Dynamical symmetries of time-periodic Hamiltonians

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The interaction of atoms, molecules, crystals, and nanotubes with time-periodic laser fields can lead to high-order dynamical symmetries (DS's). Here we employ group theoretical methods to study the DS-related properties (quantum numbers, nonaccidental degeneracies) of quantum systems possessing high-order DS's. As explicit examples we take finite-order rotation symmetry in point and plane groups, and a circularly polarized laser field. We find that nonaccidental degeneracies induced by spatial and time-reversal symmetries may not be lifted inside the laser field. A general result of this work is that the time-evolution operator needs to be computed *only* up to $1/N$ [or, even, $1/(2N)$] of the optical cycle, where N is the order of the DS. This allows a substantial reduction of the computational effort required for studying the time-dependent dynamics of such systems.

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Symmetry plays an important role in atomic, molecular, and solid-state physics [1–3]. In particular, it provides the labeling of energy levels and their corresponding *stationary* eigenstates with ''good'' quantum numbers. Most importantly, it allows us to make *a priori* assertions on matrix elements and thus provides a general method for determining selection rules. For instance, the interpretation of electronic and vibrational spectra strongly benefits from such symmetry-based predictions [4]. As is well known, the states of atoms and molecules in time-periodic laser fields can be conveniently expanded in terms of Floquet [quasienergy (QE)] states [5,6]. Within this picture, harmonic generation spectroscopy is described as the transition between *time-dependent* (Floquet) states induced by suitable *time-dependent* operators [7]. The effects of Hamiltonian symmetry on the polarization and order of harmonics emitted in various theoretical setups have been discussed by several authors (see, e.g., Ref. [8]). In particular, the spatiotemporal symmetries [hereafter referred to as dynamical symmetries (DS's)] of the Floquet Hamiltonian have been shown to govern the selection rules for the harmonic generation spectra [7,9]; thus they fulfill a similar role to that discussed above for symmetry in ''ordinary'' spectroscopy.

The DS's are basic characteristics of the time-dependent quantum system under investigation, just as the spatial symmetries are in the stationary case. The corelation between the DS's and their field-free partners constitutes a group-theoretical-based tool to quantify, even for *nonperturbative* intensities, to what extent the time-dependent interaction reduces the symmetry of the stationary system. A systematic study of DS-related properties of Floquet Hamiltonians is, therefore, not only of fundamental interest from the group theoretical point of view. Rather, it should encourage researchers to classify and explore the fingerprints of (high-order) DS's in field-induced physical phenomena, such as dynamical tunneling and localization, photoionization and stabilization of atoms and oriented molecules, generation and coherent control of photocurrents in spatially periodic materials, and so on. It is the purpose of this work to explore the DS-related properties of quantum systems possessing (high-order) DS's. To do so, we follow a similar path to that used when group theory is applied to the study of symmetry-related properties of stationary systems. Specifically, we first identify the DS's comprising the (unitary) DS group of a given time-dependent system. Secondly, we assign ''good'' quantum numbers to the QE states. Thirdly, we inquire whether the QE spectrum possesses nonaccidental degeneracies, first due to the structure of the (unitary) DS group and subsequently due to the (possible) existence of nonunitary DS's. Finally, we show how DS can be employed to reduce the computational effort associated with integrating the time-dependent problem. As mentioned above, the utilization of DS's to evaluate matrix elements (formulating selection rules) has been introduced and applied before [7,9], and therfore it will not be presented here. Being embedded in group theory, the ideas and methods employed in this work can be applied to Floquet Hamiltonians possessing *various* DS groups, such as those associated with field-free point, plane, layer, space, and rod groups. However, to make our exposition both coherent and transparent we choose to study specific examples of systems with high-order DS's, in particular those associated with finite-order rotation symmetry in point and plane groups, and a circularly polarized laser field. The generality or possible generalization of our results is briefly discussed.

