Electron scattering by trapped fermionic atoms

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Considering the Fermi gases of alkali-metal atoms that are trapped in a harmonic potential, we study theoretically the elastic and inelastic scattering of the electrons by the trapped Fermi atoms and present the corresponding differential cross sections. We also obtain the stopping power for the cases that the electronic state as well as the center-of-mass state are excited both separately and simultaneously. It is shown that the elastic scattering process is no longer coherent in contrast to the electron scattering by the atomic Bose-Einstein condensate (BEC). For the inelastic scattering process, on the other hand, the differential cross section is found to be proportional to the 2/3 power of the number of the trapped atoms. In particular, the trapped fermionic atoms display the effect of ''Fermi surface,'' that is, only the energy levels near the Fermi energy have dominant contributions to the scattering process. Moreover, it is found that the stopping power scales as the 7/6 power of the atomic number. These results are fundamentally different from those of the electron scattering by the atomic BEC, mainly due to the different statistics obeyed by the trapped atomic systems.

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I. INTRODUCTION

The recent experimental observations of Bose-Einstein condensation of trapped cold alkali-metal gases such as rubidium, sodium, lithium, and hydrogen [1] have stimulated numerous new areas in theoretical as well as experimental physics. This impressive progress not only opened the way to investigate several macroscopic quantum phenomena but also offered the chance to apply relevant techniques in various fields. In particular, the study of the trapped ultracold fermions of alkali-metal atomic gases with one spin state and two spin states has become an interesting topic [2,3]. The relevant investigations on the optical properties [4–6], statistical properties [7–9], collective excitations [10], superfluid phase transition [11,12], and other properties [13] of trapped fermions have been undertaken. Like the trapped bosonic atoms, the trapped Fermi gases of alkali-metal atoms provide a very intriguing path to study several basic physical problems such as the phase separation, phase transition, atomic optics, and some statistical topics in condensed matter physics.

It is well known that the bosons and fermions are subject to the corresponding Bose-Einstein and Fermi-Dirac statistics, which state that all the bosons can occupy a single quantum state when the temperature meets the condition of phase transition, however, fermions do not due to the Pauli's exclusion principle. Therefore a very low temperature for the trapped fermionic atoms, at least below the Fermi temperature T_F of the trapped atoms, is desired to investigate the bound quasiparticles and the superfluid phase transition in the fermionic system of alkali-metal atoms. To date, the experimental condition of quantum degeneracy has been obtained in the ^{40}K system [2], where the lowest temperature is about $0.2T_F$ [14]. The experimental realization of trapped fermionic atoms due to DeMarco and Jin in 1999 have now opened a new way of investigating the quantum degenerate gases. Needless to say, study of the trapped fermionic atoms is a challenging topic for both experimental and theoretical physics.

In this paper we consider the electron scattering by the fermionic alkali-metal atoms trapped in a three-dimensional (3D), spherically symmetric harmonic potential. As an effective method, the scattering techniques have been used to determine the structure as well as to extract relevant information of the object. In general, scattering refers to the scattering of light, electron, neutron, and atom. As pointed out by Javanainen [4] and Ruostekoski [5] in the off-resonant light scattering of the trapped atoms, the spectrum of the scattered light is closely related to the statistics of the trapped atoms, and the quantum degeneracy of the atoms or the atomic velocity distribution can be reflected in the scattered spectrum. Moreover, the statistical correlation and the collision interaction between the atoms play an important role in many problems. In the present work, the elastic and inelastic scattering processes of the electrons by the trapped fermionic atoms are studied. We obtain the differential cross sections and then calculate the stopping power for the cases when the state of center of mass (c.m.) and the internal state are excited both separately and simultaneously. It is shown that the differential cross sections for the elastic scattering process are essentially different from those of the trapped Bose-Einstein condensate (BEC), which displays the effect of ''Fermi surface'' and reflects the intrinsic difference between fermions and bosons. Interestingly, the experiment of electron scattering from atoms confined in a magnetic optical trap (MOT) have been undertaken in 1995 [15], thus using the trapped fermionic atoms as a target, from which the process of electron scattering can be studied.

The remainder of this article is organized as follows. In Sec. II the statistical property of the fermionic alkali-metal atoms trapped in a 3D spherical harmonic potential is outlined to obtain the equations satisfied by the wave function

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of a single particle in the Hartree-Fock approximation. In Sec. III, the scattering of an incident electron beam from the trapped atoms is introduced. In Sec. IV, the differential cross sections for the elastic and the inelastic scattering processes are calculated. In Sec. V, the stopping power of the electrons is obtained for the cases that the c.m. and the internal states are excited separately or simultaneously. In Sec. VI, some problems associated with experimental consideration are discussed and the main results of this paper are summarized.

II. HARTREE-FOCK DESCRIPTIONS OF TRAPPED FERMIONIC ATOMS

In this section we concentrate our attention on considering the statistical properties of the fermionic gases consisting of N alkali-metal atoms trapped in a 3D spherical harmonic potential. The Hamiltonian of this system can be written as

$$H=\sum_{\alpha}\int d\vec{r}\,\psi_{\alpha}^{\dagger}(\vec{r})\left[-\frac{\hbar^2}{2m_a}\nabla^2+\frac{1}{2}m_a\omega^2r^2\right]\psi_{\alpha}(\vec{r}) +\frac{1}{2}\sum_{\alpha,\alpha'}\int d\vec{r}d\vec{r}'\,\psi_{\alpha}^{\dagger}(\vec{r})\psi_{\alpha'}^{\dagger}(\vec{r}')V(\vec{r}-\vec{r}') \times\psi_{\alpha'}(\vec{r}')\psi_{\alpha}(\vec{r}), \tag{1}$$

where ω is the angular frequency of the trap, m_a is the atomic mass, the sum is over the possible spin states, and $V(\vec{r}-\vec{r}')$ is the interaction potential between the atoms at the positions $\vec{r}$ and $\vec{r}'$. Here $\psi_{\alpha}^{\dagger}(\vec{r})$ and $\psi_{\alpha}(\vec{r})$ are the field operators of atoms that create or annihilate an atom with the spin state α at a position $\vec{r}$ and satisfy the anticommutation relations,

$$\{\psi_{\alpha}(\vec{r}),\psi_{\alpha'}^{\dagger}(\vec{r}')\}=\delta(\vec{r}-\vec{r}')\,\delta_{\alpha,\alpha'},$$

$$\{\psi_{\alpha}(\vec{r}),\psi_{\alpha'}(\vec{r}')\}=\{\psi_{\alpha}^{\dagger}(\vec{r}),\psi_{\alpha'}^{\dagger}(\vec{r}')\}=0. \tag{2}$$

