Nonlinear interferometer as a resource for maximally entangled photonic states: Application to interferometry

Christopher C. Gerry,[1] Adil Benmoussa,[1] and R. A. Campos[2]
[1]*Department of Physics and Astronomy, Lehman College, Bronx, New York 10468-1589*
[2]*4 Perry Street, Cortlandt Manor, New York 10567*
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Nonlinear interferometers are Mach-Zehnder interferometers with Kerr media in either one or both arms. We refer to these devices, respectively, as the asymmetric and symmetric nonlinear interferometers. In the asymmetric case, with one input mode in the vacuum, it is possible to generate maximally entangled photonic states or superpositions of such states. We consider the device as a resource of entangled states for applications to Heisenberg-limited interferometry. Interferometry with the maximally entangled states cannot be performed by simply subtracting the output photocounts as in standard interferometry. Instead, one must perform parity measurements on only one of the output beams. We show that the symmetric nonlinear interferometer, with one input mode in the vacuum, may be used to perform such parity measurements. The same device is shown to produce, with an input coherent state and upon projective measurements, even or odd coherent states, examples of the Schrödinger-cat states.

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I. INTRODUCTION

Recently, there has been increasing interest in the prospects for generating a certain class of two-mode field states consisting of a superposition of the photonic states in which all the photons are in one mode or all are in the other. If the two modes are labeled *a* and *b* and if a total of *N* photons are involved, the states of interest have the generic form

$$|N::0\rangle_{a,b}^{\Phi_N} \equiv \frac{1}{\sqrt{2}}(|N\rangle_a|0\rangle_b + e^{i\Phi_N}|0\rangle_a|N\rangle_b), \tag{1.1}$$

where we have used the notation recently introduced by Kok, Lee, and Dowling [1]. A more general state with *N* photons entangled with *M* photons is

$$|N::M\rangle_{a,b}^{\Phi_{NM}} \equiv \frac{1}{\sqrt{2}}(|N\rangle_a|M\rangle_b + e^{i\Phi_{NM}}|M\rangle_a|N\rangle_b). \tag{1.2}$$

States of the form of Eq. (1.1) are often referred to as maximally entangled states (MES) of a two-mode field or, in cases where the two modes are spatially separated, as in a Mach-Zehnder interferometer, as path-entangled states [1]. Aside from their intrinsic interest, such states have potentially important applications to interferometry [2], where they may realize the ultimate level of sensitivity, the so-called Heisenberg limit [3] of phase measurement uncertainty, $\Delta\varphi_{\mathrm{HL}} = 1/N$. They may also have applications in quantum photolithography where the Rayleigh diffraction limit may be breached [4].

But a practical problem remains: How can we generate states of the form of Eq. (1.1)? For $N=1$ and 2, 50:50 beam splitters may be used. The $N=1$ entangled state is commonplace and results from the input of a vacuum state in one port and single photon, usually obtained from a downconversion process, in the other. For $N=2$, the MES can be generated by simultaneously injecting single photons at each of the input ports of a beam splitter. This has been demonstrated experimentally by Hong, Ou, and Mandel [5]. But for $N>2$, beam splitters, which are linear devices, cannot alone produce the MES [6]. This suggests that perhaps nonlinear devices may be used. One such possibility is a nonlinear four-wave mixing device operating in a nonlinear regime, discussed some years ago by Yurke and Stoler [7] and more recently by two of the present authors (C.C.G. and A.B.) [8] in connection with Heisenberg-limited interferometry. The required interaction consists of competing two-mode four-wave mixing and cross-Kerr processes. This nonlinear four-wave mixer (NFWM) device, for a particular interaction time, creates MES of the form of Eq. (1.1) as long as *N* is even. By formulating the NFWM in terms of the Schwinger realization of the angular momentum operators using a pair of field mode operators [9], one recognizes that the interaction involved has the same mathematical form as a nonlinear spin interaction recently proposed by Mølmer and Sørensen [10] for the creation of MES associated with a system of *N* two-level trapped ions (*N* even), superpositions of states with all ions excited and of all ions deexcited. (Their proposal has since been implemented experimentally to entangle the internal states of a set four two-level trapped ions [11].) To create the required even photon number states another NFWM may be used, since, as shown in Ref. [7], the NFWM may be used as a filter with respect to parity with an appropriate interaction time (different than the time required to generate the MES). However, state reduction is required at one of the output modes of this NFWM. Furthermore, large nonlinear susceptibilities in both NFWM devices will be required. Another approach to creating photonic MES, recently discussed by two of us (C.C.G. and R.A.C.) [12], is based on the quantum optical Fredkin gate [13]. The device described in Ref. [12] is essentially a pair of Mach-Zehnder interferometers (MZIs) coupled through a cross-Kerr interaction. The entire device acts as a conditional beam splitter. (A sequence of interactions, whose collective mathematical form is similar to that described in Ref. [12], has been described in the context of a system of *N* two-level trapped ions [14]. The sequence generates an MES of the internal states of the ions.)

The Kerr medium in the coupled interferometer device also requires a large nonlinear susceptibility and, furthermore, state reduction at the outputs of one of the interferometers is required. But on the other hand, Kok, Lee, and Dowling [1] and Lee *et al.* [15] have shown that linear optics along with projective measurements are sufficient to produce MES for large photon numbers. The procedure in Ref. [15], though requiring only linear optical elements (beam splitters) is limited to photon numbers $N \leqslant 4$ while that in Ref. [1] appears to be more general. Nevertheless, both methods do rely on numerous projective state reduction measurements whose outcomes are probabilistic.