Our starting point is the following field-free Hamiltonian, describing an electron's motion in a hindered quantum ring possessing N-fold rotational symmetry:

$$\hat{H}(\mathbf{r})=\frac{\hat{\mathbf{p}}^2}{2m}+V_N(\mathbf{r}),\quad [\hat{H}(\mathbf{r}),\hat{C}_N]=0,$$

$$\hat{C}_N=\left(\varphi\rightarrow\varphi+\frac{2\pi}{N}\right). \tag{1}$$

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The single-electron eigenstates of the quantum ring (1) can be classified with respect to the rotation operator $\hat{C}_N$,

$$\hat{H}(\mathbf{r})\psi_{E,p}(\mathbf{r})=E\psi_{E,p}(\mathbf{r}),$$
$$\hat{C}_N\psi_{E,p}(\mathbf{r})=e^{+i(2\pi/N)p}\psi_{E,p}(\mathbf{r}),\quad p=0,1,\ldots,N-1. \tag{2}$$

It is well known that there are nonaccidentally degenerate energy levels in the system (1). In order to characterize them, one distinguishes between two cases. If, in addition to the N-fold rotation symmetry $\hat{C}_N$, the Hamiltonian is invariant under the reflection symmetry $\hat{\sigma}_\varphi=(\varphi\rightarrow-\varphi)=(y\rightarrow-y)$, one obtains the familiar result reflected by the character table of the point group $\mathcal{C}_{Nv}$ [4]: For $p\neq\{0,N/2\}$, $\hat{\sigma}_\varphi\psi_{E,p}$ and $\psi_{E,p}$ are degenerate, while for $p=\{0,N/2\}$, $\hat{\sigma}_\varphi\psi_{E,p}\propto\psi_{E,p}$. In the second case (for $[\hat{H}(\mathbf{r}),\hat{\sigma}_\varphi]\neq0$), one can make use of the time-reversal symmetry (TRS) $\hat{R}=(t\rightarrow-t,*)$ (for real Hamiltonians $[\hat{H}(\mathbf{r}),\hat{R}]=0$). Upon applying the TRS $\hat{R}$ onto $\psi_{E,p}$, the result now reflects the character table of the point group $\mathcal{C}_N$ [4]: For $p\neq\{0,N/2\}$, $\hat{R}\psi_{E,p}$ and $\psi_{E,p}$ are degenerate, while for $p=\{0,N/2\}$, $\hat{R}\psi_{E,p}\propto\psi_{E,p}$.

Let us add a circularly polarized laser field and enquire what happens to the energy levels of the quantum ring. In the presence of a time-periodic field, the field-free stationary states become QE states [5,6]. The Floquet Hamiltonian is given by

$$\hat{H}_f(\mathbf{r},t)=\hat{H}(\mathbf{r})+eE_0\rho\cos(\varphi-\omega t)-i\hbar\frac{\partial}{\partial t}, \tag{3}$$

where E_0 and $\omega=2\pi/T$ are the incident field strength and frequency, respectively. The field propagation direction in Eq. (3) is assumed to be parallel to the N-fold symmetry axis of the quantum ring. A look at the Floquet Hamiltonian (3) reveals that it is invariant under the following Nth order unitary DS [10]:

$$\hat{P}_N=\left(\varphi\rightarrow\varphi+\frac{2\pi}{N},t\rightarrow t+\frac{T}{N}\right). \tag{4}$$

The DS operator acts on Floquet states analogously to the action of symmetry operators on stationary states. Thus, a complete set of QE states can be found, such that

$$\hat{\mathcal{H}}_f(\mathbf{r},t)\Phi_{\varepsilon,p}(\mathbf{r},t)=\varepsilon\Phi_{\varepsilon,p}(\mathbf{r},t),$$
$$\hat{P}_N\Phi_{\varepsilon,p}(\mathbf{r},t)=e^{+i2\pi p/N}\Phi_{\varepsilon,p}(\mathbf{r},t),$$
$$p=0,1,\ldots,N-1. \tag{5}$$

Recently, Ceccherini *et al.* have utilized such an assignment of quantum numbers to analyze the sidebands in high-harmonic generation spectra calculated for benzene [9].