In this system the trapped fermionic atoms are just like the special ''artificial molecules,'' for which the trap can be considered as a ''nucleus,'' whereas the atoms play the role of the bound ''electrons.'' Without loss of generality, choosing the following Hartree-Fock state as the ground state of the N-particle system

$$\phi_{\mathrm{HF}}=\frac{1}{\sqrt{N!}}\sum_{\mathcal{P}}\nu_{\mathcal{P}}\hat{\mathcal{P}}[\phi_{1,\alpha}^{e_0}(\vec{r}_1)\phi_{2,\alpha}^{e_0}(\vec{r}_2)\cdots\phi_{n,\alpha}^{e_0}(\vec{r}_n) \times\phi_{1,\beta}^{e_0}(\vec{r}_{n+1})\phi_{2,\beta}^{e_0}(\vec{r}_{n+2})\cdots\phi_{m,\beta}^{e_0}(\vec{r}_N)], \tag{3}$$

where the sum is over all the atoms and all the permutations of indices $\mathcal{P}[p_1,p_2\cdots p_N]$, $\nu_{\mathcal{P}}$ is the corresponding parity of the permutation, $\phi_{i,\alpha}^{e_0}(r_i)$ denotes the ith single-particle wave function occupied by the ith atom having the spin state $|\alpha\rangle$ ($|\alpha\rangle$ and $|\beta\rangle$ denote the two spin states of an atom) and the internal ground state $|e_0\rangle$ labeled by e_0, and the relation $(n+m)=N$ holds true. This configuration of the ground state can be described by the well-known Slater determinant that guarantees the antisymmetry of the wave function of the trapped fermionic atoms.

Starting with this equation, for the case that all trapped atoms possess the same spin state, which will be considered in this paper, we get the equation satisfied by the wave function of a single particle,

$$\left\{\hat{h}_0+\sum_{i=1}^{N}[\hat{J}_i(\vec{r})-\hat{K}_i(\vec{r})]\right\}\phi_i^{e_0}(r)=\varepsilon_i\phi_i^{e_0}(\vec{r}), \tag{4}$$

where $\hat{h}_0=-(\hbar^2/2m_a)\nabla^2+\frac{1}{2}m_a\omega^2r^2$, and the operators $\hat{J}_i(\vec{r})$ and $\hat{K}_i(\vec{r})$ are defined as

$$\hat{K}_i(\vec{r})\phi_l^{e_0}(\vec{r})=\int d\vec{r}'[\phi_i^{e_0}(\vec{r}')]^*V(\vec{r}-\vec{r}')\phi_l^{e_0}(\vec{r}')\phi_i^{e_0}(\vec{r}) \tag{5}$$

$$\hat{J}_i(\vec{r})=\int d\vec{r}'\,V(\vec{r}-\vec{r}')|\phi_i^{e_0}(\vec{r}')|^2. \tag{6}$$

In Eqs. (4)–(6), the spin index is omitted. (Here, for simplicity, we describe the detailed procedure in the Appendix, where the equations which allow the trapped atoms to possess two spin states are derived. The atoms with different spin states trapped in a MOT are an interesting topic as shown by DeMarco and Jin [2].)

For the low temperature system under study, the interaction potential between the atoms is usually represented by

$$V(\vec{r}-\vec{r}')=\frac{4\pi\hbar^2a_s}{m_a}\delta(\vec{r}-\vec{r}'), \tag{7}$$

with the negative s-wave scattering length a_s. Such an interaction makes the terms $\Sigma_{i=1}^{N}[\hat{J}_i(\vec{r})-\hat{K}_i(\vec{r})]\phi_i^{e_0}(r)$ exactly cancelled in the Hartree-Fock approximation and thus higher-order correlation interactions should be taken into account. When the interatomic interactions are weak, the ground state configuration of the trapped atoms can always be expressed as

$$|\phi_g\rangle=\frac{1}{\sqrt{N!}}\sum_{\mathcal{P}}\nu_{\mathcal{P}}\hat{\mathcal{P}}[\varphi_1^{e_0}(\vec{r}_1)\varphi_2^{e_0}(\vec{r}_2)\times\cdots\times\varphi_N^{e_0}(\vec{r}_N)], \tag{8}$$

where $\varphi_n^{e_0}(\vec{r})=\langle\vec{r}|\tilde{n}\rangle\otimes|e_0\rangle$ denotes that the internal state of an atom is in the internal ground state $|e_0\rangle$ and its c.m. state $\langle\vec{r}|\tilde{n}\rangle$ is in the $(n-1)$th state of the harmonic oscillator denoted by $\langle\vec{r}|n-1\rangle$. Here we use the symbol $\langle\vec{r}|\tilde{n}\rangle$ to stand for $\langle\vec{r}|n-1\rangle$, i.e., $\langle\vec{r}|\tilde{n}\rangle=\langle\vec{r}|n-1\rangle$, for convenience.

The nth energy level of a particle trapped in the potential $\frac{1}{2}m\omega^2r^2$ is known as $\varepsilon_n=(n+\frac{3}{2})\hbar\omega$ with $n=n_x+n_y+n_z$ and its degeneracy (atoms have same spin state) is

$$g_n=\frac{1}{2}(n+1)(n+2). \tag{9}$$

When the temperature of the system is so low that the energy levels below n^* are filled by atoms, where n^* is the lowest unoccupied energy level, i.e., the trapped atoms possess a closed shell structure, then n^* should be subject to the equation

$$\sum_{n=0}^{n^*-1} g_n = \frac{1}{6} n^*(n^*+1)(n^*+2) = N. \tag{10}$$

In this paper we consider that N is large enough that $n^* \cong (6N)^{1/3}$ and thus the corresponding Fermi energy ϵ_F becomes

$$\epsilon_F = (6N)^{1/3} \hbar \omega, \tag{11}$$

which provides an estimation of the energy shell for the ground state configuration, and hence the energy level $(n^* - 1)$ can be called the Fermi energy level of the trapped atoms.

III. SCATTERING OF AN ELECTRON BY TRAPPED FERMIONIC GASES

Now let us consider the scattering of an electron by the trapped fermionic gases of alkali-metal atoms with nuclear charge Ze. Suppose that an incident electron with mass m_e and momentum $\hbar \vec{k}_i$ (or, velocity $\vec{v}_i = \hbar \vec{k}_i / m_e$) is scattered by the trapped atoms into the state of momentum $\hbar \vec{k}_f$. The interaction between the incident and the trapped atoms is then described by $H_{int} = \Sigma_{i=1}^{N} V_i$, where

$$V_i = -\frac{Ze^2}{|\vec{r} - \vec{X}_i|} + \sum_{j=1}^{Z} \frac{Ze^2}{|\vec{r} - \vec{x}_{ij}|}. \tag{12}$$

Here $\vec{X}_i$ is the position of the ith nucleus and $\vec{x}_{ij}$ is the position of the jth electron bounded by the atom. Note that the spin-orbit and spin-spin interactions are ignored.

