Stationary cantilever vibrations in oscillating-cantilever-driven adiabatic reversals: Magnetic-resonance-force-microscopy technique

G. P. Berman,[1] D. I. Kamenev,[1] and V. I. Tsifrinovich[2]
[1]*Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545*
[2]*IDS Department, Polytechnic University, Six Metrotech Center, Brooklyn, New York 11201*
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We consider theoretically the novel technique in magnetic-resonance-force microscopy that is called "oscillating-cantilever-driven adiabatic reversals." We present an analytical and numerical analysis for the stationary cantilever vibrations in this technique. For reasonable values of parameters, we estimate the resonant frequency shift as 6 Hz per the Bohr magneton. We analyze also the regime of small oscillations of the paramagnetic moment near the transversal plane and the frequency shift of the damped cantilever vibrations.

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I. INTRODUCTION

Magnetic-resonance-force microscopy (MRFM) based on a cyclic adiabatic inversion (CAI) is considered one of the most promising roads to the ultimate goal of a single-spin detection in solids (see, for example, [1,2]). Typically, CAI is generated using the frequency modulation of the external radio-frequency (rf) field. In this case, a paramagnetic moment of a sample follows the effective magnetic field in the rotating system of coordinates (RSC), and influences the cantilever vibrations.

Recently, a new technique called "oscillating-cantilever-driven adiabatic reversals" (OSCAR) has been suggested and implemented in [3]. In this technique, the cantilever driven by an external force causes the CAI of the paramagnetic moment of a sample. The backreaction of the paramagnetic moment causes the frequency shift of the cantilever vibrations, which is supposed to be detected. The main purpose of this paper is a theoretical analysis of the stationary vibrations of the cantilever in the OSCAR technique. Our consideration is based on the classical equations of motion for the spin-cantilever system.

The paper is organized as follows. In Sec. II we introduce the model. The linear OSCAR regime is considered in Sec. III, and the nonlinear regime is analyzed in Sec. IV. A perturbative approach and numerical results are presented in Sec. V. In Sec. VI, we analyze the damped oscillations of the cantilever in the absence of the external force. In Sec. VII, we give a brief summary of our results.

II. HAMILTONIAN AND EQUATIONS OF MOTION

A schematic setup of the studied system is shown in Fig. 1. A spherical ferromagnetic particle with magnetic moment, $\vec{m}_F$, is attached to the cantilever tip. A small paramagnetic cluster with magnetic moment, $\vec{\mu}$, which must be detected, is placed on the surface of a nonmagnetic sample beneath the tip of the cantilever. The whole system is placed into the high permanent magnetic field, $\vec{B}_0+\Delta\vec{B}$, oriented in the positive z direction. The external force, $F(t)$, drives the cantilever vibrations along the z axis. The transversal rotating magnetic field, $\vec{B}_1(t)$, is applied to the paramagnetic cluster. We place the origin of our coordinate system at the equilibrium position of the cantilever tip.

We consider the cantilever tip as an oscillator with the effective mass, m^*, and the effective spring constant, k_s. The classical Hamiltonian for the cantilever with the ferromagnetic particle and the paramagnetic cluster has the form

$$\mathcal{H}=\frac{p_z^2}{2m^*}+\frac{k_s z^2}{2}-zF(t)-\frac{\mu_0 m_F}{2\pi(d+z)^3}\mu_z -\vec{\mu}(\vec{B}_0+\Delta\vec{B}+\vec{B}_1), \tag{1}$$

where p_z and z are the momentum and coordinate of the cantilever tip, and μ_0 is the permeability of the free space. Putting $F(t)=F_0\cos(\nu t+\vartheta_0)$ and taking into consideration the finite quality factor, Q, of the cantilever, we write the equation of motion for the cantilever,

$$\ddot{z}+\omega_c^2 z+\frac{q\mu_z}{(d+z)^4}+\frac{\omega_c}{Q}\dot{z}=f_0\cos(\nu t+\vartheta_0), \tag{2}$$

where $\omega_c=(k_s/m^*)^{1/2}$ is the unperturbed cantilever frequency, $f_0=F_0/m^*$, and