In the methods for photonic MES generation described in the previous paragraph, two obstacles stand out: either the requirement of nonlinear media with large nonlinear susceptibilities, or a number of projective state reduction measurements, or both. Here we have not even taken into account the fact that many of the state reduction measurements require resolving photon numbers at the level of a single photon. Clearly, there is motivation to seek alternatives to these approaches. If we restrict ourselves to linear optics, it appears that state reduction is mandated. But for nonlinear optics, there is now the real possibility that the requisite large nonlinearities may soon become available through the use of the techniques of electromagnetically induced transparency [16] as exemplified in the recent experiment of Hau *et al.* and Turukhin *et al.* [17]. As for the elimination of state reductive measurements, it so happens, as has been shown by Sanders [18] and Sanders and Rice [19], that a type of nonlinear interferometer (NLI), an MZI with a Kerr medium one arm, what we shall call the asymmetric nonlinear interferometer (ANLI), can produce, *deterministically*, exactly the required MES by unitary transformations alone. In this paper we reexamine the nonlinear interferometer as a source of maximally entangled states of light with a particular view to applications in interferometry. The idea is to replace the first beam splitter of an MZI by an ANLI. This will produce the MES over the two modes of the MZI containing a phase shift we wish to measure. However, the usual interferometric measurements for the phase shift cannot be performed by the subtraction of the photocounts of the two output ports of the MZI as this difference always vanishes. Instead, we adopt a method originally proposed by Bollinger *et al.* [2] in the context of spectroscopic measurements on an MES of trapped ions. In the optical context, the method amounts to performing parity measurements on one of the output beams, that is, measurements of the operator $(-1)^{\hat{n}}$, where $\hat{n}$ is the photon number operator for that beam. We show that the parity may be measured by the use of another NLI, this time with Kerr media in both arms, the symmetric nonlinear interferometer (SNLI). We further show that the SNLI produces, with an input coherent state in one mode and the vacuum in the other, an output state consisting of the vacuum of one mode correlated with an even coherent state in the other superposed with the odd coherent state in the latter mode correlated with vacuum state in the former. Upon projective measurement of the vacuum state in one or the other output modes one obtains either the even coherent state or the odd coherent state (but in different modes), examples of Schrödinger-cat states. With two input coherent states it unitarily generates a four-component entanglement of coherent states.

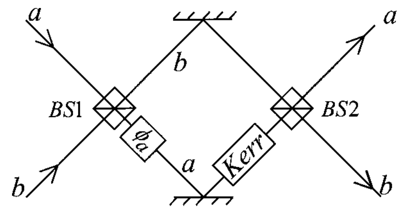

FIG. 1. The asymmetric nonlinear interferometer (ANLI). The phase shift in the lower arm is taken to be $\phi_a = -\pi/2$ as explained in the text.

The paper is organized as follows. In Sec. II, we discuss the ANLI and review how it generates MES. In Sec. III, we discuss the use of the device as a source of MES for high-resolution interferometry. In Sec. IV, we reexamine the SNLI and establish its usefulness as a parity-measuring device. We conclude the paper with some brief remarks in Sec. V.

II. THE ASYMMETRIC NONLINEAR INTERFEROMETER

In Fig. 1 we present a sketch of the ANLI where a Kerr medium is placed in the lower arm of an MZI. The beam in the counterclockwise direction we denote as a and for the clockwise b. The action of the beam splitters (taken as 50:50) is described by the transformation $\hat{U}_{\mathrm{BS}} = \exp[i\pi(a^\dagger b + ab^\dagger)/4]$ [6,20]. Such beam splitters are constructed such that the reflected beam picks up a phase shift of $\pi/2$. A phase shifter, described by the operator $\hat{U}_{\mathrm{PS}}(\phi_a) = \exp(i\phi_a \hat{a}^\dagger \hat{a})$, whose purpose will be discussed shortly, is placed in the a beam. The Kerr interaction is described by the interaction Hamiltonian [21]

$$\hat{H}_K = \hbar \chi \hat{a}^{\dagger 2} \hat{a}^2 = \hbar \chi [(\hat{a}^\dagger \hat{a})^2 - \hat{a}^\dagger \hat{a}], \tag{2.1}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the field operators for the beam and χ is proportional to the third-order nonlinear susceptibility $\chi^{(3)}$ of the medium. Many authors employ a truncated version where the term linear in the photon number operator is dropped [18,22]. We will use the above more realistic form but in order to more easily obtain our desired result we shall need to compensate for the effect of the linear term. This accounts for the presence of the phase shifter in the counterclockwise beam. Shortly we shall specify the required value of the phase shift ϕ_a. The unitary transformation associated with the Kerr interaction is

$$\begin{aligned}\hat{U}_{K,a}(\kappa) &= \exp\{-i\hat{H}_K t_K/\hbar\} = \exp\{-i\kappa[(\hat{a}^\dagger \hat{a})^2 - \hat{a}^\dagger \hat{a}]\} \\ &= \exp[-i\kappa(\hat{a}^\dagger \hat{a})^2]\exp[i\kappa \hat{a}^\dagger \hat{a}],\end{aligned} \tag{2.2}$$

where $t_K = l/\nu_K$ is the time for the light to cross the Kerr medium, l is the length of the medium, and ν_K the velocity of light in the medium. The parameter $\kappa = \chi t_K = \chi l/\nu_K$. In what follows we assume that the third-order nonlinear susceptibility is large enough to maintain the condition $\kappa = \pi/2$. It is perhaps worthwhile to recall here that many proposals for quantum nondemolition measurements of photon number require Kerr interactions, either self-interactions of the type in Eq. (2.1) [23] or cross-Kerr interactions between two modes [24], with large nonlinear susceptibilities of the magnitude required in the present work.