Under these interactions, the incident electron will be scattered by the trapped atoms and the trapped atoms either remain in the ground state configuration or become excited to one of its excited states. The former case corresponds to the elastic scattering process whereas the latter to inelastic scattering. For the latter, in particular, the atomic internal state and the c.m. state of an atom can be excited either separately or simultaneously. The initial and final states of the electron-atom system can be written, respectively, as

$$|\Phi_i\rangle = |\vec{k}_i\rangle \otimes |\phi_g\rangle \tag{13}$$

and

$$|\Phi_f\rangle = |\vec{k}_f\rangle \otimes |\phi_f\rangle, \tag{14}$$

where $\vec{k}_i = L^{-(3/2)} \exp(\imath \vec{k}_i \cdot \vec{r})$ and $\vec{k}_f = L^{-(3/2)} \exp(\imath \vec{k}_f \cdot \vec{r})$ are the normalized initial and final states of the incident electron, and $|\phi_g\rangle$ and $|\phi_f\rangle$ are the initial and final states of the trapped atoms before and after scattering. The elastic scattering process, on the other hand, corresponds to the condition that $k_f = k_i$ and $|\phi_f\rangle = |\phi_g\rangle$, where $k_f = |\vec{k}_f|$ and $k_i = |\vec{k}_i|$. The inelastic scattering corresponds to $|\phi_f\rangle \neq |\phi_g\rangle$ and $k_i \neq |\vec{k}_i|$, which means that the incident electron transfers part of its kinetic energy to the trapped atoms.

For simplicity we consider only the case of single-particle excitation. Then for the inelastic scattering, the final state can be chosen as

$$|\phi_f\rangle = \frac{1}{\sqrt{N}} \sum_{l=1}^{N} \left\{ \frac{1}{\sqrt{N!}} \sum_{\mathcal{P}} \nu_{\mathcal{P}} \hat{\mathcal{P}} [\varphi_1^{e_0}(\vec{r}_1) \varphi_2^{e_0}(\vec{r}_2) \times \cdots \times \varphi_{l-1}^{e_0}(\vec{r}_{l-1}) \varphi_{l+1}^{e_0}(\vec{r}_{l+1}) \cdots \varphi_N^{e_0}(\vec{r}_N) \varphi_n^{e_k}(\vec{r}_l)] \right\}, \tag{15}$$

where $\varphi_n^{e_k}(\vec{r}_l)$ denotes that the lth atom is in the nth $(n \geqslant n^*)$ c.m. state and the atomic internal state is in the state $|e_k\rangle$ labeled by e_k. This stands for the excitation transition of the lth atom from the state $\varphi_l^{e_0}(\vec{r}_l)$ to the state $\varphi_n^{e_k}(\vec{r}_l)$ due to the interaction. In fact, atoms in any energy level below the Fermi energy have the same possibility of being excited by the incident electron, which is reflected by the permutation operations and the factor $(1/\sqrt{N}) \Sigma_{l=1}^{N} \{\cdots\}$. Obviously, the c.m. state and the internal electric state of the trapped atom can be excited either separately or simultaneously, and the case of $n = l$ and $e_k = e_0$ just corresponds to the initial state given by Eq. (8).

Starting with the standard expression of the differential cross section, we find

$$\frac{d\sigma}{d\Omega} = \left(\frac{m_e L}{2\pi\hbar^2} \right)^2 \frac{k_f}{k_i} |T_{if}|^2, \tag{16}$$

where T_{if} describes the transition between the initial and the final states represented by operator T:

$$T = \sum_i V_i + \sum_{i,j} V_i G_0 V_j + \sum_{i,j,k} V_i G_0 V_j G_0 V_k + \cdots, \tag{17}$$

with $G_0(z) = 1/(z - H_0)$, where H_0 is the unperturbed Hamiltonian of the incident electron and the trapped atoms.

From the expression of the operator T, it can be found that the electron scattering contains direct scattering as well as multiple scattering processes. As a matter of fact, it is possible that the scattering process leads to the exchange of the incident electron with the bound electrons or the rearrangement of the atomic electrons. However, rigorous calculation of the transition matrix elements is a nontrivial task and therefore use of approximate methods is necessary. In this paper we discuss the fast electron scattering and the velocity of the incident electron v_i is considered to be large compared to that of the atomic bound electrons (typically, the velocity of a bound electron is of the order of $c/137$ with the light velocity c). For the slow electron scattering, on the other hand, the system may make many successive transitions and

thus the exchange effect and the rearrangement of atomic electrons should be taken into account.

In the high energy limit, however, the exchange and the rearrangement of the atomic electrons can be ignored due to the little overlap between the relatively fast incident electrons and the atomic electrons in the momentum space. Consequently the first Born approximation can be employed to discuss relevant problems. In fact, the main requirement for the validity of the Born approximation is $Ze^2/\hbar v_i \ll 1$ and thus in the high energy regime the Born approximation provides satisfactory results. Therefore we obtain

$$T_{if}=\frac{1}{L^3}\int d\vec{r}\,\exp(\imath\vec{q}\cdot\vec{r})\langle\Phi_f|\sum_{i=1}^{N} V_i|\Phi_i\rangle, \tag{18}$$

where $\vec{q}=\vec{k}_i-\vec{k}_f$ denotes the change of wave vector of the incident electron in the scattering process. Substituting $|\Phi_i\rangle$ in Eq. (13) and $|\Phi_f\rangle$ in Eq. (14) into Eq. (18) and integrating over $\vec{r}$, we find

$$T_{if}=\frac{4\pi e^2}{q^2L^3}\langle\Phi_f|\sum_{i=1}^{N}\left[\sum_{j=1}^{Z} e^{\imath\vec{q}\cdot\vec{x}_{ij}}-Ze^{\imath\vec{q}\cdot\vec{X}_i}\right]|\Phi_i\rangle. \tag{19}$$

Making use of the coordinate transformation between the relative motion of electrons and the c.m. motion for the ith atom, we obtain

$$\vec{r}_i=\vec{X}_i+\sum_{j=1}^{Z}\frac{m_e\vec{x}_{ij}^{e}}{m_a}, \tag{20}$$

where $\vec{r}_i$ is the c.m. position of the ith atom and $\vec{x}_{ij}^{e}=(\vec{x}_{ij}-\vec{X}_i)$ is the relative position of the jth electron.

Note that alkali-metal atoms are all hydrogenlike and they have only one valence electron outside the closed shell, and so their inner rare-gas structure is spherically symmetric. Then in the summation of Eq. (20), only the position of the valence electron can be retained. In addition, because the mass of electrons is much less than that of atoms, the retained term related to the position of the valence electron in Eq. (20) becomes negligibly small. So $\vec{r}_i\cong\vec{X}_i$ and the term $[\Sigma_{j=1}^{Z}\exp(\imath\vec{q}\cdot\vec{x}_{ij})-Z\exp(\imath\vec{q}\cdot\vec{X}_i)]$ in Eq. (19) becomes $\exp(\imath\vec{q}\cdot\vec{X}_i)[\Sigma_{j=1}^{Z}\exp(\imath\vec{q}\cdot\vec{x}_{ij}^{e})-Z]$.