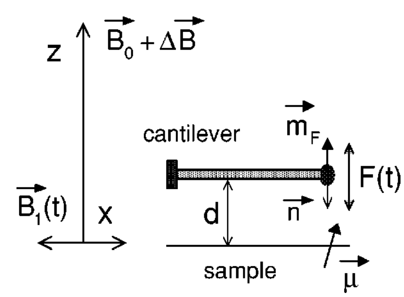

FIG. 1. A schematic setup of the system under consideration. $\vec{B}_0+\Delta\vec{B}$ is the uniform permanent magnetic field, $\vec{B}_1$ is the rotating rf magnetic field, $F(t)$ is an external force acting on the cantilever in the z direction, $\vec{m}_F$ is the magnetic moment of the ferromagnetic particle, $\vec{\mu}$ is the magnetic moment of the paramagnetic cluster, and d is the equilibrium distance between the center of the ferromagnetic particle and the cluster.

$$q = \frac{3\mu_0 m_F}{2\pi m^*}. \tag{3}$$

Next, we assume that the rf field $\vec{B}_1$ rotates in the (x,y) plane with the frequency

$$\omega_0 = \gamma[B_0 + B_d(0)]. \tag{4}$$

Here γ is the gyromagnetic ratio of the paramagnetic cluster, $B_d(z)$ is the dipole magnetic field produced by the ferromagnetic particle at the point of location of the paramagnetic cluster,

$$B_d(z) = \frac{\mu_0 m_F}{2\pi(d+z)^3}, \tag{5}$$

and $B_d(0)$ is the value of $B_d(z)$ at the equilibrium position of the cantilever, $z=0$. The equation of motion for the paramagnetic moment $\vec{\mu}$ in the RSC has the form

$$\dot{\vec{\mu}} = \gamma[\vec{\mu} \times \vec{B}_{\text{eff}}]. \tag{6}$$

Here $\vec{B}_{\text{eff}}$ is the effective magnetic field in the RSC with the x component B_1 and the z component $\Delta B + B'_d(z)$, where $B'_d(z)$ is the oscillatory part of the dipole field produced by the ferromagnetic particle on the cluster,

$$B'_d(z) = B_d(z) - B_d(0). \tag{7}$$

III. THE LINEAR OSCAR REGIME: SMALL OSCILLATIONS OF $\vec{\mu}$

In this section we consider the linear OSCAR regime. Suppose that initially an auxiliary $\pi/2$ pulse changes the direction of the paramagnetic moment, $\vec{\mu}$, from $+z$ to $+x$ of the RSC. We also assume that the oscillatory part of the dipole field, $B'_d(z)$, is small compared to the rf field, B_1. Certainly we assume that the unperturbed cantilever frequency $\omega_c \ll \gamma B_1$ to keep the conditions of CAI. In the quasistatic approximation a paramagnetic moment, $\vec{\mu}$, follows the effective field B_{eff}. Putting in Eq. (6) $\dot{\vec{\mu}}=0$, we obtain for $|z| \ll d$

$$\mu_x(t) \approx \mu, \quad \mu_y(t) = 0, \quad \mu_z(t) = \frac{\mu}{B_1}\left[\Delta B - \frac{3\mu_0 m_F z(t)}{2\pi d^4}\right]. \tag{8}$$

These equations describe small (linear) oscillations of $\vec{\mu}$ near the x axis. Substituting the last expression in Eq. (8) into Eq. (2), we derive an approximate equation for the cantilever oscillations,

$$\ddot{z} + \omega_c^{*2} z + \frac{\omega_c^*}{Q^*}\dot{z} = f_0 \cos(\nu t + \vartheta_0). \tag{9}$$

Here,

$$\omega_c^* = \omega_c + \Delta\omega_c,$$

$$\Delta\omega_c = -\frac{3\mu_0 m_F \mu}{\pi m^* \omega_c B_1 d^5}\left(\Delta B + \frac{3\mu_0 m_F}{8\pi d^3}\right),$$

$$Q^* = Q\left(1 + \frac{\Delta\omega_c}{\omega_c}\right). \tag{10}$$

Equation (9) describes the motion of the linear oscillator with the effective frequency, ω_c^*, and the effective quality factor, Q^*. Due to the backreaction of the paramagnetic moment on the cantilever, the effective frequency and the quality factor of the cantilever depend on the permanent magnetic field, ΔB (in our approximation, $\Delta B \ll B_1$). If $\Delta B > -3\mu_0 m_F/(8\pi d^3)$, then both the frequency and the quality factor of the cantilever decrease. In the opposite case, they increase.