Having identified the *unitary* DS associated with the quantum-ring–field system (3) and assigned ''good'' quantum numbers to its QE states, we follow the reasoning used in the field-free case and ask whether there are nonaccidentally degenerate Floquet states in this system. One would not expect the existence of nonaccidental degeneracies, because the circularly polarized field breaks both the reflection symmetry and the TRS utilized above to characterize such degeneracies in the field-free system. Let us *prove* with the help of group theory why this is indeed the case. We first consider the case where the field-free Hamiltonian does not possess the reflection symmetry $\hat{\sigma}_\varphi$. In this case the (full) DS group of the system, $\mathcal{G}_N\equiv\{\hat{P}_N,\hat{P}_N^2,\ldots,\hat{P}_N^{N-1},\hat{P}_N^N=\mathbf{I}\}$, is cyclic and *unitary*. Hence, no nonaccidental degeneracies can be found in the QE spectrum of the system (3). If, on the other hand, $[\hat{H}(\mathbf{r}),\hat{\sigma}_\varphi]$ then one finds that the Floquet Hamiltonian is invariant under the following generalized TRS (GTRS):

$$\hat{R}_{\varphi t}=(\varphi\rightarrow-\varphi,t\rightarrow-t,*)=\hat{R}\cdot\hat{\sigma}_\varphi. \tag{6}$$

It is therefore required to check whether this additional *antiunitary* DS can introduce nonaccidental degeneracies into the QE spectrum. To this end, one has to resort to Wigner theory of antiunitary symmetries (Ref. [1], Chap. 26) and apply it to the nonunitary DS $\hat{R}_{\varphi t}$ and the unitary DS group $\mathcal{G}_N$. In practice, one has to examine the relation between the irreducible representation (irrep) $\Gamma_p(\hat{P}_N)$ and the complex conjugate one $\overline{\Gamma}_p(\hat{P}_N)\equiv\Gamma_p^*(\hat{R}_{\varphi t}^{-1}\hat{P}_N\hat{R}_{\varphi t})$. If $\Gamma_p(\hat{P}_N)$ and $\overline{\Gamma}_p(\hat{P}_N)$ are equivalent, that is their characters are equal, it is possible to find a similarity transformation B, such that $B^{-1}\Gamma_p(\hat{P}_N)B=\overline{\Gamma}_p(\hat{P}_N)$. In this case two possibilities arise, depending on the nature of the product B^*B. If $B^*B=\Gamma_p(\hat{R}_{\varphi t}^2)$ then no additional degeneracy is introduced [case (a)]. If, on the other hand, $B^*B=-\Gamma_p(\hat{R}_{\varphi t}^2)$, there is an additional doubling of the degeneracy, and $\Gamma_p(\hat{P}_N)$ always occurs twice [case (b)]. Finally, if $\Gamma_p(\hat{P}_N)$ and $\overline{\Gamma}_p(\hat{P}_N)$ are inequivalent irreps, there is an additional doubling of the degeneracy, and $\Gamma_p(\hat{P}_N)$ and $\overline{\Gamma}_p(\hat{P}_N)$ always occur together as a pair [case (c)]. Now, in contrast to the field-free case where $\hat{R}^{-1}\hat{C}_N\hat{R}=\hat{C}_N$, one readily finds that

$$\hat{R}_{\varphi t}^{-1}\hat{P}_N\hat{R}_{\varphi t}=\hat{P}_N^{-1}\Rightarrow\overline{\Gamma}_p(\hat{P}_N)=\Gamma_p^*(\hat{R}_{\varphi t}^{-1}\hat{P}_N\hat{R}_{\varphi t})$$
$$=\Gamma_p^*(\hat{P}_N^{-1})=\Gamma_p(\hat{P}_N) \tag{7}$$

[the last step is valid since $\Gamma_p(\hat{P}_N)$ is a one-dimensional (1D) irrep], meaning that all the irreps $\Gamma_p(\hat{P}_N),p=0,1\ldots,N-1$, belong to case (a) described above. Thus, unlike the role played by TRS in the field-free case (1), the GTRS $\hat{R}_{\varphi t}$ does not introduce nonaccidental degeneracies into the QE spectrum of the Floquet Hamiltonian (3).

