Chaotic dynamics of a single two-level atom in the field of a plane standing electromagnetic wave

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In this work we undertake an analytical and numerical investigation of the motion of a neutral two-level atom in a plane standing electromagnetic wave using a semiclassical approximation. In our study we neglect the energy loss from radiation. We show that, for experimentally achievable parameter values, the dynamics of the atom can be chaotic.

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I. INTRODUCTION

The interaction of two-level atoms with a single mode of an electromagnetic field is one of the basic problems in laser and atomic physics. The simplest models are based on a semiclassical approximation. It is well known that these models can exhibit dynamical chaos. One of the first systems showing chaotic behavior was proposed by P. I. Belobrov *et al.* in 1976 [1] and considers a group of N atoms in a resonant cavity. A similar system, based on the semiclassical Jaynes-Cummings model [2], was considered later [3]. The onset of chaos in these models is due to the break down of the rotating-wave approximation (RWA) and is possible only for very high densities of atoms. However, such an approximation would fail here since we assume noninteracting atoms. A more physically relevant situation is described in [4,5] where a group of N monoenergetic, noninteracting two-level atoms moves through a single-mode cavity. In this case chaos occurs in the RWA. It is essential that (i) the number of particles should be sufficiently high ($N=10^{10}$) in order to change the electromagnetic field in the cavity effectively; (ii) the speed of the group should be sufficiently high to neglect the effect of spatial heterogeneity of the field on the motion of atoms. Further considerations and generalizations of this system can be found in [6].

The motion of atoms under the influence of a cavity field was investigated in depth in connection with laser cooling and trapping [7]. A system of a small number of laser-cooled ions confined in an electromagnetic trap (usually a Paul trap) was considered in a series of papers [8]. It was shown that the dynamics of the ions can become chaotic due to nonlinearity arising from the ion-ion Coulomb repulsion. In [8] ions are treated as classical particles and the *internal dynamics* is not taken into consideration. The interaction of the internal dynamics with spatial motion of an ion can lead to dynamical chaos as well. In [9] the authors analyze motion of a two-level ion in a Paul trap under the influence of a resonant laser field. The dynamics were shown to be chaotic for a wide range of parameter values. However, the resonant approximation, which simplifies the system significantly, rules out the effect of *internal dynamics* (described by Bloch-type equations) on the motion of the ion (described classically).

In this paper we analyze the motion of a single neutral two-level atom in the field of a standing electromagnetic wave. We do not make any assumptions about either the initial speed of the atom or the detuning between the atomic transition frequency and field frequency (except that it is presumed to be sufficiently large to neglect radiation effects).

The rest of the paper is organized as follows. In Sec. II we introduce the model and derive the equations governing the dynamics of the atom. In Sec. III we study the motion of the atom analytically under certain conditions. In Sec. IV the applicability of the analytical solution is investigated and the numerical results are presented. Finally, the conclusion and discussion are presented in Sec. V.

II. MODEL

We consider the dynamics of a two-level neutral atom in the field of a plane standing electromagnetic wave $\vec{E}(\vec{r},t)=\vec{E}_0\cos kx\cos\omega t$, where $\vec{E}_0=(0,0,E_0)$ is the amplitude of the electric field strength vector, k is the wave number, and ω is the frequency of the electromagnetic wave. The dynamics of the atom is described by the wave function

$$\psi(\vec{r},t)=A(\vec{r},t)|1\rangle+B(\vec{r},t)|2\rangle, \tag{1}$$

where $\vec{r}$ denotes the position of the atom (center of mass), and $|1\rangle$ and $|2\rangle$ are the ground and excited states of the atom, respectively. By introducing the variables

$$\begin{aligned} u&=P_+\cos\omega t+P_-\sin\omega t,\\ v&=-P_+\sin\omega t+P_-\cos\omega t,\\ w&=|B|^2-|A|^2, \end{aligned} \tag{2}$$

where $P_\pm=A^*B\pm AB^*$, we can write the Bloch-type equations for the internal dynamics of the atom in the RWA as

$$\begin{aligned} \dot{u}&=-\Delta v,\\ \dot{v}&=\Delta u+\Omega w\cos kx,\\ \dot{w}&=-\Omega v\cos kx, \end{aligned} \tag{3}$$

where $\Delta=\omega_{21}-\omega$ is the detuning between the atomic transition frequency ω_{21} and the frequency of the electromagnetic wave, and $\Omega=|\vec{E}_0\vec{d}_{12}|/\hbar$ denotes the Rabi frequency.

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The matrix element of dipole momenta $\vec{d}_{12}$ is considered to be real. We can always choose the common phase of the wave function (1) in such way that the latter condition is satisfied.