IV. NONLINEAR ADIABATIC REGIME: ADIABATIC REVERSALS OF $\vec{\mu}$

To increase the backreaction of $\vec{\mu}$, it is important to provide large oscillations (adiabatic reversals) of the paramagnetic moment. In this section, we consider stationary vibrations of the cantilever in the nonlinear OSCAR regime. It is convenient to write the equations of motion in the dimensionless form,

$$Z'' + Z + \frac{\lambda M_z}{(1+\alpha Z)^4} + \frac{1}{Q}Z' = \frac{1}{Q}\cos[(1+\rho)\tau + \vartheta_0],$$

$$M_x' = (\delta - \chi Z) M_y, \tag{11}$$

$$M_y' = \varepsilon M_z - (\delta - \chi Z) M_x,$$

$$M_z' = -\varepsilon M_y,$$

where we introduced the dimensionless time $\tau = \omega_c t$; a prime denotes differentiation over τ, $Z = z/A$ is the dimensionless coordinate, $A = f_0 Q/\omega_c^2$ is the unperturbed (in the absence of the magnetic moment $\vec{M}$) amplitude of the stationary cantilever vibrations in the resonant regime (when $\nu = \omega_c$), $\vec{M} = \vec{\mu}/\mu$ is the dimensionless magnetic moment, and $\delta = \gamma\Delta B/\omega_c$. The parameter $\alpha = A/d$ is small, $\alpha \sim 0.01$. The dynamics is controlled by the following dimensionless parameters:

$$\lambda = \frac{3\mu_0 m_F \mu}{2\pi d^4 Q F_0},$$

$$\chi = \frac{3\gamma\mu_0 m_F Q f_0}{2\pi\omega_c^3 d^4}, \tag{12}$$

$$\varepsilon = \frac{\gamma B_1}{\omega_c}, \quad \rho = \nu/\omega_c - 1.$$

Parameter λ can be expressed in a simpler form,

$$\lambda = \frac{F_m}{k_s A}, \tag{12a}$$

where $F_m = 3\mu_0 m_F \mu / 2\pi d^4$ is the magnetic force between the ferromagnetic particle at equilibrium ($z=0$) and the paramagnetic cluster.

Suppose that the paramagnetic moment, $\vec{M}$, points initially in the direction of the effective magnetic field, $\vec{B}_{\rm eff}$, and the cantilever points in the opposite direction, $Z(0) = -1$. In this case the quasistatic motion of $\vec{M}$ is given by the expressions

$$M_x(\tau) = \frac{\varepsilon}{\sqrt{\varepsilon^2 + (\delta - \chi Z)^2}},$$

$$M_y = 0, \tag{13}$$

$$M_z(\tau) = \frac{\delta - \chi Z}{\sqrt{\varepsilon^2 + (\delta - \chi Z)^2}}.$$

Substituting Eqs. (13) into the first equation in Eqs. (11), we obtain the nonlinear equation for Z,

$$Z'' + Z - \frac{\lambda \chi Z}{\sqrt{\varepsilon^2 + (\chi Z)^2}} + \frac{1}{Q} Z' = \frac{1}{Q} \cos[(1+\rho)\tau + \vartheta_0], \tag{14}$$

where we neglected the term αZ in the denominator in the third term on the left-hand side and put $\delta = 0$. The third term in Eq. (15) corresponds to the modification of the potential energy of the cantilever, due to the interaction with the magnetic moment, $\vec{M}$, by the value

$$\delta U(Z) = -\frac{\lambda}{\chi} \sqrt{\varepsilon^2 + (\chi Z)^2}.$$

Now, we present an approximate "semiquantitative" analysis of the stationary oscillations described by Eq. (14). The approximate solution for the stationary driven oscillations of the cantilever, described by Eq. (14), can be written in the form