We take as our input a coherent state in mode a with the vacuum in mode b: $|\text{in}\rangle = |\alpha\rangle_a |0\rangle_b$. At the first beam splitter, BS1, we have the transformation

$$|\alpha\rangle_a |0\rangle_b \xrightarrow{\text{BS1}} |\alpha/\sqrt{2}\rangle_a |i\alpha/\sqrt{2}\rangle_b \tag{2.3}$$

just as we expect for a classical field. The phase shifter in the a beam causes the transformation

$$|\alpha/\sqrt{2}\rangle_a |i\alpha/\sqrt{2}\rangle_b \xrightarrow{\text{PS}} |\alpha e^{i\phi_a}/\sqrt{2}\rangle_a |i\alpha/\sqrt{2}\rangle_b. \tag{2.4}$$

We now apply the operator $\hat{U}_{K,a}$ of Eq. (2.2):

$$\hat{U}_{K,a}\left(\frac{\pi}{2}\right) |\alpha e^{i\phi_a}/\sqrt{2}\rangle_a |i\alpha/\sqrt{2}\rangle_b = \exp\left[-i\frac{\pi}{2}(a^\dagger a)^2\right] |\alpha e^{i(\phi_a+\pi/2)}/\sqrt{2}\rangle_a |i\alpha/\sqrt{2}\rangle_b. \tag{2.5}$$

At this point we choose to set the phase shift $\phi_a = -\pi/2$, thus compensating for the effect of the second factor in $\hat{U}_{K,a}(\pi/2)$. As shown some time ago by Yurke and Stoler [25], for an arbitrary coherent state $|\beta\rangle$,

$$\exp\left[-i\frac{\pi}{2}(\hat{a}^\dagger \hat{a})^2\right]|\beta\rangle = \frac{1}{\sqrt{2}}[e^{-i\pi/4}|\beta\rangle + e^{i\pi/4}|-\beta\rangle], \tag{2.6}$$

a form of Schrödinger-cat state usually known as the Yurke-Stoler state. Applying this result to Eq. (2.5) we obtain

$$\frac{1}{\sqrt{2}}[e^{-i\pi/4}|\alpha/\sqrt{2}\rangle_a + e^{i\pi/4}|-\alpha/\sqrt{2}\rangle_a]|i\alpha/\sqrt{2}\rangle_b. \tag{2.7}$$

The second beam splitter performs the transformations

$$|\alpha/\sqrt{2}\rangle_a |i\alpha/\sqrt{2}\rangle_b \xrightarrow{\text{BS2}} |0\rangle_a |i\alpha\rangle_b,$$
$$|-\alpha/\sqrt{2}\rangle_a |i\alpha/\sqrt{2}\rangle_b \xrightarrow{\text{BS2}} |-\alpha\rangle_a |0\rangle_b, \tag{2.8}$$

and thus the output state of the nonlinear interferometer is

$$|\text{out}\rangle = |\alpha :: 0\rangle_{a,b} \equiv \frac{1}{\sqrt{2}}[|0\rangle_a |i\alpha\rangle_b + i|-\alpha\rangle_a |0\rangle_b], \tag{2.9}$$

where an overall irrelevant $e^{-i\pi/4}$ factor has been dropped [26]. We have introduced the symbol $|\alpha :: 0\rangle_{a,b}$ in an obvious way to represent the entanglement of a coherent state and the vacuum as defined by Eq. (2.9). In terms of the number basis we have

$$|\alpha :: 0\rangle_{a,b} = \frac{1}{\sqrt{2}} e^{-|\alpha|^2/2} \sum_{N=0}^{\infty} \frac{\alpha^N}{\sqrt{N!}} (-1)^N \times [|N\rangle_a |0\rangle_b + e^{i\Phi_N}|0\rangle_a |N\rangle_b], \tag{2.10}$$

where $\Phi_N = -(N+1)\pi/2$ and where an irrelevant overall $e^{i\pi/2}$ factor has been dropped. We may deduce from Eq. (2.10) that for the input number states $|\text{in}\rangle = |N\rangle_a |0\rangle_b$ the output state will be the MES,

$$|\text{out}\rangle = \frac{1}{\sqrt{2}}(-1)^N[|N\rangle_a |0\rangle_b + e^{i\Phi_N}|0\rangle_a |N\rangle_b] = (-1)^N |N :: 0\rangle_{a,b}^{\Phi_N}, \tag{2.11}$$

Eq. (1.1) apart from an irrelevant overall phase factor. Clearly, the result in Eq. (2.10) is a superposition of these MES weighted with the probability amplitudes of a coherent state. By extension, for any input state of the form

$$|\text{in}\rangle = \sum_N c_N |N\rangle_a |0\rangle_b, \quad \sum_N |c_N|^2 = 1, \tag{2.12}$$

the output will be

$$|\text{out}\rangle = \frac{1}{\sqrt{2}} \sum_N c_N (-1)^N [|N\rangle_a |0\rangle_b + e^{i\Phi_N}|0\rangle_a |N\rangle_b]. \tag{2.13}$$