As a result, Eq. (19) can be rewritten as

$$T_{if}=\frac{4\pi e^2}{q^2L^3}\langle\Phi_f|\sum_{i=1}^{N}\left[e^{\imath\vec{q}\cdot\vec{r}_i}\left(\sum_{j=1}^{Z} e^{\imath\vec{q}\cdot\vec{x}_{ij}^{e}}-Z\right)\right]|\Phi_i\rangle. \tag{21}$$

In Eq. (21), the position of electrons and atomic c.m. are decoupled, which is the starting point of calculating the differential cross sections for the elastic and the inelastic scattering processes.

IV. DIFFERENTIAL SCATTERING CROSS SECTIONS AND FERMI SURFACE EFFECT

The differential scattering cross section is an important physical quantity to describe the scattering characteristics. In this section we will present the corresponding expressions of the differential cross sections for the elastic as well as the inelastic electron scattering by the trapped fermionic atoms.

For the elastic scattering process, no energy is transferred from the incident electron to the trapped atoms. Thus all the atoms remain in the ground state configuration and $k_f=k_i$, this leads to the equation

$$T_{if}=\frac{4\pi e^2 Z}{q^2L^3}\sum_{l=1}^{N}\langle\tilde{l}|\exp(\imath\vec{q}\cdot\vec{r})|\tilde{l}\rangle[F_0(\vec{q})-1], \tag{22}$$

where $F_k(\vec{q})$ is the form factor of the alkali-metal atoms defined by

$$F_k(\vec{q})=\frac{1}{Z}\langle e_k|\sum_{j=1}^{Z}\exp(\imath\vec{q}\cdot\vec{x}_j^{e})|e_0\rangle, \tag{23}$$

with $|e_0\rangle$ and $|e_k\rangle$ being the configurations of the ground state and the kth excited state of the bounded electrons, respectively. For the inelastic scattering process, on the other hand, the atomic internal state and the c.m. state can be excited separately or simultaneously and from Eq. (21) we find

$$T_{if}=\frac{4\pi e^2 Z}{q^2L^3\sqrt{N}}\sum_{l=1}^{N}\langle\tilde{n}|\exp(\imath\vec{q}\cdot\vec{r})|\tilde{l}\rangle F_k(\vec{q})$$

$$(n\geqslant n^{*}\ \text{ or }\ k\neq 0). \tag{24}$$

Substituting Eqs. (22) and (24) into Eq. (16), the differential cross sections for the elastic and the inelastic scattering processes can be obtained immediately,

$$\frac{d\sigma_{\mathrm{el}}}{d\Omega}=\frac{4Z^2a_0^2}{(qa_0)^4}\left|\sum_{l=1}^{N}\langle\tilde{l}|\exp(\imath\vec{q}\cdot\vec{r})|\tilde{l}\rangle\right|^2|F_0(\vec{q})-1|^2, \tag{25}$$

$$\frac{d\sigma_{\mathrm{inel}}}{d\Omega}=\frac{4Z^2a_0^2k_f}{N(qa_0)^4k_i}\left|\sum_{l=1}^{N}\langle\tilde{n}|\exp(\imath\vec{q}\cdot\vec{r})|\tilde{l}\rangle\right|^2|F_k(\vec{q})|^2$$

$$(n\geqslant n^{*}\ \text{ or }\ k\neq 0), \tag{26}$$

where $a_0=\hbar^2/m_ee^2$ is the Bohr radius. It is obvious that the differential cross sections for the elastic and inelastic scattering processes are closely related to the matrix elements $\mathcal{A}_{nm}=\langle n|\exp(\imath\vec{q}\cdot\vec{r})|m\rangle$ and therefore we must execute further calculations in order to describe the scattering in detail. Below the theory of the displaced number state [16] will be used to carry out these results.

Here, for this purpose, we outline some properties of the displaced number states by considering an example of a one-dimensional harmonic oscillator with the Hamiltonian $\overline{H}=\hbar\omega(a^{\dagger}a+\frac{1}{2})$, where $a^{\dagger}$ and a are the creation and annihilation operators. The displaced number state associated with this harmonic oscillator is

$$|\eta,m\rangle=\hat{\mathcal{D}}(\eta)|m\rangle, \tag{27}$$

where $|m\rangle$ is the mth eigenstate of $\bar{H}$ and $\hat{\mathcal{D}}(\eta)$ is the displaced operator, which has the form

$$\hat{\mathcal{D}}(\eta)=\exp(\eta a^{\dagger}-\eta^{*}a) \tag{28}$$

and satisfies the relations

$$\hat{\mathcal{D}}(\eta)a\hat{\mathcal{D}}^{\dagger}(\eta)=a-\eta, \quad \hat{\mathcal{D}}(\eta)\hat{\mathcal{D}}^{\dagger}(\eta)=1,$$

$$\hat{\mathcal{D}}(\eta)a^{\dagger}\hat{\mathcal{D}}^{\dagger}(\eta)=a^{\dagger}-\eta^{*}, \quad \hat{\mathcal{D}}^{\dagger}(\eta)\hat{\mathcal{D}}(\eta)=1. \tag{29}$$

Here the quantity η is called the translational parameter that may be real or complex. In particular, when $\eta=0$, the displaced number state reduces to the well-known coherent state. As an interesting application of the theory of the displaced number state [16], the following matrix elements

$$\langle n|\hat{\mathcal{D}}(\eta)|m\rangle=\left(\frac{m!}{n!}\right)^{1/2}\eta^{n-m}\exp\left(-\frac{1}{2}|\eta|^{2}\right)\mathcal{L}_{m}^{n-m}(|\eta|^{2}) \tag{30}$$

can be obtained when the condition $n\geqslant m$ holds, where $\mathcal{L}_{m}^{n-m}$ is the associated Lagurerre polynomial with the argument $|\eta|^{2}$.

In order to evaluate the matrix element $\langle n|\exp(\imath\vec{q}\cdot\vec{r})|m\rangle$, we rewrite it in the following form:

$$\mathcal{A}_{nm}=\prod_{j}\langle n_{j}|e^{\imath q_{j}r_{j}}|m_{j}\rangle \quad (j=1,2,3), \tag{31}$$

where the index $j(j=1,2,3)$ corresponds to the x, y, and z component of the corresponding physical quantity. For the 3D spherically symmetric harmonic potential, we have

$$r_{j}=(a_{j}^{\dagger}+a_{j})\lambda, \quad \lambda=\sqrt{\frac{\hbar}{2m_{a}\omega}},$$

$$\exp(\imath q_{j}r_{j})=\hat{\mathcal{D}}_{j}(\eta_{j}), \quad \eta_{j}=\imath q_{j}\lambda. \tag{32}$$

Combining Eqs. (30)–(32), we then find

$$\mathcal{A}_{nm}=\prod_{j}\left(\frac{m_{j}!}{n_{j}!}\right)^{1/2}\eta_{j}^{n_{j}-m_{j}}e^{-(1/2)|\eta_{j}|^{2}}\mathcal{L}_{m_{j}}^{n_{j}-m_{j}}(|\eta_{j}|^{2}). \tag{33}$$

