is most distinct for weak pumping. The values of $\mathbf{B}_{max}$ are largest for $N_{ex}=0.01$ and they are, for some values of Θ, very close to the maximal possible value. $\mathbf{B}_{max}$ decreases for the next considered values of the pumping parameter. It is interesting that, in this case, for $N_{ex}=200$ the values of $\mathbf{B}_{max}$ are constantly smaller than the critical value. Again $\mathbf{B}_{max}$ weakly depends upon the delay time and is slightly smaller for larger $d\Theta$.

Up until now, we have analyzed the maximal value of the quantity appearing in the Bell inequality. $\mathbf{B}_{max}$ does not depend upon the configuration of vectors $\vec{a}$, $\vec{a}'$, $\vec{b}$, and $\vec{b}'$, i.e., it is independent of any measurement details. This makes this quantity very convenient and useful when the upper limit of the quantum correlations present in the system is investigated. However, it is also interesting to know how much $\mathbf{B}_{max}$ overestimate the vector-dependent $\mathbf{B}(\vec{a},\vec{a}',\vec{b},\vec{b}')\equiv\mathbf{B}$, which is measured in experiments. In Fig. 3, we show $\mathbf{B}(\vec{a},\vec{a}',\vec{b},\vec{b}')$ calculated for $r=1$, $d\Theta=0$, and $N_{ex}=0.01$, 1, 20, and 200. We compare $\mathbf{B}(\vec{a},\vec{a}',\vec{b},\vec{b}')$ with $\mathbf{B}_{max}$, which has been considered previously (Fig. 1). The values of the pumping parameter N_{ex} are selected in such a way as to observe the most characteristic relations between considered quantities.

The configuration of the vectors for which $\mathbf{B}(\vec{a},\vec{a}',\vec{b},\vec{b}')$ becomes maximal depends upon the state of the system and it is actually impossible to guess which vectors should be chosen in order to optimize experimental results. We decide to take the vectors which are optimal for the singlet state

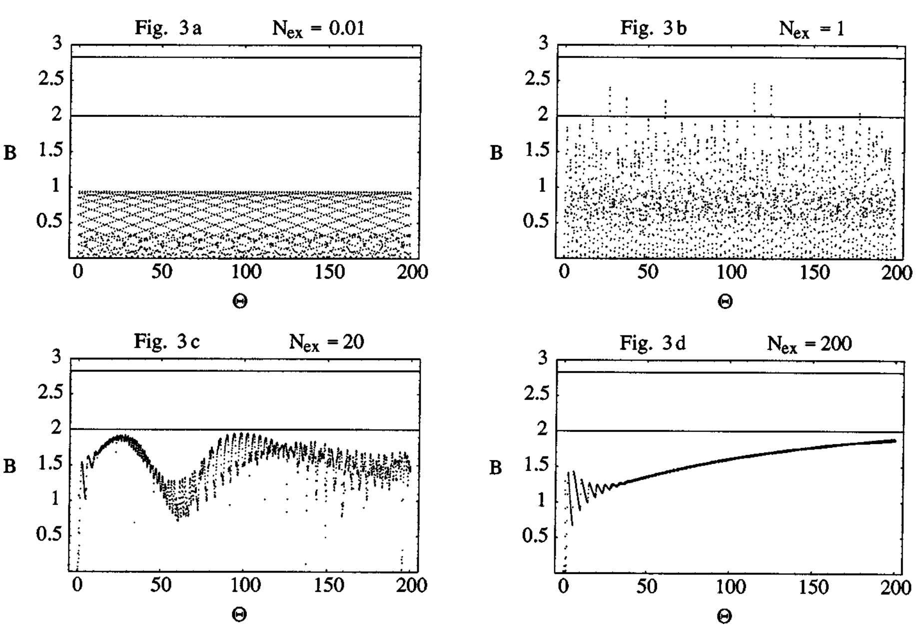

FIG. 3. The values of $\mathbf{B}(\vec{a},\vec{a}',\vec{b},\vec{b}')$ calculated for $r=1$ and $d\Theta=0$. Horizontal lines indicate the critical values: $\mathbf{B}=2$, which separates the classical and quantum regions; and $\mathbf{B}=2\sqrt{2}$, which is the upper limit of $\mathbf{B}$.