For the sake of completeness we state the result of inputing arbitrary coherent states $|\alpha\rangle$ and $|\beta\rangle$ in modes a and b, respectively, i.e., $|\text{in}\rangle = |\alpha\rangle_a |\beta\rangle_b$. Retaining the choice of phase $\phi_a = -\pi/2$, it is straightforward to obtain the output state

$$|\text{out}\rangle = |\alpha :: \beta\rangle_{a,b} \equiv \frac{1}{\sqrt{2}}[|i\beta\rangle_a |i\alpha\rangle_b + i|-\alpha\rangle_a |\beta\rangle_b], \tag{2.14}$$

an entanglement of coherent states. Entangled coherent states have been much discussed in the literature, especially in connection with applications in quantum information theory [18,27]. Expressed in terms of numbers states,

$$|\alpha :: \beta\rangle_{a,b} = \exp[-(|\alpha|^2 + |\beta|^2/2] \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{\alpha^N \beta^M}{\sqrt{N!M!}} \times (-1)^N |N :: M\rangle_{a,b}^{\Phi_{NM}}, \tag{2.15}$$

where $\Phi_{NM} = -\pi(N-M+1)/2$ and where again an irrelevant overall $e^{i\pi/2}$ factor has been dropped. Thus we have a superposition of the entangled states $|N :: M\rangle_{a,b}^{\Phi_{NM}}$ of Eq. (1.2). From this result we deduce that the input state $|N\rangle_a |M\rangle_b$ results in the output state $|N :: M\rangle_{a,b}^{\Phi_{NM}}$ apart from an irrelevant phase factor.

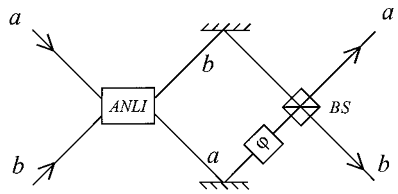

FIG. 2. A Mach-Zehnder interferometer to measure the phase φ where the first beam splitter has been replaced by an ALNI.

III. APPLICATION TO INTERFEROMETRY

We now suppose that our ANLI in its entirety replaces the first beam splitter in an otherwise ordinary MZI being used for measuring the phase difference between its two arms. This modified MZI is depicted in Fig. 2. The boxes containing φ_1 and φ_2 represent phase shifts in each of the arms of the MZI. The goal is to measure the phase shift difference $\varphi=\varphi_2-\varphi_1$.

To aid in the analysis of the second interferometer, it will prove convenient to introduce the Schwinger [9] realization of the angular momentum operators in terms of the set of Bose operators representing the beams of the interferometer. The usefulness of the description of interferometers in terms of the angular momentum operators, or equivalently, the su(2) Lie algebra, is well established [6,20]. So, in terms of the a- and b-mode operators, the angular momentum operators are

$$\hat{J}_1=\frac{1}{2}(\hat{a}^\dagger\hat{b}+\hat{a}\hat{b}^\dagger),\quad \hat{J}_2=\frac{1}{2i}(\hat{a}^\dagger\hat{b}-\hat{a}\hat{b}^\dagger),$$

$$\hat{J}_3=\frac{1}{2}(\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}) \tag{3.1}$$

and they satisfy the su(2) algebra $[\hat{J}_i,\hat{J}_j]=i\varepsilon_{ijk}\hat{J}_k$. The Casimir operator, the square of the angular momentum, may be written as

$$\hat{J}_1^2+\hat{J}_2^2+\hat{J}_3^2=\hat{J}_0(\hat{J}_0+1),\quad \hat{J}_0=\tfrac{1}{2}(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}), \tag{3.2}$$

where the operator $\hat{J}_0$, half the total photon number, commutes with the operator of Eq. (3.1), i.e., $[\hat{J}_0,\hat{J}_i]=0$, $i=1$, 2, 3. The number states $|N\rangle_a|M\rangle_b$ correspond to the angular momentum states $|j,m\rangle$ provided $j=\frac{1}{2}(N+M)$ and $m=\frac{1}{2}(N-M)$ where $\hat{J}_3|j,m\rangle=m|j,m\rangle$ and $\hat{J}_0|j,m\rangle=j|j,m\rangle$. In terms of the angular momentum states Eqs. (1.1) and (1.2), respectively, read

$$|N::0\rangle_{a,b}^{\Phi_N}=\frac{1}{\sqrt{2}}[|j,j\rangle+e^{i\Phi_{2j}}|j,-j\rangle],\quad j=N/2 \tag{3.3}$$

and

$$|N::M\rangle_{a,b}^{\Phi_{NM}}=\frac{1}{\sqrt{2}}[|j,m\rangle+e^{i\Phi_{j+m,j-m}}|j,-m\rangle], \tag{3.4}$$

In terms of the operators of Eq. (3.1), the beam splitter transformation has the form $\hat{U}_{\rm BS}=\exp[i\pi J_1/2]$. If φ_1 and φ_2, respectively, represent the phase shifts in the a and b arms of the interferometer as shown in Fig. 2, then we may write $\exp(i\varphi_1\hat{a}^\dagger\hat{a})\exp(i\varphi_2\hat{b}^\dagger\hat{b})=\exp[i(\varphi_1+\varphi_2)\hat{J}_0]\exp[i(\varphi_1-\varphi_2)\hat{J}_3]$. The first factor contains only the total phase shift and hence contributes another overall irrelevant phase factor (so we drop this factor) while the second factor contains the phase difference. Thus the operator representing the relative phase difference between the two arms is $\hat{U}_{\rm PD}=\exp(-i\varphi\hat{J}_3)$, where $\varphi=\varphi_2-\varphi_1$.

In standard interferometry, phase measurements are obtained by photocounting on the output beams and subtracting, i.e., measuring $\langle\hat{J}_3\rangle$. (Of course, by adding one can also obtain $\langle\hat{J}_0\rangle$.) However, such a procedure will not work in the case of the MES: one finds that $\langle\hat{J}_3\rangle=0$. To circumvent this problem, we adopt the suggestion of Bollinger *et al.* [2] of measuring the parity of one of the output beams [28]. Choosing the output b mode, this amounts to measuring the operator

$$\hat{O}\equiv(-1)^{\hat{b}^\dagger\hat{b}}=\exp[i\pi(\hat{J}_0-\hat{J}_3)]. \tag{3.5}$$

The technique amounts to a direct detection of the photon count at the output b mode and raising -1 to that power. This is equivalent to measuring all the moments of the number operator $\hat{b}^\dagger\hat{b}$. Clearly, photon detectors with resolutions at the level of a single photon will be required but we hasten to point out that the same requirement must also hold for standard measurements for $\langle\hat{J}_3\rangle$. Of course, all that really needs to be determined is the parity, not necessarily the value of $\hat{b}^\dagger\hat{b}$. This allows us to relax the requirement for single-photon-resolving photodetectors provided we find devices able to resolve parity without the need for high-resolution photon detection. We shall address this issue in Sec. IV where we show that the SNLI is capable of performing such a task.