$$Z = a(\rho) \sin[(1+\rho)\tau + \vartheta_0]. \tag{15}$$

We define the frequency shift as the shift of the maximum, $a_{\max}(\rho_1)$, of the amplitude, $a = a(\rho)$, caused by the paramagnetic cluster. In order to estimate $a_{\max}$, we replace

$$Z^2 \sim \sin^2[(1+\rho)\tau + \vartheta_0]$$
$$= \tfrac{1}{2}\{1 - \cos[2(1+\rho)\tau + 2\vartheta_0]\} \rightarrow \tfrac{1}{2}$$

in the denominator in the third term in Eq. (14), and neglect the term $\cos[2(1+\rho)\tau + 2\vartheta_0]$ because it is nonresonant. Then Eq. (14) takes the form

$$Z'' + \left[1 - \frac{\lambda \chi}{\sqrt{\varepsilon^2 + \frac{\chi^2}{2}}} \right] Z + \frac{1}{Q} Z' = \frac{1}{Q} \cos[(1+\rho)\tau + \vartheta_0]. \tag{16}$$

The position, ρ_1, of the maximum of the amplitude, $a_{\max}$, of the driven oscillations (the frequency shift) is

$$\rho_1 - \rho_0 = -\frac{\lambda \chi}{2\sqrt{\varepsilon^2 + \frac{\chi^2}{2}}} \approx -\frac{\lambda}{\sqrt{2}}, \tag{17}$$

where $\rho_0 = -1/4Q^2$ is the position of the maximum of the amplitude in the absence of the paramagnetic sample, and we suppose that $\chi \gg \varepsilon$. The frequency shift (18) can be considered in connection with the change of the effective spring constant, Δk_s, which can be expressed in a very simple form,

$$\Delta k_s \approx 2m^* \omega_c \Delta \nu = \sqrt{2} F_m / A, \tag{17a}$$

where $\Delta \nu = \nu - \omega_c$.

For estimation of the value of the frequency shift, the following parameters were used: $D = 1.5 \times 10^{-7}$ m is the diameter of the ferromagnetic particle with the volume $V = 1.8 \times 10^{-21}$ m^3, $\mu_0 m_F / V \approx 1.1$ T, $k_c \approx 10^{-3}$ N/m, $\omega_c / 2\pi \approx 10^5$ Hz, $A \approx 1$ nm, $d \approx 100$ nm, and $B_1 \approx 10^{-3}$ T. For these values of parameters, we obtain

$$\varepsilon \approx 280, \quad \chi \approx 2.5 \times 10^3,$$

$$\lambda \approx 8.5 \times 10^{-5} (\mu/\mu_B), \quad \alpha = 0.01, \tag{18}$$

where μ/μ_B is the paramagnetic moment expressed in units of the Bohr magneton. The corresponding frequency shift is

$$\rho_1 - \rho_0 \approx -6 \times 10^{-5} (\mu/\mu_B). \tag{19}$$

This gives the frequency shift -6 Hz per one Bohr magneton.

V. PERTURBATION APPROACH

The qualitative estimation presented above can be supported by application of the approach based on the perturbation theory developed by Bogoliubov and Mitropolskii in [4]. We look for the solution of Eq. (14) in the form

$$Z = a(\tau) \cos[\psi] + \lambda u_1(a, \psi), \tag{20}$$

where $\psi = (1+\rho)\tau + \vartheta(\tau)$. The function $u_1(a,\psi)$ is the sum of the Fourier terms with the phases 3ψ, 5ψ, 7ψ, ... The amplitudes of these terms decrease with increasing Fourier number, n, as $1/(2n+1)^2$. The first nonvanishing term is small and equals $u_1(a,\psi) \approx 0.02 \cos(3\psi)$. This allows us to neglect the contribution of $u_1(a,\psi)$ into the expression for Z in Eq. (20).