One may suspect that nonaccidental degeneracies, in particular those that are associated with antiunitary DS's, cannot exist in systems with high-order DS's. Certainly, accidental degeneracies can occur at specific values of the field strength and intensity. For instance, the appearance of accidental degeneracies when model systems are subject to linearly polarized fields [such systems possess the second-order DS $\hat{P}_x=(x\rightarrow-x,t\rightarrow t+T/2)$] has been shown to play an important

role in coherent destruction of tunneling and even-harmonic generation [12]. Below we analyze the more intricate case of thin crystals in circularly polarized fields, where it is demonstrated that GTRS *can* induce nonaccidental degeneracies in the crystal modified energy bands (EB's). We mention that thin crystals in linearly polarized fields (invariant under the second-order DS $\hat{P}_x$) were studied by several authors (see, e.g., Ref. [13]). In particular, the symmetry of the modified EB's in this case was studied, e.g., in Ref. [14].

Let us consider a thin crystal (e.g., a 2D sheet of graphite) possessing the rotation $\hat{C}_N$ and reflection $\hat{\sigma}_\varphi$ symmetries. Upon shining the thin crystal with (monochromatic) circularly polarized laser field, propagating in parallel to the *N*-fold rotation axes, the single-electron Floquet-Bloch (FB) Hamiltonian is given by [$V_{cr}(\mathbf{r})$ stands for the lattice spatially-periodic potential]

$$\hat{\mathcal{H}}_f(\mathbf{r},t)=\frac{\hat{\mathbf{p}}^2}{2m}+V_{cr}(\mathbf{r})-\frac{eE_0}{m\omega}[\hat{p}_x\sin(\omega t)-\hat{p}_y\cos(\omega t)]-i\hbar\frac{\partial}{\partial t},$$

$$\hat{\mathcal{H}}_f(\mathbf{r},t)\Phi_{\varepsilon(\mathbf{k}),\mathbf{k}}(\mathbf{r},t)=\varepsilon(\mathbf{k})\Phi_{\varepsilon(\mathbf{k}),\mathbf{k}}(\mathbf{r},t). \tag{8}$$

Here the EB's of the field-free Hamiltonian have been transformed into the quasienergy bands (QEB's) $\varepsilon(\mathbf{k})$ of the FB Hamiltonian. It is, therefore, natural to investigate the symmetries of the QEB's and the relation between these symmetries and those of the EB's. For this purpose let us apply the DS operators $\hat{P}_N$ and $\hat{R}_{\varphi t}$ to the FB eigenstate $\Phi_{\varepsilon(\mathbf{k}),\mathbf{k}}(\mathbf{r},t)=e^{+i\mathbf{k}\cdot\mathbf{r}}\phi_{\varepsilon(\mathbf{k}),\mathbf{k}}(\mathbf{r},t)$ which gives

$$\hat{P}_N\Phi_{\varepsilon(\mathbf{k}),\mathbf{k}}(\mathbf{r},t)=e^{+i(C_N^{-1}\mathbf{k})\cdot\mathbf{r}}\phi_{\varepsilon(\mathbf{k}),\mathbf{k}}(C_N\mathbf{r},t+T/N)\equiv\Phi_{\varepsilon(\mathbf{k}),C_N^{-1}\mathbf{k}}(\mathbf{r},t), \tag{9}$$

and [$\sigma_{\bar{\varphi}}=(\varphi_k\to-\varphi_k+\pi)=(k_x\to-k_x),\varphi_k\equiv\arctan(k_y/k_x)$]

$$\hat{R}_{\varphi t}\Phi_{\varepsilon(\mathbf{k}),\mathbf{k}}(\mathbf{r},t)=e^{+i(-k_xx+k_yy)}\phi^*_{\varepsilon(\mathbf{k}),\mathbf{k}}(\sigma_\varphi\mathbf{r},-t)\equiv\Phi_{\varepsilon(\mathbf{k}),\sigma_{\bar{\varphi}}\mathbf{k}}(\mathbf{r},t). \tag{10}$$