For convenience, we define the following quantities

$$\mathcal{B}_{\mathrm{el}}=\left|\sum_{l=1}^{N}\langle\tilde{l}|e^{\imath\vec{q}\cdot\vec{r}}|\tilde{l}\rangle\right|^{2}=\left|\sum_{m=0}^{n^{*}-1}g_{m}\mathcal{A}_{mm}\right|^{2},$$

$$\mathcal{B}_{\mathrm{inel}}=\left|\sum_{l=1}^{N}\langle\tilde{n}|e^{\imath\vec{q}\cdot\vec{r}}|\tilde{l}\rangle\right|^{2}=\left|\sum_{m=0}^{n^{*}-1}g_{m}\mathcal{A}_{nm}\right|^{2}, \tag{34}$$

where the degeneracy of the trapped energy levels in the 3D potential has been taken into account. Thus the differential cross sections for the elastic and the inelastic scattering processes can be explicitly given by

$$\frac{d\sigma_{\mathrm{el}}}{d\Omega}=\frac{4Z^{2}a_{0}^{2}}{(qa_{0})^{4}}|F_{0}(\vec{q})-1|^{2}\mathcal{B}_{\mathrm{el}} \tag{35}$$

and

$$\frac{d\sigma_{\mathrm{inel}}}{d\Omega}=\frac{4Z^{2}a_{0}^{2}k_{f}}{N(qa_{0})^{4}k_{i}}|F_{k}(\vec{q})|^{2}\mathcal{B}_{\mathrm{inel}} \quad (n\geqslant n^{*} \text{ or } k\neq 0), \tag{36}$$

where the factors $\mathcal{B}_{\mathrm{el}}$ and $\mathcal{B}_{\mathrm{inel}}$ can be explicitly expressed as

$$\mathcal{B}_{\mathrm{el}}=\left|\sum_{m=0}^{n^{*}-1}g_{m}\prod_{j}\mathcal{L}_{m_{j}}^{0}(|\eta_{j}|^{2})\right|^{2}\exp(-q^{2}\lambda^{2}), \tag{37}$$

$$\mathcal{B}_{\mathrm{inel}}=\left|\sum_{m=0}^{n^{*}-1}g_{m}\prod_{j}\left(\frac{m_{j}!}{n_{j}!}\right)^{1/2}\eta_{j}^{n_{j}-m_{j}}\mathcal{L}_{m_{j}}^{n_{j}-m_{j}}(|\eta_{j}|^{2})\right|^{2}\times\exp(-q^{2}\lambda^{2}). \tag{38}$$

Let us suppose that η_{j} and $|\eta_{j}|^{2}$ are small quantities. Then, making use of the expression of the associated Laguerre polynomial with the argument x,

$$\mathcal{L}_{n}^{k}(x)=\sum_{i=0}^{n}\frac{(n+k)!}{(n-i)!(k+i)!i!}(-x)^{i}, \tag{39}$$

and ignoring high-order small quantities, we have

$$\mathcal{B}_{\mathrm{el}}\cong\left|N-\sum_{m=0}^{n^{*}-1}g_{m}\left(\sum_{j}m_{j}|\eta_{j}|^{2}\right)\right|^{2}\exp(-q^{2}\lambda^{2}). \tag{40}$$

Without loss of generality, setting $q_{j}^{2}=\frac{1}{3}q^{2}$, we find $\mathcal{B}_{\mathrm{el}}\cong N^{2}|1-(q^{2}\lambda^{2}/3N)\sum_{m=0}^{n^{*}-1}mg_{m}|^{2}\exp(-q^{2}\lambda^{2})$, which can be further reduced to $N^{2}|1-(q^{2}\lambda^{2}n^{*}[(n^{*})^{2}-1](n^{*}+2)/24N)|^{2}$. Recalling that $n^{*}\cong(6N)^{1/3}$, we can finally obtain

$$\mathcal{B}_{\mathrm{el}}\cong N^{2}|1-\tfrac{1}{4}(6N)^{1/3}q^{2}\lambda^{2}|^{2}\exp(-q^{2}\lambda^{2}). \tag{41}$$

In the following we will derive the relationship between the factor $\mathcal{B}_{\mathrm{inel}}$ and the number of trapped atoms. Similarly, discussion will be made assuming that η_{j} and $|\eta_{j}|^{2}$ are small. From Eq. (38), it is obvious that for small η_{j} and $|\eta_{j}|^{2}$, as the difference between the quantum numbers n_{j} and m_{j} is increased, the magnitude of the term becomes small. After a direct calculation, neglecting the high order terms, one obtains

$$\mathcal{A}_{nm}\cong e^{-(q^{2}\lambda^{2}/2)}\left(\frac{\imath q\lambda}{\sqrt{3}}\right)^{n-m}\left(1-\frac{mq^{2}\lambda^{2}}{6}\right)\prod_{j}\left(\frac{n_{j}!}{m_{j}!}\right)^{1/2}, \tag{42}$$

where we have used $q_{j}^{2}=\frac{1}{3}q^{2}$. From this equation, we find that the main contributions come from the energy levels near the level n due to the factor $(\imath q\lambda/\sqrt{3})^{n-m}$. The maxima of $\mathcal{A}_{nm}$ can therefore be obtained by arranging the terms in such a way that the difference of quantum number is $(n-m)$ at

one direction and zero at another two directions, the corresponding maxima for given n and m can be expressed as

$$\mathcal{A}_{nm}^{\max}=e^{-(q^2\lambda^2/2)}\sqrt{\frac{n!}{m!}}\left(\frac{\imath q\lambda}{\sqrt{3}}\right)^{n-m}\left(1-\frac{mq^2\lambda^2}{6}\right). \quad (43)$$

This result shows that for a given energy level n, when the other energy levels lie nearby, the corresponding contributions to $\mathcal{A}_{nm}^{\max}$ are increased.

Note that $n\geqslant n^*$, $m\leqslant(n^*-1)$, and the Fermi energy level (n^*-1) is the nearest energy level to n. Then n should take the value of n^* and the energy level $m=n^*-1$ offers the crucial contribution, which is the leading term and also the most important term in the factor $\mathcal{B}_{\rm inel}$. This indicates that atoms distributed in the Fermi energy level (n^*-1) or nearby play a key role in the inelastic scattering process. This is just like the effect of the ''Fermi surface'' in the condensed matter physics, in which it has been manifested that only the electrons near the ''Fermi surface'' are closely related to the electric, optical, and transport properties of a solid. In fact, the system consisting of trapped atoms can be thought of as an ''artificial molecule'' from the viewpoint of the orbital theory in chemistry, which states that the highest occupied orbit and the lowest unoccupied orbit play a key role in chemical reaction and determine the chemical properties of matter. In a word, this is the intrinsic nature of fermions and the trapped fermionic atoms again display this kind of feature in the mesoscopic level.