The slow varying amplitude, $a(\tau)$, and the phase, $\vartheta(\tau)$, in the first order of the perturbation theory satisfy the two coupled differential equations,

$$\frac{da}{d\tau}=-\frac{\lambda}{2\pi}\int_0^{2\pi}\frac{\chi a\cos\psi\sin\psi d\psi}{\sqrt{\varepsilon^2+(\chi a\cos\psi)^2}}-\frac{a}{2Q}-\frac{1}{Q(2+\rho)}\sin\vartheta, \tag{21}$$

$$\frac{d\vartheta}{d\tau}=-\frac{1}{8Q^2}-\rho-\frac{\lambda}{2\pi a}\int_0^{2\pi}\frac{\chi a\cos^2\psi d\psi}{\sqrt{\varepsilon^2+(\chi a\cos\psi)^2}}-\frac{1}{aQ(2+\rho)}\cos\vartheta. \tag{22}$$

Note that the integral on the right-hand side of Eq. (21) is equal to zero. The integral on the right-hand side of Eq. (22) can be expressed through the elliptic integrals as (see the Appendix)

$$4\int_0^{\pi/2}\frac{\chi a\cos^2\psi d\psi}{\sqrt{\varepsilon^2+(\chi a\cos\psi)^2}}=4\left[\frac{1}{k}E(k)-p^2kK(k)\right], \tag{23}$$

where $k=1/\sqrt{1+p^2}$, and $K(k)$ and $E(k)$ are the complete elliptic integrals, respectively, of the first and second kind, $p=\varepsilon/(a\chi)$. When $p^2\ll 1$, one can decompose $K(k)$ and $E(k)$ as

$$K(k)\approx C+(C-1)\frac{k'^2}{4}+\cdots,$$

$$E(k)\approx 1+\left(C-\frac{1}{2}\right)\frac{k'^2}{2}+\cdots, \tag{24}$$

where $k'^2=1-k^2\approx p$, $C=\ln(4/k')\approx\ln(4/p)$. From Eqs. (23) and (24) we find the value of the integral in Eq. (22) for $p\ll 1$,

$$-\frac{\lambda}{2\pi a}\int_0^{2\pi}\frac{\chi a\cos^2\psi d\psi}{\sqrt{\varepsilon^2+(\chi a\cos\psi)^2}}\approx-\frac{2\lambda}{\pi a}\left[1-\frac{p^2}{4}\left(2\ln\frac{4}{p}-1\right)\right]. \tag{25}$$

Substituting Eq. (25) into Eq. (22), we obtain

$$\frac{da}{d\tau}=-\frac{a}{2Q}-\frac{1}{Q(2+\rho)}\sin\vartheta, \tag{26}$$

$$\frac{d\vartheta}{d\tau}=-\frac{1}{8Q^2}-\rho-\frac{2\lambda}{\pi a}\left[1-\frac{p^2}{4}\left(2\ln\frac{4}{p}-1\right)\right]-\frac{1}{aQ(2+\rho)}\cos\vartheta.$$

We now calculate the position of the maximum of the amplitude, $a(\rho)$, in the stationary regime of driven oscillations using Eq. (26), and compare it with the results obtained in Sec. IV. In the regime of driven oscillations $a=\text{const}$, $\vartheta=\text{const}$, and we must solve the system of two equations (26), where $da/d\tau=0$ and $d\vartheta/d\tau=0$. Canceling the phase, ϑ, we have

$$\frac{1}{a^2(2+\rho)^2}=\frac{1}{4}+Q^2\left(\frac{1}{8Q^2}+\rho+\frac{2\lambda}{\pi a}\right)^2, \tag{27}$$

where we neglected the term proportional to $p^2\ll 1$. The amplitude, a, can be written as $a=1+\beta$, where $\beta\ll 1$, so that

$$\frac{1}{a(2+\rho)}=\frac{1}{(1+\beta)(2+\rho)}\approx\frac{1}{2}\left(1-\beta-\frac{\rho}{2}\right). \tag{28}$$

Taking the square root from both sides of Eq. (27) and using Eq. (28), we obtain

$$-\beta-\frac{\rho}{2}\approx 2Q^2\left(\frac{1}{8Q^2}+\rho+\frac{2\lambda}{\pi}\right)^2, \tag{29}$$

where we put $a\approx 1$ in the denominator of the term proportional to λ (i.e., we neglected the term of the order of $\beta\lambda$). The maximum of the function, $\beta=\beta(\rho)$, can be found from the condition $d\beta(\rho_1)/d\rho=0$, which yields

$$\rho_1=-\frac{1}{4Q^2}-\frac{2\lambda}{\pi}. \tag{30}$$

This is approximately the same value as that given by Eq. (18), obtained from the qualitative considerations. The second term in Eq. (30) describes the influence of the paramagnetic moment reversals on the resonance frequency of the cantilever.