Thus, excluding the nonlinear interferometer replacing the first beam splitter in Fig. 2, the rest of the Mach-Zehnder interferometer is represented by the operator

$$\hat{U}_{\rm MZI}=\hat{U}_{\rm Bs}\hat{U}_{\rm PD}=\exp\left(i\frac{\pi}{2}\hat{J}_1\right)\exp(-i\varphi\hat{J}_3). \tag{3.6}$$

If $|\text{out}\rangle$ represents the output of the NLI, then the expectation value of $\hat{O}$ is given by

$$\begin{aligned}\langle\hat{O}\rangle&=\langle\text{out}|\hat{U}_{\rm MZI}^\dagger\hat{O}\hat{U}_{\rm MZI}|\text{out}\rangle\\&=\langle\text{out}|e^{i\pi\hat{J}_0}e^{i\varphi\hat{J}_3}e^{-i(\pi/2)\hat{J}_1}e^{-i\pi\hat{J}_3}e^{i(\pi/2)\hat{J}_1}e^{-i\varphi\hat{J}_3}|\text{out}\rangle\\&=\langle\text{out}|e^{i\pi\hat{J}_0}e^{i\varphi\hat{J}_3}e^{i\pi\hat{J}_2}e^{-i\varphi\hat{J}_3}|\text{out}\rangle,\end{aligned} \tag{3.7}$$

where in the last line we have used the results

$$e^{-i(\pi/2)\hat{J}_1}\hat{J}_3 e^{i(\pi/2)\hat{J}_1} = -\hat{J}_2,$$

$$e^{-i(\pi/2)\hat{J}_1} e^{-i\pi\hat{J}_3} e^{i(\pi/2)\hat{J}_1} = e^{i\pi\hat{J}_2}. \tag{3.8}$$

We now suppose that the output of the NLI is the state $|N::0\rangle_{a,b}^{\Phi_N}$ of Eq. (1.1). In this case, using the result [6]

$$\exp(i\pi\hat{J}_2)|N\rangle_a|M\rangle_b = (-1)^N|M\rangle_a|N\rangle_b, \tag{3.9}$$

we obtain

$$\langle\hat{O}\rangle_{N::0} = \tfrac{1}{2}i^N[e^{i(N\varphi+\Phi_N)} + (-1)^N e^{-i(N\varphi+\Phi_N)}]$$
$$= \begin{cases} (-1)^{N/2}\cos(N\varphi+\Phi_N), & N \text{ even} \\ (-1)^{(N+1)/2}\sin(N\varphi+\Phi_N), & N \text{ odd}. \end{cases} \tag{3.10}$$

The phase uncertainty for the state $|N::0\rangle_{a,b}^{\Phi_N}$, for N either even or odd, is found to be

$$\Delta\varphi = \Delta\hat{O} \bigg/ \left|\frac{\partial\langle\hat{O}\rangle}{\partial\varphi}\right| = \frac{1}{N}, \tag{3.11}$$

exactly the Heisenberg limit of sensitivity, where $\Delta\hat{O} = (1 - \langle\hat{O}\rangle^2)^{1/2}$ owing to the fact that $\hat{O}^2 = \hat{I}$. Note that the result is independent of the phase difference φ.

Of course, to obtain this ultimate limit of sensitivity one must be able to generate number states $|N\rangle$, to inject into the NLI; and the higher N the better. Some progress has been made in that direction [29] but there is as yet no available source of optical number states for arbitrary N. This begs the question: Are there other states, perhaps more conventional ones, such as the coherent states, that may be used to obtain sensitivity at the level $1/N$ or $1/\bar{N}$, $\bar{N}$ being an average photon number? We have addressed this issue in the earlier work [8,12] in regard to different proposals for generating MES and have found the answer to be in the affirmative. Indeed, we have already noted that with an input coherent state to the ANLI, an entangled coherent state $|\alpha::0\rangle_{a,b}$, a superposition of the states $|N::0\rangle_{a,b}^{\Phi_N}$, is obtained at the output. So we now consider the phase uncertainty in the case of the entangled coherent state. In order to calculate ${}_{a,b}\langle\alpha::0|\hat{O}|\alpha::0\rangle_{a,b}$ we need the following easily derived result:

$$\exp(i\pi\hat{J}_0)\exp(i\varphi\hat{J}_3)\exp(i\pi\hat{J}_2)\exp(-i\varphi\hat{J}_3)|\alpha::0\rangle_{a,b}$$
$$= \frac{1}{\sqrt{2}}[|-\alpha e^{i\varphi}\rangle_a|0\rangle_b + i|0\rangle_a|i\alpha e^{-i\varphi}\rangle_b], \tag{3.12}$$

where Eq. (3.7) has been used. We thus obtain

$$\langle\hat{O}\rangle = {}_{a,b}\langle\alpha::0|\hat{O}|\alpha::0\rangle_{a,b}$$
$$= e^{-\bar{N}}[1 + e^{\bar{N}\cos\varphi}\sin(\bar{N}\sin\varphi)], \tag{3.13}$$

and

$$\frac{d\langle\hat{O}\rangle}{d\varphi} = \bar{N}e^{-\bar{N}(1-\cos\varphi)}\cos(\varphi + \bar{N}\sin\varphi). \tag{3.14}$$