Based on the considerations mentioned above, we have

$$\begin{aligned}\mathcal{B}_{\rm inel}&\cong\frac{q^2\lambda^2}{12}\left|(n^*)^{5/2}\left(1-\frac{1}{6}n^*q^2\lambda^2\right)\right|^2\exp(-q^2\lambda^2)\\&=\frac{N(6N)^{2/3}q^2\lambda^2}{2}\left[1-\frac{1}{6}q^2\lambda^2(6N)^{1/3}\right]^2\exp(-q^2\lambda^2).\end{aligned} \quad (44)$$

Substituting this equation into Eq. (36) yields

$$\begin{aligned}\frac{d\sigma_{\rm inel}}{d\Omega}&=\frac{2Z^2a_0^2k_f}{(qa_0)^4k_i}|F_k(\vec{q})|^2q^2\lambda^2(6N)^{2/3}\exp(-q^2\lambda^2)\\&\quad\times\left[1-\frac{1}{6}q^2\lambda^2(6N)^{1/3}\right]^2,\end{aligned} \quad (45)$$

which is proportional to the $\frac{2}{3}$ power of the number of the trapped atoms. Comparing the differential cross sections of the elastic and inelastic scattering process of trapped fermionic atoms with that of atomic BEC [17], we find

$$\frac{d\sigma_{\rm el}/d\Omega|_{\rm F}}{d\sigma_{\rm el}/d\Omega|_{\rm B}}=\left[1-\frac{1}{4}(6N)^{1/3}q^2\lambda^2\right]^2, \quad (46)$$

$$\frac{d\sigma_{\rm inel}/d\Omega|_{\rm F}}{d\sigma_{\rm inel}/d\Omega|_{\rm B}}=\frac{6^{2/3}}{12}\left[1-\frac{1}{6}(6N)^{1/3}q^2\lambda^2\right]^2N^{-(1/3)}, \quad (47)$$

where all the conditions such as the atomic number and the initial and final states of the incident electrons are set to be identical.

It can be observed that both the elastic and the inelastic differential cross sections of trapped fermionic atoms are smaller than those of atomic BEC. The elastic scattering of electrons by the atomic BEC is a kind of coherent scattering because the differential cross section is proportional to the square of atomic number N. However, for the electron scattering by the trapped fermionic atoms, the differential cross section is proportional to the factor $N^2|1-(6N)^{1/3}q^2\lambda^2/4|^2$, for which this kind of coherence is absent. For the inelastic scattering process, the ratio of the corresponding differential cross sections displays a scaling behavior of $N^{-(1/3)}$. This result arises due to the different statistics that the Fermi and the Bose gases obey. In the study of off-resonant light scattering by the trapped atoms, Javanainen [4] and Ruostekoski [5] have pointed out that the spectrum of the scattered light is closely related to the statistics of atoms. Indeed, this conclusion can be similarly recovered in the electron scattering by trapped atoms.

V. STOPPING POWER OF ELECTRON SCATTERING

In the inelastic scattering process, the kinetic energy of the incident electron is transferred to the trapped fermionic atoms and this leads to the excitation of the c.m. state or the atomic internal state. Hence the stopping power or the energy loss of the incident electron is an important quantity, which can be derived in terms of the differential cross section of the inelastic scattering given by Eq. (45). As is usually the case in scattering problems, we are interested in the energy loss of the incident electron along the travel path denoted by $-dE/dx$, which can be expressed as

$$-\frac{dE}{dx}=\rho_a\sum_{k,n}(E_{e_k,n}-E_{e_0,g})\int_{q_{\min}}^{q_{\max}}\frac{d\sigma_{\rm inel}}{dq}dq, \quad (48)$$

where ρ_a is the density of the trapped atoms, $q_{\min}$ and $q_{\max}$ are the corresponding integral limits, and $E_{e_0,g}$ and $E_{e_k,n}$ denote the energies of the initial state and the excited state, respectively.

Note that

$$q^2=|\vec{q}|^2=k_i^2+k_f^2-2k_ik_f\cos\theta, \quad (49)$$

with θ being the angle between $\vec{k}_i$ and $\vec{k}_f$. This then leads to

$$\frac{d\sigma_{\rm inel}}{dq}=\frac{2\pi q}{k_ik_f}\frac{d\sigma_{\rm inel}}{d\Omega}. \quad (50)$$

Therefore we obtain

$$-\frac{dE}{dx}=\frac{8\pi\rho_aZ^2}{N(k_ia_0)^2}\sum_{k,n}(E_{e_k,n}-E_{e_0,g})\int_{q_{\min}}^{q_{\max}}|F_k(\vec{q})|^2\mathcal{B}_{\rm inel}\frac{dq}{q^3}. \quad (51)$$

For the case that only the atomic internal state is excited, the energy difference $(E_{e_k,n}-E_{e_0,g})$ is equal to $(E_{e_k}-E_{e_0})$.

Moreover, the factor $\mathcal{B}_{\rm inel}$ becomes equal to $\mathcal{B}_{\rm el}$ since the c.m. state does not change. Thus the stopping power under this condition becomes

$$-\left.\frac{dE}{dx}\right|_{n=g}=\frac{4\pi\rho_a Ze^4}{Nm_e v_i^2}\int_{q_{\min}}^{q_{\max}}\mathcal{B}_{\rm el}\frac{dq}{q}. \tag{52}$$

In obtaining this equation, the well-known result

$$\sum_k (E_{e_k}-E_{e_0})|F_k(\vec{q})|^2=\frac{1}{Z}\frac{\hbar^2q^2}{2m_e}, \tag{53}$$

proposed by Gottfried and Bethe [18], has been used.

For the case that only the trapped energy level is excited, no change occurs in the atomic internal state and thus the total energy difference is the same as the energy difference between the c.m. states. Then from the result $\Sigma_n(E_n-E_g)|\Sigma_{l=1}^N\langle\tilde{n}|e^{i\vec{q}\cdot\vec{r}}|\tilde{l}\rangle|^2$, as stated in Sec. IV, we can only consider the Fermi energy level so that

$$\sum_n (E_n-E_g)\left|\sum_{l=1}^N\langle\tilde{n}|e^{i\vec{q}\cdot\vec{r}}|\tilde{l}\rangle\right|^2\cong\frac{N\hbar\omega(6N)^{2/3}q^2\lambda^2}{2}, \tag{54}$$

where the term $\exp(-q^2\lambda^2)$ has been expanded in a series form and the terms containing the high order of $q^2\lambda^2$ has been omitted. Therefore for the case that only the c.m. state is excited, we have

$$-\left.\frac{dE}{dx}\right|_{k=0}=\frac{2\pi\rho_a Z^2e^4}{m_e v_i^2}\frac{m_e}{m_a}(6N)^{2/3}\int_{q_{\min}}^{q_{\max}}|F_0(\vec{q})|^2\frac{dq}{q}. \tag{55}$$

For the case that the c.m. as well as the internal states are excited simultaneously, on the other hand, with the total energy difference $(E_{e_k,n}-E_{e_0,g})$ being decomposed as $(E_{e_k}-E_{e_0})+(E_n-E_g)$, we obtain

$$-\frac{dE}{dx}=\frac{2\pi\rho_a Ze^4(6N)^{2/3}}{m_e v_i^2}\left(1+\frac{Zm_e}{m_a}\right)\int_{q_{\min}}^{q_{\max}}e^{-q^2\lambda^2}\frac{dq}{q}. \tag{56}$$

In obtaining this we have used the following relations

$$\sum_n \mathcal{B}_{\rm inel}\cong\frac{N(6N)^{2/3}q^2\lambda^2}{2}\exp(-q^2\lambda^2) \tag{57}$$

and

$$\sum_k |F_k(\vec{q})|^2=1. \tag{58}$$

Note that Eq. (57) can be derived from Eq. (44) by considering the effect of ''Fermi surface'' mentioned above and with the terms containing high order of $q^2\lambda^2$ being omitted. Equation (58) is an identity because the interpretation of the left-hand side of this equation stands for the sum of probabilities that the ground internal state of an atom is excited to all possible excited states, which must be equal to 1.