To verify our analytical results, we solved numerically the exact equations of motion (11). Figure 2 (solid line) demonstrates the dependence of the stationary amplitude of the cantilever vibrations, a, on the frequency detuning, ρ. (The stationary amplitude is achieved at $\tau\gg Q$.) The initial conditions are taken in the form

$$Z(0)=-1,\quad \dot{Z}(0)=0,\quad M_x(0)=\frac{\varepsilon}{\sqrt{\varepsilon^2+\chi^2}},$$

$$M_y(0)=0,\quad M_z(0)=\frac{\chi}{\sqrt{\varepsilon^2+\chi^2}}. \tag{31}$$

For these initial conditions at $\tau=0$, the paramagnetic moment, $\vec{M}$, points in the direction of the effective magnetic field, while the cantilever is displaced in the $-z$ direction ($\vartheta_0=3\pi/2$) from its equilibrium position.

Figure 3 demonstrates the motion of the paramagnetic moment, $\vec{M}(\tau)$. One can see the close correspondence between the analytical and numerical solutions. Note that the frequency shift caused by the adiabatic reversals changes its sign if the paramagnetic moment points initially in the direction opposite to the effective magnetic field [while $Z(0)$

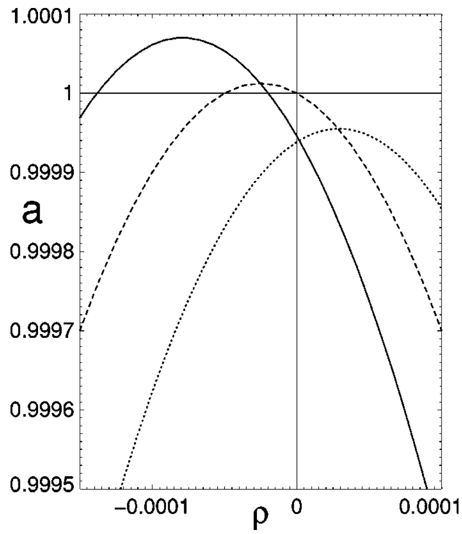

FIG. 2. The dependence of the amplitude of the driven oscillations of the cantilever on the frequency detuning, ρ, obtained using numerical solution of exact equations of motion (11). The solid line corresponds to the initial conditions (31) and the values of the parameters $\lambda=8.5\times10^{-5}$, $\chi=2500$, $\varepsilon=280$, $\alpha=0.05$, $Q=100$, and $\delta=0$. The dotted line corresponds to the same values of the parameters but for "inverted" initial conditions [in Eq. (31), $M_x\to-M_x$, $M_z\to-M_z$]. The dashed line represents the dependence $a(\rho)$ with no paramagnetic moment ($\lambda=0$).

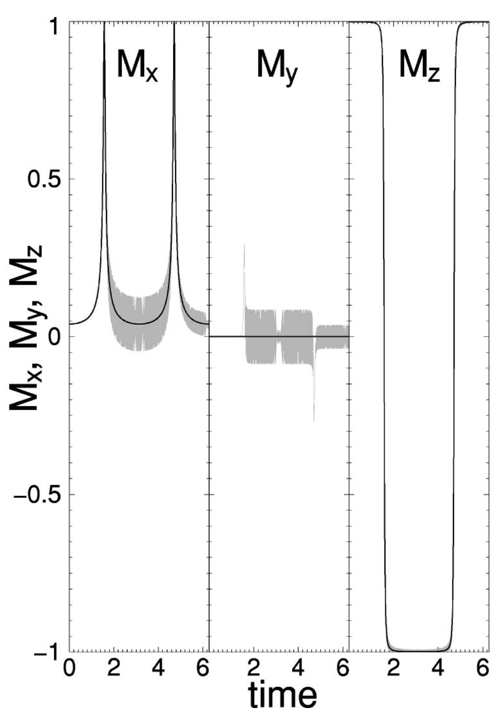

FIG. 3. The dynamics of the projections of the paramagnetic moment, $\vec{M}(\tau)$, of the sample with the initial conditions (31). The gray line is obtained as a result of the numerical integration of Eq. (11), and the black line indicates the quasistatic solution (13). For $M_z(\tau)$, both curves almost coincide. The parameters are the same as those for the solid line in Fig. 2.

