Quantum nonlocality for a three-particle nonmaximally entangled state without inequalities

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We show that it is possible to demonstrate quantum nonlocality for three particles in a nonmaximally entangled state without using inequality. The nonlocal effect might be much stronger than that which a two-particle nonmaximally entangled state can exhibit, as described by Hardy [Phys. Rev. Lett. **71**, 1665 (1993)].

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In recent years, much attention has been paid to quantum entanglement, one of the most striking features of quantum mechanics. If two subsystems are in an entangled state the Bell's inequalities [1] may be violated, supporting quantum mechanics over local hidden variable theories. A few years ago, Greenberger, Horne, and Zeilinger (GHZ) [2] showed that much stronger refutations of local realism can be provided by entangled states involving three or more particles. For the GHZ states, a single set of measurements is sufficient to demolish local realistic theories.

Proofs of quantum nonlocality without inequality for two spin-1 particles have been given [3,4]. The argument has been generalized to two spin-*s* particles [5] and *N* spin-1/2 particles [6]. Several years ago, Hardy showed that it is possible to demonstrate nonlocality without using the Bell inequality for two spin-1/2 particles prepared in nonmaximally entangled states [7]. Hardy proposed an experiment to realize his argument using the parametric down-conversion process [8]. In the context of cavity QED, Freyberger [9] presented a simple example along the lines of Ref. [7]. The probability of obtaining a nonlocality proof of the Hardy type is no larger than 0.09. In an experiment it might be difficult to distinguish an event with a probability of 0.09 from errors. So far nonlocality without inequality has not been proved for a maximally entangled state of two spin-1/2 particles [10]. On the other hand, a proof of the Hardy type based on two copies of maximally entangled states of two spin-1/2 particles has been presented [11]. It works for 100% of the runs of an experiment. In this paper, we show that a three-particle nonmaximally entangled state may also reveal quantum nonlocality without using inequality, and the probability of obtaining the nonlocality proof might be much larger than 0.09.

We assume that we cannot get a maximally entangled state for three spin-1/2 particles due to some experimental problems; instead, we can prepare these particles in the nonmaximally entangled state

$$|\psi_{1,2,3}\rangle=\cos\theta|+_1\rangle|+_2\rangle|+_3\rangle+i\sin\theta|-_1\rangle|-_2\rangle|-_3\rangle, \tag{1}$$

where $|+\rangle$ and $|-\rangle$ are the spin-up and -down states along the z axis, and the superscripts 1,2,3 characterize the three particles. We here assume that $0<\theta<\pi/4$. We define the following bases:

$$|\varphi_i\rangle=i\sin\theta|+_i\rangle+\cos\theta|-_i\rangle, \tag{2}$$

$$|\varphi_i^\dagger\rangle=\cos\theta|+_i\rangle+i\sin\theta|-_i\rangle \tag{3}$$

and

$$|\phi_i\rangle=\tfrac{1}{2}(|+_i\rangle+|-_i\rangle), \tag{4}$$

$$|\phi_i^\dagger\rangle=\tfrac{1}{2}(|+_i\rangle-|-_i\rangle). \tag{5}$$

Consider the physical observables E_i and U_i $(i=1,2,3)$ corresponding to the operators

$$\hat{E}_i=|\varphi_i\rangle\langle\varphi_i|-|\varphi_i^\dagger\rangle\langle\varphi_i^\dagger|, \tag{6}$$

$$\hat{U}_i=|\phi_i\rangle\langle\phi_i|-|\phi_i^\dagger\rangle\langle\phi_i^\dagger|. \tag{7}$$

In order to measure E_i and U_i we have to perform state-selective measurement on the ith particle with respect to the corresponding bases. The physical quantities E_i and U_i can take values 1 or -1 corresponding to the eigenvalues of $\hat{E}_i$ and $\hat{U}_i$.

We can expand $|\psi_{1,2,3}\rangle$ in terms of $|\varphi_1\rangle$ and $|\varphi_1^\dagger\rangle$:

$$\begin{aligned}|\psi_{1,2,3}\rangle=&-i\sin\theta\cos\theta|\varphi_1\rangle(|+_2\rangle|+_3\rangle-|-_2\rangle|-_3\rangle)\\&+|\varphi_1^\dagger\rangle(\cos^2\theta|+_2\rangle|+_3\rangle+\sin^2\theta|-_2\rangle|-_3\rangle).\end{aligned} \tag{8}$$

On the other hand, we can expand $|\psi_{1,2,3}\rangle$ in terms of $|\varphi_2\rangle$ and $|\varphi_2^\dagger\rangle$:

$$\begin{aligned}|\psi_{1,2,3}\rangle=&-i\sin\theta\cos\theta|\varphi_2\rangle(|+_1\rangle|+_3\rangle-|-_1\rangle|-_3\rangle)\\&+|\varphi_2^\dagger\rangle(\cos^2\theta|+_1\rangle|+_3\rangle+\sin^2\theta|-_1\rangle|-_3\rangle).\end{aligned} \tag{9}$$