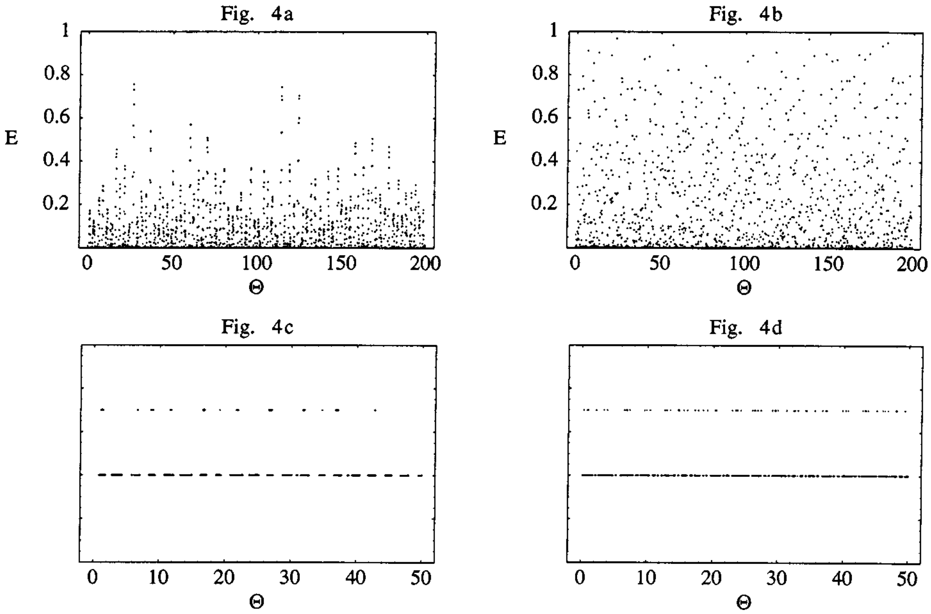

FIG. 4. In (a),(b) the entanglement of formation calculated for the values of parameters $r=1$, $d\Theta=0$, $N_{ex}=1$ and $r=1.5$, $d\Theta=0$, $N_{ex}=0.01$ is shown. In (c),(d) dashed lines indicate the values of Θ for which the states of atoms are entangled (lower line) and for which the Bell inequality is violated (upper line).

$|\Psi^-\rangle$. We see that for very weak pumping, values of $\mathbf{B}$ are much smaller than their counterparts corresponding to $\mathbf{B}_{max}$, and they do not even approach them. The situation changes when the pumping parameter increases, and for $N_{ex}=1$ the values of $\mathbf{B}(\vec{a},\vec{a}',\vec{b},\vec{b}')$ are closer to $\mathbf{B}_{max}$. They exceed the border value 2 for some values of Θ. The values of the N_{ex} belonging to an interval [1–10] seem to be optimal because already for $N_{ex}=20$ the values of $\mathbf{B}(\vec{a},\vec{a}',\vec{b},\vec{b}')$ are smaller and they do not approach the quantum limit. $\mathbf{B}(\vec{a},\vec{a}',\vec{b},\vec{b}')$ was investigated also for $r>1$, but we do not present appropriate pictures. We just notice that the relation between comparing quantities changes slightly when the ratio of the two interaction times is greater than 1. The values of $\mathbf{B}(\vec{a},\vec{a}',\vec{b},\vec{b}')$ are, as previously, smaller than values of $\mathbf{B}_{max}$, but the separation between values of these two quantities, distinctly seen for $r=1$, does not appear in this case. Generally, the values of $\mathbf{B}(\vec{a},\vec{a}',\vec{b},\vec{b}')$ and $\mathbf{B}_{max}$ are relatively similar for $r>1$, except for the situation when $N_{ex}=200$. In this case, $\mathbf{B}(\vec{a},\vec{a}',\vec{b},\vec{b}')<\mathbf{B}_{max}$ for almost all considered values of the parameter Θ.