Following the Bethe's treatment with respect to the integral limits, that is, $q_{\max}=2k_i$ and $q_{\min}=k_i-k_f$, one can expand the exponential function in Eq. (56) in the series form and then integrate over q to obtain

$$-\frac{dE}{dx}=\frac{2\pi\rho_a Ze^4(6N)^{2/3}}{m_e v_i^2}\left(1+\frac{Zm_e}{m_a}\right)\ln\left(\frac{2m_e v_i^2}{I}\right), \tag{59}$$

where I is a semiempirical parameter that is related to the average excitation energy and in this equation only the leading term is listed.

For the fermionic atoms trapped in the 3D potential discussed in this paper, the density ρ_a of atoms can be expressed as [7,20]

$$\rho_a=\frac{8}{\pi^2}\frac{N}{R_{\rm F}^3}\left(1-\frac{r^2}{R_{\rm F}^2}\right)^{3/2}, \tag{60}$$

where $R_{\rm F}=(48N)^{1/6}L$ with $L=\sqrt{\hbar/m_a\omega}$ is called the Fermi radius of the density distribution. Substituting the expression of atomic density into Eq. (59) yields

$$-\frac{dE}{dx}=\frac{4\sqrt{2}Ze^4 6^{1/6}}{\pi L^3 m_e v_i^2}\left(1-\frac{r^2}{R_{\rm F}^2}\right)^{3/2}N^{7/6}\left(1+\frac{Zm_e}{m_a}\right)\times\ln\left(\frac{2m_e v_i^2}{I}\right). \tag{61}$$

This shows that the dependence of the stopping power on the number of trapped atoms is $N^{7/6}$, which is proportional to the $\frac{7}{6}$ power of the atomic number. Note that for the trapped atomic BEC [17], the stopping power is proportional to the $\frac{7}{5}$ power of the atomic number. The different dependence of the stopping power between the trapped fermionic atoms and the atomic BEC on the atomic number is a characteristic property, which essentially originates from the different statistics that the atoms obey. It should be noted that some elaborate methods and techniques are necessary to probe the difference between components of $\frac{7}{5}$ and $\frac{7}{6}$ when one measures the stopping power since distinguishing them is not easy, and thus these components keep only theoretical interest.

VI. DISCUSSION AND SUMMARY

Having obtained several quantities related to the electron scattering by the trapped fermionic atoms, an important and direct problem arises: how to test these theoretical results through an experiment? As is well known, the current trapped potentials used to confine atoms are almost MOT, then one should consider how to effectively avoid the influence of the magnetic field on the incident electron beam when one tries to realize such an experiment.

In 1995, Schappe [15] and his co-workers first performed the electron scattering experiment using the ground-state Rb atoms trapped in a MOT as a target, they measured the total scattering cross section for the collision between electron and

Rb atoms with the incident energy in the range of 7–500 eV by controlling the time sequence. Since the atoms trapped in MOT are very sensitive to the atomic recoil, some atoms therefore would escape from the trap due to electron-atom collision when the incident electrons are applied on the system. In their excellent experiment, in order to ensure that the incident electrons are not distorted by the magnetic field, they first turned off the magnetic field and shifted the modulation frequency of trapping lasers, after a short time the electron beam is pulsed on the system. Second, the trap effectively remains off after the end of the electron beam for a long time (t_{off}) to allow some recoiling atoms due to the electron-atom collision to leave the trap. Third, the trap is turned on again for a time (t_{on}), which allows the remaining atoms to be recaptured, and finally, they measured the number of the trapped atoms and obtained some information.

After this experiment, this kind of technique has been used to study the electron-impact ionization using Rb atoms trapped in a MOT as a target [19]. Likewise, it is possible that through this method with a proper time sequence, the experiments of electron scattering from trapped bosonic and fermionic atoms can be performed, in particular for the dilute trapped atomic system in which the interparticle interactions are rather weak, and therefore some related topics can be investigated.

In conclusion, we have studied the electron scattering by the fermionic atoms trapped in a 3D spherically symmetric harmonic potential. The corresponding differential cross sections for the elastic and the inelastic scattering processes are obtained. Moreover the stopping power is calculated by considering the c.m. state and the atomic internal state is excited separately or simultaneously. Several interesting properties of the trapped fermionic atoms are noteworthy in comparison with the electron scattering by the atomic BEC. First, for the elastic scattering process, unlike the coherent property exhibited in the elastic scattering by the atomic BEC, such a coherence effect is absent for trapped fermionic atoms. Second, for the inelastic scattering process, the trapped fermionic atoms show the effect of ''Fermi surface'' like that in the condensed matter physics, that is, the Fermi energy level or the nearby energy levels have the most important contributions to the scattering process. In particular, the dependence of the differential cross section on the number of the trapped atoms scales as $N^{2/3}$, which is also due to the effect of ''Fermi surface'' of trapped atoms and hence the trapped fermionic atoms manifest the effect of ''Fermi surface'' in the mesoscopic level. Third, the stopping power of the electron scattering by the trapped fermionic atoms is found to be proportional to the $\frac{7}{6}$ power of the atomic number, which is obviously different from the $N^{7/5}$ power dependence of the stopping power for the electron scattering by the atomic BEC.

Note that in this paper, only the fast electron scattering process is studied, which allows the exchange effect and the rearrangement of the atomic electron to be neglected. In addition, the spin-spin and the spin-orbit interactions are not taken into account. However, these interactions may become non-negligible and they may lead to different influences on the scattering results at different velocity ranges of the incident electron. Some further works will be performed by considering the effect of these factors. Nevertheless, those results that are detailed here will be useful for experimental studies of the scattering properties of the trapped fermionic atoms.