The phase uncertainty may again be calculated according to Eq. (3.9). This time $\Delta\varphi$ will clearly depend on φ. Suppose we take $\varphi=0$ as for a balanced interferometer. In this case we find that $\langle\hat{O}\rangle = e^{-\bar{N}}$ and $d\langle\hat{O}\rangle/d\varphi = \bar{N}$ and thus

$$\Delta\varphi = \sqrt{1-e^{-2\bar{N}}}/\bar{N}, \tag{3.15}$$

which clearly goes over to $1/\bar{N}$ for large $\bar{N}$. (In fact, this result is valid for $\varphi = 2\pi k$, $k=0,1,2,\ldots$.) But suppose φ deviates from zero by only the small amount $\delta \ll 1$ as anticipated for a weak signal in the interferometer. Letting $\varphi=\delta \ll 1$ we then have

$$\langle\hat{O}\rangle \approx e^{-\bar{N}} + e^{-\bar{N}\delta^2/2}\sin(\bar{N}\delta). \tag{3.16}$$

If $\bar{N}$ is large but $\bar{N}\delta^2/2$ still small we have $\langle\hat{O}\rangle \approx \sin(\bar{N}\delta)$ and thus $\Delta\varphi \approx 1/\bar{N}$. In Fig. 3 we plot the exact phase uncertainty $\Delta\varphi$ (solid line) vs $\bar{N}$ for three values of φ a bit removed from zero: (a) $\varphi=\pi/45$, (b) $\varphi=\pi/18$, and (c) $\varphi=\pi/7$. For the sake of comparison we also include the curve for the Heisenberg limit, $\Delta\varphi_{\mathrm{HL}} = 1/\bar{N}$ (the dashed lines) and of the standard quantum limit given by $\Delta\varphi_{\mathrm{SQL}} = 1/(\sqrt{\bar{N}})$ (the dot-dashed lines). We note from Fig. 3(a) for $\varphi=\pi/45$ that apart from the spike at around $\bar{N}\approx 21$, the curve we obtain follows very closely that of the Heisenberg-limit curve. The spike occurs simply because of the periodic functions in $\langle\hat{O}\rangle$ and its derivative. In Fig. 3(b) for the larger phase difference $\varphi=\pi/18$ we find more spikes but nevertheless the same general trend that for a wide range of $\bar{N}$ our curve is close to the Heisenberg limit and considerably below that of the standard quantum limit. For the phase difference as high as $\varphi=\pi/7$, however, the correspondence to the Heisenberg limit breaks down and generally even exceeds that of the standard quantum limit as indicated in Fig. 3(c). But as we have said this breakdown may be of no practical consequence if one starts with a balanced interferometer, and if the expected signals are very weak causing only small disturbances. Finally, it is worth mentioning here that there is really an advantage in using coherent states rather than number states as inputs to the ANLI. The former could be applied essentially continuously thus yielding a higher rate of measurement on the output beams, whereas in the latter case the same number state must be prepared repeatedly, if it can be produced at all, necessarily decreasing the rate of measurement.

IV. THE SYMMETRIC NLI

In Fig. 4 we sketch the SNLI, an MZI with Kerr media in both arms (along with the phase shifters as per the discussion in Sec. II). We follow through the case of the input states $|\mathrm{in}\rangle = |\alpha\rangle_a|0\rangle_b$. After the first beam splitter we again have $|\alpha/\sqrt{2}\rangle_a|i\alpha/\sqrt{2}\rangle_b$. The phase shift operators $\exp(i\phi_a\hat{a}^\dagger\hat{a})\exp(i\phi_b\hat{b}^\dagger\hat{b})$ produce $|\alpha e^{i\phi_a}/\sqrt{2}\rangle_a|i\alpha e^{i\phi_b}/\sqrt{2}\rangle_b$.