This shows that other FB states are generated, carrying the same QE (since $\hat{P}_N$ and $\hat{R}_{\varphi t}$ are DS's of the FB Hamiltonian) but having transformed quasimomenta $\mathbf{k}$. Equation (9) shows that the *N*-fold rotation symmetry in real space is transformed to the QEB's, similarly to the known situation in EB's [2,3]. On the other hand, due to the GTRS $\hat{R}_{\varphi t}$, each reflection symmetry $\hat{\sigma}_\varphi=(y\to-y)$ in *real* space leads to the orthogonal reflection $\sigma_{\bar{\varphi}}=(k_x\to-k_x)$ in *reciprocal* space. For instance, consider a field-free Hamiltonian, the plane group of which is $p3m1[x]$ (the $[x]$ indicates that one of the reflection planes lies along the *x* axis). In this case it is the C_{6v} symmetry of the EB's which contains the $C_{3v}[x]$ point group of the plane group (in real space) and the rotated $C_{3v}[y]$ symmetry of the QEB's (in reciprocal space). This subgroup relation ''ensures'' that the application of the circularly polarized field cannot lead to a higher symmetry in the QEB's than that of the field-free EB's. The results of a similar analysis performed on the 17 plane groups and 13 families of rod groups will be presented elsewhere.

Next, let us examine whether, analogously to the common situation for EB's in thin crystals [2,3], there are nonaccidentally degenerate FB states at *specific* values of the quasimomentum $\mathbf{k}$. We discuss first the case of the $\Gamma(\mathbf{k}=0)$ point. For this value of the quasimomentum, the FB Hamiltonian (8) is equivalent to the Floquet Hamiltonian of the hindered quantum ring (3). Consequently, no nonaccidental degeneracies are possible for $\mathbf{k}=0$ [compare to the field-free case where twofold degeneracies characterize the $\Gamma(\mathbf{k}=0)$ point in plane groups with $N>2$ [2,3]]. The edge of the Brillouin zone (BZ) constitutes the second example. Below we show that sticking together of QEB's can occur along the whole side of the BZ. To this end, consider a field-free Hamiltonian, the nonsymmorphic plane group of which is $p2mg$ (see, e.g., Fig. 38 in Ref. [2] for an illustration of $p2mg$). Denoting the glide reflection by $\hat{\mathrm{g}}=(x\to x+a/2,y\to-y)$, we learn that the corresponding FB Hamiltonian [see Eq. (8)] is invariant under the GTRS operator

$$\hat{R}_{gt}=\left(x\to x+\frac{a}{2},y\to-y,t\to-t,*\right)=\hat{R}\cdot\hat{\mathrm{g}}. \tag{11}$$

Let us see why $\hat{R}_{gt}$ does not lift the sticking together *all* EB's exhibit along the edge $\mathbf{k}_Z=(\pi/a,k_y),k_y\in(\pi/b,\pi/b]$. For that, we pick up a FB state $\Phi_{\varepsilon(Z),Z}(\mathbf{r},t)$ at an edge point *Z*. It is evident that the FB state $\hat{R}_{gt}\Phi_{\varepsilon(Z),Z}(\mathbf{r},t)$ possesses the quasimomentum $\mathbf{k}_Z$ as well. Therefore, one may suppose that $\hat{R}_{gt}\Phi_{\varepsilon(Z),Z}(\mathbf{r},t)=\beta\Phi_{\varepsilon(Z),Z}(\mathbf{r},t)$. Next, we calculate the FB state $\hat{R}_{gt}\hat{R}_{gt}\Psi_{\varepsilon(Z),Z}(\mathbf{r},t)$ in two different ways:

$$[\hat{R}_{gt}\hat{R}_{gt}]\Phi_{\varepsilon(Z),Z}(\mathbf{r},t)=(x\to x+a)\Phi_{\varepsilon(Z),Z}=-\Phi_{\varepsilon(Z),Z}(\mathbf{r},t),$$

$$\hat{R}_{gt}[\hat{R}_{gt}\Phi_{\varepsilon(Z),Z}(\mathbf{r},t)]=\hat{R}_{gt}\beta\Phi_{\varepsilon(Z),Z}(\mathbf{r},t)=\beta^*\beta\Phi_{\varepsilon(Z),Z}(\mathbf{r},t). \tag{12}$$

However, since $\beta^*\beta$ cannot be equal to -1 we encounter a contradiction. Thus, our assumption must be false and $\hat{R}_{gt}\Phi_{\varepsilon(Z),Z}(\mathbf{r},t)$ and $\Phi_{\varepsilon(Z),Z}(\mathbf{r},t)$ do represent two *independent* and degenerate FB states. In other words, *all* QEB's at the edge *Z* exhibit sticking together, as is the situation for the EB's of a $p2mg$ lattice [2]. To systematically classify all effects of GTRS's on the QEB's for the 17 plane groups and 13 families of rod groups one has to employ heavier tools than the simple analysis given in Eq. (12). For instance, one has to combine the concept of magnetic groups with that of DS groups. Such a study is beyond the scope of the present manuscript.

We saw above that labeling of Floquet states with quantum numbers and classifying nonaccidental degeneracies of QE's depend on the *detailed* structure of the unitary DS group and the antiunitary GTRS. In comparison to these case-sensitive properties, the following conclusion is generic. Below we show how in systems with high-order DS's the computational effort of the time-evolution operator computed to one optical cycle, written *symbolically* as $\hat{U}(0\rightarrow T)=\exp[-i/\hbar\int_0^T H(t')dt']$, can be substantially reduced. Specifically, when the Floquet Hamiltonian is invariant under an Nth-order DS and under a GTRS, this effort can be reduced to about $1/(2N)$. It is instructive to mention that if the Floquet Hamiltonian is invariant under the (ordinary) TRS $\hat{R}$, it has been shown that the computational effort can be reduced to about half [6]. Similarly, it can be reduced to about half for a system in linearly polarized fields invariant under the second-order DS $\hat{P}_x$ [15].

Without loss of generality, suppose that the (Floquet) Hamiltonian is invariant under $\hat{P}_N$. Thus, we may write

$$\hat{H}(\varphi,t)=\hat{C}_N\hat{H}\left(\varphi,t+\frac{T}{N}\right)\hat{C}_N^{-1}. \tag{13}$$

Consequently,

$$\hat{U}\left(\frac{T}{N}\rightarrow\frac{2T}{N}\right)=\hat{C}_N^{-1}\hat{U}\left(0\rightarrow\frac{T}{N}\right)\hat{C}_N, \tag{14}$$

and by induction,

$$\hat{U}\left(\frac{jT}{N}\rightarrow\frac{(j+1)T}{N}\right)=\hat{C}_N^{-j}\hat{U}\left(0\rightarrow\frac{T}{N}\right)\hat{C}_N^{j},$$

$$j=0,1,\ldots,N-1. \tag{15}$$

Consequently, $\hat{U}(0\rightarrow T)$ can be decomposed as follows:

$$\hat{U}(0\rightarrow T)=\left[\hat{C}_N\hat{U}\left(0\rightarrow\frac{T}{N}\right)\right]^N, \tag{16}$$

meaning that the numerical effort is reduced to almost $1/N$ than that which has been originally required. When the (Floquet) Hamiltonian is invariant under $\hat{P}_N$ *and* $\hat{R}_{\varphi t}$, the following relation holds:

$$\hat{H}(\varphi,t)=\hat{\sigma}_\varphi\hat{C}_N\hat{H}^*\left(\varphi,\frac{T}{N}-t\right)\hat{C}_N^{-1}\hat{\sigma}_\varphi. \tag{17}$$