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APPENDIX: EQUATIONS FOR TRAPPED FERMIONIC ATOMS WITH TWO SPIN STATES

Here we give the equations describing trapped fermionic atoms with two spin states in a MOT. With the use of Eqs. (1) and (3), the energy corresponding to the configuration of the ground state can be given by

$$E_{\mathrm{HF}}=\sum_{i=1}^{n}\left[H_{ii}^{\alpha}+\frac{1}{2}\sum_{l=1}^{n}(J_{il}^{\alpha\alpha}-K_{il}^{\alpha\alpha})+\sum_{l=1}^{m}J_{ik}^{\alpha\beta}\right]+\sum_{j=1}^{m}\left[H_{jj}^{\beta}+\frac{1}{2}\sum_{k=1}^{m}(J_{jk}^{\beta\beta}-K_{jk}^{\beta\beta})+\sum_{l=1}^{n}J_{jl}^{\beta\alpha}\right], \tag{A1}$$

where

$$H_{ii}^{\alpha}=\int d\vec{r}[\phi_{i,\alpha}^{e_0}(\vec{r})]^*\left(-\frac{\hbar^2\nabla^2}{2m_a}+\frac{1}{2}m_a\omega^2r^2\right)\phi_{i,\alpha}^{e_0}(\vec{r}),$$

$$H_{jj}^{\beta}=\int d\vec{r}[\phi_{j,\beta}^{e_0}(\vec{r})]^*\left(-\frac{\hbar^2\nabla^2}{2m_a}+\frac{1}{2}m_a\omega^2r^2\right)\phi_{j,\beta}^{e_0}(\vec{r}),$$

$$J_{il}^{\alpha\alpha}=\int d\vec{r}d\vec{r}'|\phi_{i,\alpha}^{e_0}(\vec{r})|^2V(\vec{r}-\vec{r}')|\phi_{l,\alpha}^{e_0}(\vec{r}')|^2,$$

$$J_{jk}^{\beta\beta}=\int d\vec{r}d\vec{r}'|\phi_{j,\beta}^{e_0}(\vec{r})|^2V(\vec{r}-\vec{r}')|\phi_{k,\beta}^{e_0}(\vec{r}')|^2, \tag{A2}$$

$$K_{il}^{\alpha\alpha}=\int d\vec{r}d\vec{r}'[\phi_{i,\alpha}^{e_0}(\vec{r})\phi_{l,\alpha}^{e_0}(\vec{r}')]^*\times V(\vec{r}-\vec{r}')\phi_{i,\alpha}^{e_0}(\vec{r}')\phi_{l,\alpha}^{e_0}(\vec{r}),$$

$$K_{jk}^{\beta\beta}=\int d\vec{r}d\vec{r}'[\phi_{j,\beta}^{e_0}(\vec{r})\phi_{k,\beta}^{e_0}(\vec{r}')]^*\times V(\vec{r}-\vec{r}')\phi_{j,\beta}^{e_0}(\vec{r}')\phi_{k,\beta}^{e_0}(\vec{r}),$$

$$J_{ik}^{\alpha\beta}=\int d\vec{r}d\vec{r}'|\phi_{i,\alpha}^{e_0}(\vec{r})|^2V(\vec{r}-\vec{r}')|\phi_{k,\beta}^{e_0}(\vec{r}')|^2,$$

$$J_{jl}^{\beta\alpha}=\int d\vec{r}d\vec{r}'|\phi_{j,\beta}^{e_0}(\vec{r})|^2V(\vec{r}-\vec{r}')|\phi_{l,\alpha}^{e_0}(\vec{r}')|^2.$$

Minimizing the energy given by Eq. (A1), one can obtain the equations satisfied by the wave function of a single atom [21] with different spin states

$$\hat{H}^{\alpha}_{\mathrm{HF}}\phi^{e_0}_{i,\alpha}(r)=\varepsilon^{\alpha}_i\phi^{e_0}_{i,\alpha}(\vec{r}),$$
$$\hat{H}^{\beta}_{\mathrm{HF}}\phi^{e_0}_{j,\beta}(r)=\varepsilon^{\beta}_j\phi^{e_0}_{j,\beta}(\vec{r}), \tag{A3}$$

where

$$\hat{H}^{\alpha}_{\mathrm{HF}}=\hat{h}_0+\sum_{i=1}^{n}[\hat{J}^{\alpha}_i(\vec{r})-\hat{K}^{\alpha}_i(\vec{r})]+\sum_{l=1}^{m}\hat{J}^{\beta}_i(\vec{r}),$$
$$\hat{H}^{\beta}_{\mathrm{HF}}=\hat{h}_0+\sum_{j=1}^{m}[\hat{J}^{\beta}_j(\vec{r})-\hat{K}^{\beta}_j(\vec{r})]+\sum_{i=1}^{n}\hat{J}^{\alpha}_i(\vec{r}), \tag{A4}$$
$$\hat{h}_0=-\frac{\hbar^2}{2m_a}\nabla^2+\frac{1}{2}m_a\omega^2r^2.$$

Here $\hat{H}^{\alpha}_{\mathrm{HF}}$ ($\hat{H}^{\beta}_{\mathrm{HF}}$) is the Hamiltonian of atoms with the spin state $|\alpha\rangle$ ($|\beta\rangle$) in the Hartree-Fock approximation. In obtaining Eq. (A3), we also have used the following definitions of the operators

$$\hat{K}^{\alpha}_i(\vec{r})\phi^{e_0}_{l,\alpha}(\vec{r})=\int d\vec{r}'[\phi^{e_0}_{i,\alpha}(\vec{r}')]^*V(\vec{r}-\vec{r}')\phi^{e_0}_{l,\alpha}(\vec{r}')\phi^{e_0}_{i,\alpha}(\vec{r}),$$
$$\hat{K}^{\beta}_j(\vec{r})\phi^{e_0}_{k,\beta}(\vec{r})=\int d\vec{r}'[\phi^{e_0}_{j,\beta}(\vec{r}')]^*V(\vec{r}-\vec{r}')\phi^{e_0}_{k,\beta}(\vec{r}')\phi^{e_0}_{j,\beta}(\vec{r}),$$
$$\hat{J}^{\alpha}_i(\vec{r})=\int d\vec{r}'V(\vec{r}-\vec{r}')|\phi^{e_0}_{i,\alpha}(\vec{r}')|^2,$$
$$\hat{J}^{\beta}_j(\vec{r})=\int d\vec{r}'V(\vec{r}-\vec{r}')|\phi^{e_0}_{j,\beta}(\vec{r}')|^2, \tag{A5}$$

where ε^{α}_i and ε^{β}_j are the single particle energies given, respectively, by

$$\varepsilon^{\alpha}_i=\sum_{i=1}^{n}\left[H^{\alpha}_{ii}+\sum_{l=1}^{n}(J^{\alpha\alpha}_{il}-K^{\alpha\alpha}_{il})+\sum_{l=1}^{m}J^{\alpha\beta}_{ik}\right],$$
$$\varepsilon^{\beta}_j=\sum_{j=1}^{m}\left[H^{\beta}_{jj}+\sum_{k=1}^{m}(J^{\beta\beta}_{jk}-K^{\beta\beta}_{jk})+\sum_{l=1}^{n}J^{\beta\alpha}_{jl}\right]. \tag{A6}$$

In general, Eq. (A6) does not have exact solutions and thus can be only solved in a self-consistent manner. With the help of this equation, it is straightforward to give Eq. (4) in Sec. II.

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