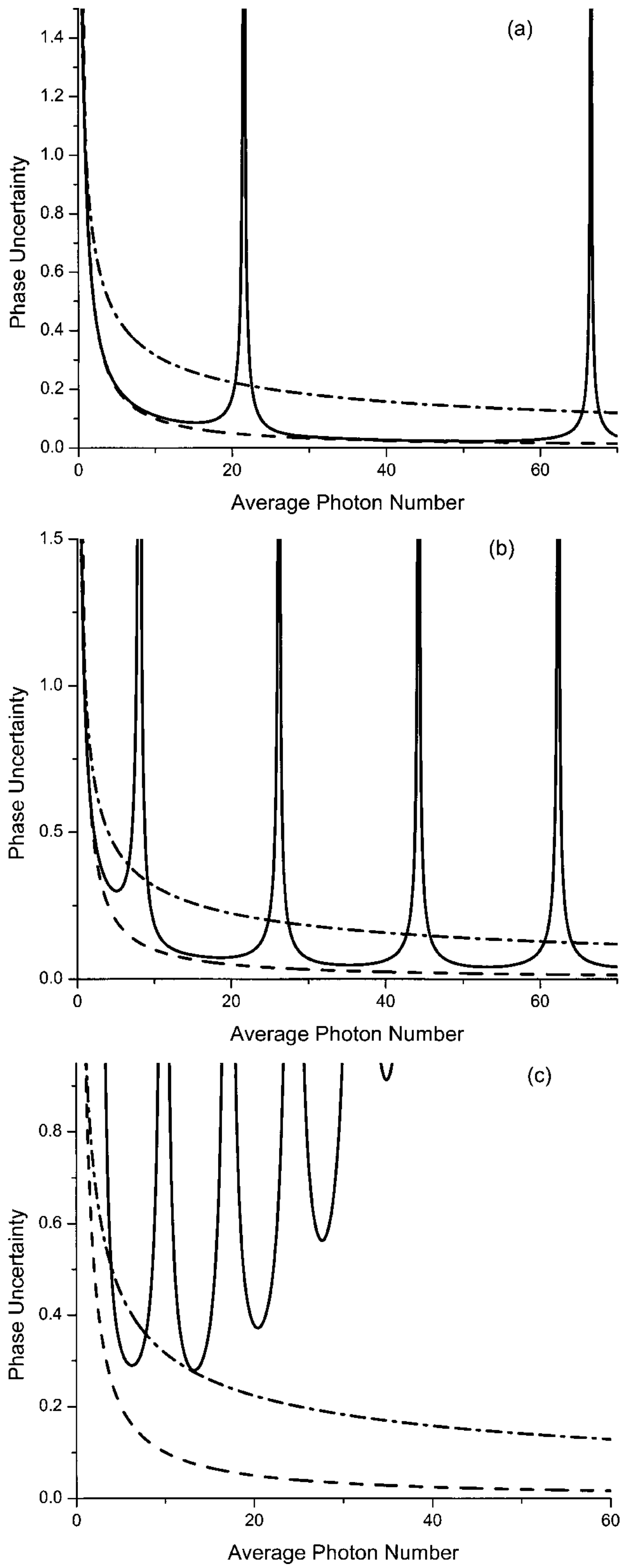

FIG. 3. The phase uncertainty $\Delta\varphi$ vs the average photon number $\bar{N}=|\alpha|^2$ for an input coherent state $|\alpha\rangle$ in the a mode and a vacuum state in the b mode of the ANLI for (a) $\varphi=\pi/45$, (b) $\varphi=\pi/18$, and (c) $\varphi=\pi/7$. The solid line is our $\Delta\varphi$ obtained from Eqs. (3.13) and (3.14), the dashed line for the Heisenberg limit $\Delta\varphi_{\mathrm{HL}}=1/\bar{N}$, and the dash-dot line is for the standard quantum limit $\Delta\varphi_{\mathrm{SQL}}=1/\sqrt{\bar{N}}$.

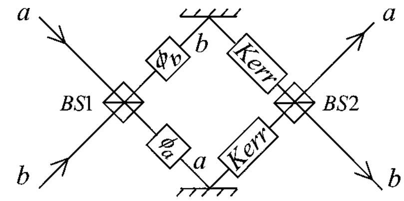

FIG. 4. The symmetric nonlinear interferometer (SNLI). The required phase shifts are $\phi_a=-\pi/2=\phi_b$ as explained in the text.

With the choices $\phi_a=-\pi/2=\phi_b$, the Kerr media, described by the operator $\exp(-i\kappa\hat{a}^{\dagger 2}\hat{a}^2)\exp(-i\kappa\hat{b}^{\dagger 2}\hat{b}^2)$, with $\kappa=\pi/2$, produce

$$\frac{1}{2}[e^{-i\pi/2}|\alpha/\sqrt{2}\rangle_a|i\alpha/\sqrt{2}\rangle_b+e^{i\pi/2}|-\alpha/\sqrt{2}\rangle_a|-i\alpha/\sqrt{2}\rangle_b$$
$$+|-\alpha/\sqrt{2}\rangle_a|i\alpha/\sqrt{2}\rangle_b+|\alpha/\sqrt{2}\rangle_a|-i\alpha/\sqrt{2}\rangle_b]. \quad (4.1)$$

Lastly, the second beam splitter produces the output state

$$|\mathrm{out}\rangle=\frac{1}{2}[-i|i\alpha\rangle_a|0\rangle_b+i|0\rangle_a|-i\alpha\rangle_b$$
$$+|-\alpha\rangle_a|0\rangle_b+|\alpha\rangle_a|0\rangle_b]. \quad (4.2)$$

We may rewrite this more transparently as

$$|\mathrm{out}\rangle=\frac{1}{2}[-i|0\rangle_a(|i\alpha\rangle_b-|-i\alpha\rangle_b)+(|\alpha\rangle_a+|-\alpha\rangle_a)|0\rangle_b]. \quad (4.3)$$

Coherent state superpositions of the form $|\beta\rangle\pm|-\beta\rangle$ are just single-mode examples of the Schrödinger-cat states, the even $(+)$ and odd $(-)$ coherent states [30]. Evidently, the symmetric NLI generates an entanglement of the vacuum with the even and odd coherent states. Suppose that a photodetector is placed in the output of mode a, with no detector in the output of mode b. If the vacuum is detected (i.e., a "no count" detection) in the output a mode, the b mode is projected into the odd coherent state $|i\alpha\rangle-|-i\alpha\rangle$. Conversely, if the detector is placed in output mode b and if a vacuum state is detected then a mode is projected into an even coherent state $|\alpha\rangle+|-\alpha\rangle$ (apart from the normalizations). Notice that only one detector (in either output beam) is used to make these projections. So we have found yet another procedure for generating single-mode Schrödinger-cat states.