IV. ENTANGLEMENT OF FORMATION

In the preceding section we investigated the quantum correlations of atoms interacting with the field in the two-atom micromaser analyzing the violation of the Bell inequality. For some values of the control parameters the Bell inequality is violated, but for others it is fulfilled. The fulfillment of the Bell inequality does not mean, however, that the atoms are not correlated. Unfortunately, the violation of the Bell inequality, an effect that can be directly observed in the experiments, is not completely satisfactory as the measure of the quantum nonlocality [22]: it is not as sensitive as other measures of the entanglement. There are other quantities that have been developed as measures of the quantum correlations. In the case of a pair of correlated two-level systems, one such measure is the entanglement of formation [23]. This quantity is much more sensitive to the quantum correlations than the Bell inequality. The entanglement of formation $\mathbf{E}(\hat{\rho})$ is defined in the following way [23]:

$$\mathbf{E}(\hat{\rho})=\mathcal{E}(C(\hat{\rho})), \tag{27}$$

where

$$C(\hat{\rho})=\max\{0,\lambda_1-\lambda_2-\lambda_3-\lambda_4\}, \tag{28}$$

$$\mathcal{E}(y)=h\left(\frac{1+\sqrt{1-y^2}}{2}\right), \tag{29}$$

and $h(x)=-x\log_2(x)-(1-x)\log_2(1-x)$. In these expressions, λ_i's are the square roots of eigenvalues, in decreasing order, of the non-Hermitian matrix $\hat{\rho}\hat{\tilde{\rho}}$, where $\hat{\tilde{\rho}}=(\hat{\sigma}_y\otimes\hat{\sigma}_y)\hat{\rho}^*(\hat{\sigma}_y\otimes\hat{\sigma}_y)$. We check whether and in how many cases the quantum correlated states of the atoms are omitted in the analysis when the Bell inequality is used. We calculate the entanglement of formation of the states of the atoms leaving the cavity of the micromaser. We choose for a comparison the most important cases when the violation of the Bell inequality is the largest. In Figs. 4(a) and 4(b), we show the entanglement of formation calculated for two sets of parameters $r=1$, $d\Theta=0$, $N_{ex}=1$ and $r=1.5$, $d\Theta=0$, $N_{ex}=0.01$.

We see that actually almost all pairs of the atoms leave the micromaser's cavity in entangled states. The degree of the entanglement of the atoms strongly depends upon the values of the interaction time Θ. It seems that in the former case [Fig. 4(a)], well-outlined peaks are present. In the latter case [Fig. 4(b)], the Θ dependence seems to be more irregular, but now the entanglement is much larger for almost all values of Θ and approaches, for some Θ, the maximal value equal to 1. The atoms are entangled for almost all values of Θ. This is even more evident when we look at Figs. 4(c) and

4(d), where we compare regions of the parameter Θ for which the atoms are in the entangled states (lower line), and for which the atoms can violate the Bell inequality, i.e., the values of $\mathbf{B}_{\mathrm{max}}$ are greater than 2 (upper line). We see that the atoms in the pairs are quantum correlated much more often than can be detected in the experiments testing the violation of the Bell inequality. The micromaser turns out to be an effective source of the entangled pairs of atoms. We think that the micromaser can produce equally well highly entangled multiatomic systems, when more than two atoms at the same time interact with the electromagnetic field in the cavity.

V. CONCLUSIONS

In this paper, we have investigated properties of the two-atom micromaser, pumped by the Poissonian beam of the atomic pairs. We have shown that the atoms, after interaction with the electromagnetic field, leave the micromaser's cavity in the quantum correlated states violating the Bell inequality. We have analyzed the violation of the Bell inequality using two quantities: the maximal value $\mathbf{B}_{\mathrm{max}}$ and vector-dependent $\mathbf{B}(\vec{a},\vec{a}\,',\vec{b},\vec{b}\,')$ depending upon experimental setup. We have investigated the degree of violation of the Bell inequality for the different values of the micromaser parameters. We have shown that the Bell inequality is violated in many different regimes of work of the micromaser. In particular, we have found that in some cases the upper limit of $\mathbf{B}$ is attainable. We have checked also when the values of $\mathbf{B}(\vec{a},\vec{a}\,',\vec{b},\vec{b}\,')$ approach the values of $\mathbf{B}_{\mathrm{max}}$, i.e., when the given experimental configuration of the parameters can be considered as optimal and exploited in practice. Finally, we have considered the entanglement of the formation of the pairs of atoms that fly out from the cavity. It turns out that actually all pairs of atoms are entangled even if they do not violate the Bell inequality. The atoms are entangled in the micromaser very effectively. The entanglement approaches the maximal value, for some values of the parameters, and it is much greater than zero in wide ranges of the values of the pumping N_{ex} and the interaction time Θ.

ACKNOWLEDGMENT

We wish to thank Professor K. Rzążewski for fruitful discussions.

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