We can also expand $|\psi_{1,2,3}\rangle$ in terms of $|\varphi_2\rangle$ and $|\varphi_2^\dagger\rangle$:

$$\begin{aligned}|\psi_{1,2,3}\rangle=&-i\sin\theta\cos\theta|\varphi_3\rangle(|+_1\rangle|+_2\rangle-|-_1\rangle|-_2\rangle)\\&+|\varphi_3^\dagger\rangle(\cos^2\theta|+_1\rangle|+_2\rangle+\sin^2\theta|-_1\rangle|-_2\rangle).\end{aligned} \tag{10}$$

We now measure E_1 on particle 1. Suppose we obtain the result $E_1=1$. In this case particles 2 and 3 will collapse onto the state

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$$|\psi_{2,3}\rangle=\frac{1}{\sqrt{2}}(|+_2\rangle|+_3\rangle-|-_2\rangle|-_3\rangle). \tag{11}$$

If we now measure U_2 on particle 2 and U_3 on particle 3 we will have the result $U_2U_3=-1$. This means

$$\text{if } E_1=1, \text{ then } U_2U_3=-1. \tag{12}$$

By symmetry, if we measure E_2 on particle 2, U_1 on particle 1, and U_3 on particle 3, we have

$$\text{if } E_2=1, \text{ then } U_1U_3=-1. \tag{13}$$

On the other hand, if we measure E_3 on particle 3, U_1 on particle 1, and U_3 on particle 3, we obtain

$$\text{if } E_3=1, \text{ then } U_1U_2=-1. \tag{14}$$

We now measure E_1, E_2, and E_3. The probability of obtaining the result $E_1=E_2=E_3=1$ is

$$\begin{aligned}P&=\mathrm{Tr}_{1,2,3}\{|\varphi_1\varphi_2\varphi_3\rangle\langle\varphi_1\varphi_2\varphi_3||\psi_{1,2,3}\rangle\langle\psi_{1,2,3}|\}\\&=\cos^2\theta\sin^2\theta\\&=\frac{1}{4}\sin^2(2\theta).\end{aligned} \tag{15}$$

Consider a run of measurements in which the predictions (12), (13), and (14) are verified and $E_1=E_2=E_3=1$ is obtained. According to local hidden theory, from the result $E_1=1$ and Eq. (12) one can conclude that if U_2 and U_3 had been measured one should have obtained $U_2U_3=-1$. On the other hand, from the result $E_2=1$ and Eq. (13) one can conclude that if U_1 and U_3 had been measured one should have obtained $U_1U_3=-1$. From the result $E_3=1$ and Eq. (14) one can conclude that if U_1 and U_2 had been measured one should have obtained $U_1U_2=-1$. This leads to

$$(U_2U_3)(U_1U_2)(U_1U_3)=-1. \tag{16}$$

However, according to local hidden theory these elements of reality U_i have values 1 or -1 and thus $U_iU_i=1$. This leads to

$$(U_2U_3)(U_1U_2)(U_1U_3)=1, \tag{17}$$

contradicting Eq. (15). We thus have revealed the inconsistency hidden in the local hidden variable theory. The self-contradiction arises from the assumption that there exists some element of reality corresponding to each of U_i even when these quantities are not measured and regardless of what is done to other systems.

We note that the probability P of obtaining $E_1=E_2=E_3=1$ depends on the degree of entanglement we can get. Even if $\cos^2\theta$ is much larger than $\sin^2\theta$ such a probability might be much larger than the probability of revealing the nonlocality using a two-particle nonmaximally entangled state without inequality [7]. We here give an example. Assume that we can get a nonmaximally entangled state of Eq. (1) with $\cos^2\theta=4/5$ and $\sin^2\theta=1/5$. Then we have $P=0.16$. Such a probability of success is almost twice the maximal probability of getting the proof of nonlocality using a two-particle nonmaximally entangled state.

We now turn to another kind of three-particle nonmaximally entangled state, i.e., the W states [12]

$$\begin{aligned}|\psi_{1,2,3}\rangle=\frac{1}{\sqrt{3}}[&|+_1\rangle|-_2\rangle|-_3\rangle+|-_1\rangle|+_2\rangle|-_3\rangle\\&+|-_1\rangle|-_2\rangle|+_3\rangle].\end{aligned} \tag{18}$$

Now we consider the physical observables E_i and U_i ($i=1,2,3$) corresponding to the operators

$$\hat{E}_i=|+_i\rangle\langle+_i|, \tag{19}$$

$$\hat{U}_i=\tfrac{1}{2}(|+_i\rangle+|-_i\rangle)(\langle+_i|+\langle-_i|). \tag{20}$$

We have the following predictions. If we measure E_1, E_2, and E_3 we have

$$E_iE_j=0,\ \ i\neq j. \tag{21}$$

This is due to the fact that there is only one particle in the state $|+\rangle$.