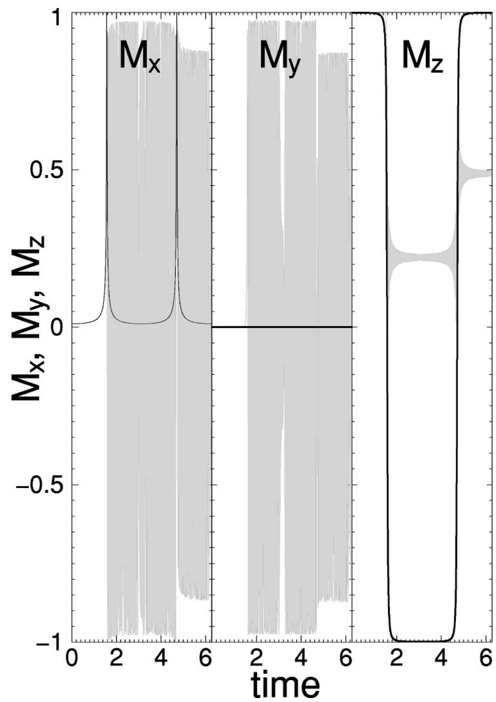

FIG. 4. The same as in Fig. 3 but for $\varepsilon=28$.

$=-1$]. The dotted line in Fig. 2 depicts this case. We should also note that decreasing the parameter ε (the x component of the effective magnetic field) leads to the violation of the CAI conditions. Figure 4 demonstrates this situation for $\varepsilon=28$.

VI. DAMPED OSCILLATIONS OF THE CANTILEVER

The influence of the sample on the cantilever can be measured if one turns off the external force acting on the cantilever, and measures the frequency of small damped oscillations of the cantilever. In the absence of the paramagnetic moment $\vec{M}$, the frequency of the oscillations is independent of time and equals $\sqrt{1-1/(4Q^2)}$.

We look for the solution of the cantilever vibrations in the form $Z=a(\tau)\cos[\tau+\vartheta(\tau)]$. Then the dynamical equations for the slow varying amplitude, $a(\tau)$, and phase, $\vartheta(\tau)$, in the presence of the sample and in the absence of the external force take the form

$$\frac{da}{d\tau}=-\frac{a}{2Q}, \tag{32}$$

$$\frac{d\vartheta}{d\tau}=-\frac{1}{8Q^2}-\frac{\lambda}{2\pi a}\int_0^{2\pi}\frac{\chi a\cos^2\psi d\psi}{\sqrt{\varepsilon^2+(\chi a\cos\psi)^2}}. \tag{33}$$

For $p\ll1$, Eq. (33) can be written as

$$\frac{d\vartheta}{d\tau}=-\frac{1}{8Q^2}-\frac{2\lambda}{\pi a}\left[1-\frac{p^2}{4}\left(2\ln\frac{4}{p}-1\right)\right]. \tag{34}$$

The last term on the right-hand side of Eq. (34) describes a change of the frequency of small oscillations of the cantile-