In terms of the number states the output state of Eq. (4.3) may be written as

$$|\mathrm{out}\rangle=e^{-|\alpha|^2/2}\left[-i\sum_{N,\mathrm{odd}}^{\infty}\frac{(i\alpha)^N}{\sqrt{N!}}|0\rangle_a|N\rangle_b\right.$$
$$\left.+\sum_{N,\mathrm{even}}^{\infty}\frac{\alpha^N}{\sqrt{N!}}|N\rangle_a|0\rangle_b\right]. \quad (4.4)$$

The probability of detecting the vacuum in mode b is

$$\mathrm{Prob}(|0\rangle_b)=\langle\mathrm{out}|0\rangle_b\,{}_b\langle 0|\mathrm{out}\rangle=e^{-|\alpha|^2}\sum_{N,\mathrm{even}}^{\infty}\frac{|\alpha|^{2N}}{N!}, \quad (4.5)$$

while the probability of detecting the vacuum in mode a is

$$\mathrm{Prob}(|0\rangle_a)=\langle \mathrm{out}|0\rangle_a\ {}_a\langle 0|\mathrm{out}\rangle=e^{-|\alpha|^2}\left(\sum_{N,\mathrm{odd}}^{\infty}\frac{|\alpha|^{2N}}{N!}+1\right). \tag{4.6}$$

In this last expression the $+1$ appears in the parentheses because the vacuum state $|0\rangle_a$ appears not only correlated with all the odd number states of mode *b*, but also with the vacuum state $|0\rangle_b$. Of course, for large $|\alpha|$ that contribution is nil and to a good approximation $\mathrm{Prob}(|0\rangle_a)\approx\frac{1}{2}\approx\mathrm{Prob}(|0\rangle_b)$.

From Eq. (4.4) we deduce that if the input to the SNLI is $|\mathrm{in}\rangle=|N\rangle_a|0\rangle_b$ the output will be

$$|\mathrm{out}\rangle=\begin{cases}|N\rangle_a|0\rangle_b, & N \text{ even}\\ -i^{N+1}|0\rangle_a|N\rangle_b, & N \text{ odd.}\end{cases} \tag{4.7}$$

Evidently, the SNLI acts as a filter with respect to parity. All one needs to do is place photodetectors in *both* the output modes, and by noting which detector fires (and/or which does not fire) the parity can be determined. It will not be necessary to employ detectors with resolutions at the level of a single photon. The action of the SNLI is very similar to that of the nonlinear four-wave mixing device described by Yurke and Stoler [7], which produces essentially identical output states for the given inputs as the symmetric NLI. However, the former requires two competing nonlinear interactions, a cross-Kerr and four-wave mixing, while the latter requires only single-mode Kerr interactions. Evidently, the symmetric NLI is capable of performing the parity measurements required for the interferometric measurements discussed in Sec. III.

For the sake of completeness, we state the output of the symmetric NLI with coherent state inputs in both modes. If $|\mathrm{in}\rangle=|\alpha\rangle_a|\beta\rangle_b$ then

$$|\mathrm{out}\rangle=\tfrac{1}{2}[-i|i\beta\rangle_a|i\alpha\rangle_b+i|-i\beta\rangle_a|-i\alpha\rangle_b +|\alpha\rangle_a|-\beta\rangle_b+|-\alpha\rangle_a|\beta\rangle_b]. \tag{4.8}$$

Thus we obtain a four-component entanglement of coherent states of the two modes. In terms of the number states we have

$$|\mathrm{out}\rangle=\frac{1}{2}e^{-(|\alpha|^2+|\beta|^2)/2}\sum_{M=0}^{\infty}\sum_{N=0}^{\infty}\frac{\alpha^N\beta^M}{\sqrt{N!M!}} \times\{i^{M+N+1}[-1+(-1)^{M+N}]|M\rangle_a|N\rangle_b +[(-1)^N+(-1)^M]|N\rangle_a|M\rangle_b\}. \tag{4.9}$$

Thus for input number states $|N\rangle_a|M\rangle_b$ we obtain the output state

$$|\mathrm{out}\rangle=\tfrac{1}{2}\{i^{M+N+1}[-1+(-1)^{M+N}]|M\rangle_a|N\rangle_b +[(-1)^N+(-1)^M]|N\rangle_a|M\rangle_b\}. \tag{4.10}$$

If *M* and *N* are both even or both odd, then apart from irrelevant phase factors we have $|N\rangle_a|M\rangle_b \xrightarrow{\mathrm{SNLI}} |N\rangle_a|M\rangle_b$; the photons do not exchange modes. But if one of them is even and the other odd we have $|N\rangle_a|M\rangle_b \xrightarrow{\mathrm{SNLI}} |M\rangle_a|N\rangle_b$; the photons collectively exchange modes. In this case it will not be possible to determine the parity of the states of the individual modes, only that they are different. In summary, if either *M* or *N* is zero we can determine the parity; if they are nonzero we can determine if they have different or the same parity if they exchange modes or not. We cannot determine the parity of the individual number states except in the case where one of the input modes is in the vacuum.

V. CONCLUSIONS

In this paper we have studied nonlinear interferometers with Kerr media in one (the ANLI) or both (the SNLI) arms. In the case of the ANLI, maximally entangled photonic states are generated if one of the input states is the vacuum. The principal advantage of using this device is that the MES may be generated unitarily and therefore deterministically; no projective measurements need to be performed. The alternatives with either nonlinear or linear optical devices require numerous projective measurements and are not deterministic. Another advantage of our proposal over those requiring multiple projective measurements [1,15] is that the former does not require number state inputs as does the latter and thus readily available coherent states may be used. The principal disadvantage is that large nonlinearities will be required. But on the other hand, there is the likelihood that such nonlinearities will eventually become available through the techniques of electromagnetically induced transparency. We have shown that the device could be used to replace the first beam splitter of a Mach-Zehnder interferometer to achieve high-resolution phase-shift measurements with sensitivity at the Heisenberg limit. With MES in the interferometer, it is necessary to perform parity measurements on one of the output beams. But the SNLI acts as a filter with respect to parity and therefore could be used as a detector in the interferometry experiments using MES or in any other experiment where photon parity must be measured.

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