We now measure U_1 and U_2 and E_3. If we find $U_1=1$ and $U_2=0$, the density operator of particle 3 will collapse onto

$$\hat{\rho}_3=\frac{\mathrm{Tr}_{1,2}\{\hat{U}_2^\dagger\hat{U}_1|\psi_{1,2,3}\rangle\langle\psi_{1,2,3}|\}}{\mathrm{Tr}_{1,2,3}\{\hat{U}_2^\dagger\hat{U}_1|\psi_{1,2,3}\rangle\langle\psi_{1,2,3}|\}}, \tag{22}$$

where

$$\hat{U}_2^\dagger=\tfrac{1}{2}(|+_2\rangle-|-_2\rangle)(\langle+_2|-\langle-_2|). \tag{23}$$

$\mathrm{Tr}_{1,2}$ denotes the trace over particles 1 and 2, while $\mathrm{Tr}_{1,2,3}$ denotes the trace over particles 1, 2, and 3. Substituting Eqs. (20) and (23) into Eq. (22) we obtain

$$\hat{\rho}_3=|+_3\rangle\langle+_3|. \tag{24}$$

Therefore, we have

$$\text{if } U_1=1,\ \ U_2=0, \text{ then } E_3=1. \tag{25}$$

By symmetry we can draw the conclusion that, if we measure U_i, U_j, and E_k ($i\neq j\neq k$) on atoms i, j, and k, respectively, we have

$$\text{if } U_i=1,\ \ U_j=0, \text{ then } E_k=1. \tag{26}$$

We now measure U_1, U_2, and U_3. The probability of obtaining the result $U_1=1$, $U_2=0$, $U_3=0$ is

$$P_{1,0,0}=\mathrm{Tr}_{1,2,3}\{\hat{U}_3^\dagger\hat{U}_2^\dagger\hat{U}_1|\psi_{1,2,3}\rangle\langle\psi_{1,2,3}|\}=\tfrac{1}{24}. \tag{27}$$

By symmetry we obtain $P_{0,1,0}=P_{0,0,1}=\frac{1}{24}$. The probability of obtaining the result $U_1=1$, $U_2=1$, and $U_3=0$ is

$$P_{1,1,0}=\mathrm{Tr}_{1,2,3}\{\hat{U}_3^{\dagger}\hat{U}_2\hat{U}_1|\psi_{1,2,3}\rangle\langle\psi_{1,2,3}|\}=\tfrac{1}{24}. \tag{28}$$

Again, we have $P_{1,0,1}=P_{0,1,1}=\frac{1}{24}$.

Consider a run of measurements, in which the predictions (21) and (26) are verified and $U_i=1, U_j=0, U_k=0$ is obtained. According to local hidden theory, from the result $U_i=1, U_j=0$, and Eq. (26) one can conclude that if E_k had been measured one should have obtained $E_k=1$. On the other hand, from the result $U_i=1, U_k=0$, and Eq. (26) one can conclude that if E_j had been measured one should have obtained $E_j=1$. This means that, if we had measured E_j and E_k, instead of U_j and U_k, we would have obtained $E_jE_k=1$, contradicting Eq. (21). Consider another case, in which prediction (26) is verified and $U_i=1, U_j=1, U_k=0$ are obtained. Then the local hidden theory claimed that if E_i and E_j had been measured the result should have been $E_jE_k=1$, again contradicting Eq. (21). Thus, the local hidden theory is demolished. The probability of obtaining the outcome violating the local hidden variable theory is

$$P=P_{1,0,0}+P_{0,1,0}+P_{0,0,1}+P_{1,1,0}+P_{1,0,1}+P_{0,1,1}=0.25. \tag{29}$$

We now discuss the possibility of getting an even stronger nonlocality for the state of Eq. (18) using more general physical observables U_i, with the corresponding operators $\hat{U}_i$ given by

$$\hat{U}_i=(\cos\theta|+_i\rangle+\sin\theta e^{i\phi}|-_i\rangle)(\langle+_i|\cos\theta+\langle-_i|\sin\theta e^{-i\phi}), \tag{30}$$

where $0\leqslant\theta<\pi/2$ and $0\leqslant\phi<\pi$. In order to verify prediction (26) we should satisfy $\theta=\pi/4$ and $\phi=0$. Therefore, $\hat{U}_i$ should be given by Eq. (20) and we cannot get a stronger nonlocality.

In summary, we have shown that the inconsistency in the local hidden theory can be revealed using two types of three-particle nonmaximally entangled states. In the first type of state, either each particle is in the spin-up state or it is in the spin-down state along some axis. The second type of states are the *W* states. The quantum nonlocality is revealed probabilistically. The probability of obtaining an outcome violating the local hidden variable theory is much larger than that using two-particle nonmaximally entangled states.

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