Equation (17) allows us to express $\hat{U}(0\rightarrow T/N)$ itself by $\hat{U}(0\rightarrow T/2N)$. The final result for the time-evolution operator [see Eq. (16)] is given by $[U^t\equiv(\hat{U}^\dagger)^*]$

$$\hat{U}(0\rightarrow T)=\left[\hat{\sigma}_\varphi\hat{U}^t\left(0\rightarrow\frac{T}{2N}\right)\hat{\sigma}_\varphi\hat{C}_N\hat{U}\left(0\rightarrow\frac{T}{2N}\right)\right]^N. \tag{18}$$

Therefore, when the Floquet Hamiltonian is invariant under $\hat{P}_N$ and $\hat{R}_{\varphi t}$ the numerical effort required to integrate $\hat{U}(0\rightarrow T)$ can be reduced to about $1/(2N)$ of that which was originally required. For instance, for benzene and graphite ($N=6$) in a circularly polarized field the computational effort can be reduced to less than 10%. For the (10,10) armchair carbon nanotube ($N=20$) [16] in such a field it is less than 4%. This substantial reduction of the numerical effort would enable one to calculate more accurate time-dependent dynamics for such large systems.

Our results extend the ones relating to atoms in circularly polarized fields, where knowledge of the infinitesimal time-evolution operator $\hat{U}(0\rightarrow\delta t)$ is sufficient to construct $\hat{U}(0\rightarrow T)$ [17]. This can be readily understood in the elegant context of DS, since atoms in a circularly polarized field possess the *infinite* order DS $\hat{P}_\infty=(\varphi\rightarrow\varphi+\omega\delta t, t\rightarrow t+\delta t)$ [9,11]. This observation suggests that whenever a time-dependent system is invariant under an ∞-th order DS, its time-evolution operator can be constructed from the infinitesimal time-evolution operator.

Finally, it is interesting to mention that the DS properties of the Floquet Hamiltonian (3) (see also Ref. [10]) are gauge invariant (we refer to the ''conventional'' gauges in laser-atom physics: the length gauge, the momentum gauge, and the acceleration representation [6]). This can be readily proved by examining the transformation from the Floquet Hamiltonian expressed in one gauge to another one. For instance, to transform the length-gauge Floquet Hamiltonian (3) to its partner expressed in the momentum gauge the following transformation is required: $\hat{\mathcal{O}}_{l\rightarrow p}(\mathbf{r},t)=\exp[-i(eE_0/\hbar\omega)\rho\sin(\omega t-\varphi)]$. Since $\hat{\mathcal{O}}_{l\rightarrow p}$ commutes with $\hat{P}_N$ and with $\hat{R}_{\varphi t}$, the Floquet Hamiltonian expressed in the momentum gauge is also invariant under these DS's.

In conclusion, we have demonstrated that DS analysis of time-periodic Hamiltonians is analogous to symmetry analysis of stationary Hamiltonians. As such, one can label Floquet states, determine symmetry properties of QEB's, and analyze nonaccidental degeneracies and, especially, the effect of GTRS on QE and QEB spectra. We have found that nonaccidental degeneracies induced by spatial and time-reversal symmetries (in the field-free case) may not be lifted inside the laser field. The substantial reduction in the computational effort to about $1/(2N)$ (or only $1/N$) due to DS-based considerations opens the door to tackling the time-dependent dynamics of large systems with high-order DS's.

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$$\hat{H}_f^{\pm}(\mathbf{r},t)=\hat{H}(\mathbf{r})+eE_1\rho\cos(\varphi-\omega t)$$
$$+eE_{N\mp1}\rho\cos[\varphi\pm(N\mp1)\omega t]-i\hbar\frac{\partial}{\partial t}$$

are invariant under $\hat{P}_N$ of Eq. (4) [9,11].
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