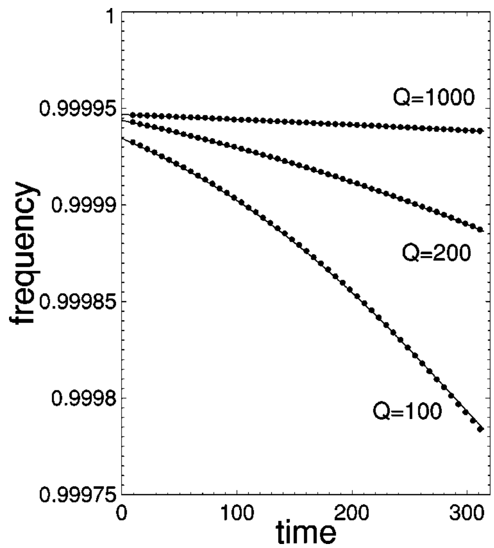

FIG. 5. The frequency of small damped oscillations of the cantilever for the initial conditions (33) as a function of time, τ, for three values of the quality factor, Q. The solid lines are obtained using Eqs. (34) and (36). The results of exact numerical solution are plotted by the filled circles, $\lambda=8.5\times10^{-5}$, $\chi=2500$, $\varepsilon=280$, $\delta=0$, and $\alpha=0.05$.

ver caused by the adiabatic reversals of $\vec{M}$. From Eq. (32) we have $a(\tau)=a(0)\exp[-\tau/(2Q)]$. One can see from Eq. (34) that for the initial conditions (31) the influence of $\vec{M}$ results in a decrease of the frequency of small oscillations of the cantilever in comparison with the case $\lambda=0$. For small p [the value of $p=\varepsilon/(a\chi)\sim\exp(t/2Q)$ increases with time], the frequency of oscillations decreases when time increases, as shown in Fig. 5, while in the absence of the sample this frequency remains independent of time. We should note that in the studied approximation the sample does not influence the amplitude of the cantilever oscillations.

VII. SUMMARY

We have studied theoretically and numerically the stationary cantilever vibrations in the novel OSCAR MRFM technique. Our results are based on the application of the classical theory for the motion of the cantilever and the paramagnetic moment of a cluster on the surface of the sample. We have estimated the resonant frequency shift for the cantilever vibrations. For the reasonable values of parameters, our estimate is about 6 Hz per Bohr magneton. The sign of the shift depends on the initial direction of the paramagnetic moment relative to the initial position of the cantilever. We supported our estimation by the analytical analysis based on the perturbation theory and by the numerical solution of the equations of motion. Our perturbative approach is based on the fact that the influence of the paramagnetic moment on the sample is weak ($\lambda\ll1$). We considered also the regime of small oscillations of the paramagnetic moment near the transversal plane (linear OSCAR regime). Finally, we analyzed the damped oscillations of the cantilever (without the external force). We have shown that the frequency of the damped oscillations becomes time-dependent due to the adiabatic reversals of the paramagnetic moment.

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APPENDIX

Here we express the integral in Eq. (22) in terms of complete elliptic integrals,

$$\int_0^{\pi/2}\frac{\chi a\cos^2\psi d\psi}{\sqrt{\varepsilon^2+(\chi a\cos\psi)^2}}=\int_0^{\pi/2}\frac{\cos^2\psi d\psi}{\sqrt{p^2+\cos^2\psi}}=\int_0^{\pi/2}\frac{(1-\sin^2\psi)d\psi}{\sqrt{p^2+1-\sin^2\psi}}=\int_0^{\pi/2}\frac{(p^2+1)\left(1-\dfrac{1}{p^2+1}\sin^2\psi\right)-p^2}{\sqrt{p^2+1}\sqrt{1-\dfrac{1}{p^2+1}\sin^2\psi}}d\psi,$$

where we introduced the notation $p=\varepsilon/(a\chi)$. Splitting this integral in two parts, we obtain the right-hand side of Eq. (23).

[1] D. Rugar, O. Zuger, S. Hoen, C.S. Yannoni, H.-M. Vieth, and R.D. Kendrick, Science **264**, 1560 (1994).
[2] K. Wago, D. Botkin, C.S. Yannoni, and D. Rugar, Phys. Rev. B **57**, 1108 (1998).
[3] B.C. Stipe, H.J. Mamin, C.S. Yannoni, T.D. Stowe, T.W. Kenny, and D. Rugar, Phys. Rev. Lett. **87**, 277602 (2001).
[4] N. N. Bogoliubov and Iu. A. Mitropolskii, *Asymptotic Methods in the Theory of Nonlinear Oscillations* (Fiz.-Mat. Lit., Moscow, 1958) (in Russian).