Let us presume that the *external evolution* (center-of-mass motion) of the atom can be treated classically. Then the spatial dynamics of the atom in the RWA satisfies the equation

$$m\ddot{x}=\frac{1}{2}\hbar\Omega ku\sin kx, \tag{4}$$

where m is the mass of the atom.

The above model appears to be rather phenomenological. However, the exact derivation is beyond the scope of this work and readers should refer to [10] for details. We can omit the equations for the dynamics of the atom along the y and z directions as they can be shown to be trivial.

Next we introduce the new variables $x'=kx$, $t'=\Omega t$, and parameters $\varepsilon=(\hbar\omega^2/2mc^2\Omega)^{1/2}$, $\Delta'=\Delta/\Omega$, and as a result Eqs. (3) and (4) can be rewritten (with primes omitted) as

$$\begin{aligned}\dot{u}&=-\Delta v,\\ \dot{v}&=\Delta u+w\cos x,\\ \dot{w}&=-v\cos x,\\ \ddot{x}&=\varepsilon^2 u\sin x.\end{aligned} \tag{5}$$

System (5) is not integrable. It has only two integrals:

$$\begin{aligned}S_1&=u^2+v^2+w^2=1,\\ S_2&=\frac{\dot{x}^2}{2}+\varepsilon^2 u\cos x-\varepsilon^2\Delta w,\end{aligned} \tag{6}$$

corresponding to unitarity and energy conservation, respectively. Note that the parameter ε can be treated as the scaled Planck constant. Consequently, the description of the external evolution by means of classical dynamics requires at least the inequality $\varepsilon\ll 1$ to be satisfied. For instance, if we take atomic characteristics $m=10^{-24}$ g, $d_{12}=10^{-18}$ erg$^{1/2}$ cm$^{3/2}$, and an electromagnetic wave with $E_0=10$ G, $\omega=10^{15}$ s^{-1}, then $\varepsilon=10^{-3}$.

III. INTERNAL VS EXTERNAL DYNAMICS

In this section we consider $\Delta\gg\varepsilon$. This indicates that for moderate values of the initial speed $|\dot{x}(0)|\leqslant\varepsilon$, the center-of-mass motion is much slower than the internal evolution governed by Bloch-type equations [see the definition of S_2 in Eqs. (6)]. We exclude this fast dynamics by appropriate change of variables and rewrite Eqs. (5) in a form that allows further analysis; namely, we introduce new variables α, β, and γ such that

$$\begin{aligned}u&=-\frac{\cos x}{\Omega_+}\gamma-\frac{\Delta}{\Omega_+}(\alpha\sin\theta-\beta\cos\theta),\\ v&=\alpha\cos\theta+\beta\sin\theta,\\ w&=\frac{\Delta}{\Omega_+}\gamma-\frac{\cos x}{\Omega_+}(\alpha\sin\theta-\beta\cos\theta),\end{aligned} \tag{7}$$

where $\Omega_+=\sqrt{\Delta^2+\cos^2 x}$ and $\theta=\int_0^t\Omega_+dt'$. the transformation (7) is invariant with respect to the integral S_1 and it converts the integral $S_2=\dot{x}^2/2-\varepsilon^2\Omega_+\gamma$.

After substitution of Eq. (7) into Eq. (5) we derive the following equations:

$$\begin{aligned}\dot{\alpha}&=\frac{\Delta}{\Omega_+^2}\dot{x}\gamma\sin x\sin\theta,\\ \dot{\beta}&=-\frac{\Delta}{\Omega_+^2}\dot{x}\gamma\sin x\cos\theta,\\ \dot{\gamma}&=-\frac{\Delta}{\Omega_+^2}\dot{x}\sin x(\alpha\sin\theta-\beta\cos\theta),\\ \ddot{x}&=-\frac{\varepsilon^2\Delta}{\Omega_+}\sin x(\gamma\cos x+\alpha\sin\theta-\beta\cos\theta).\end{aligned} \tag{8}$$

The right part of each equation in (8) is proportional to at least the first power of the small parameter ε. Systems of this type are well known and allow further analytical investigation by using an averaging method developed in [11]. According to this method we can represent the solution of Eq. (8) as a sum of a slowly *regularly* varying term (or secular term) and small fast oscillations such that

$$\begin{aligned}\alpha&=\overline{\alpha}+\tilde{\alpha},\quad \beta=\overline{\beta}+\tilde{\beta},\\ \gamma&=\overline{\gamma}+\tilde{\gamma},\quad x=\overline{x}+\tilde{x}.\end{aligned} \tag{9}$$

Both the secular and fast oscillating terms are sought in the form of a series with ε being a small parameter. The aim of the method is to find the equations describing the regular motion with accuracy to the desired order of the small parameter ε. To first order, the fast oscillating terms are not affected by the slow motion and satisfy the equations

$$\dot{\bar{\alpha}}=\frac{\Delta}{\bar{\Omega}_+^2}\dot{\bar{x}}\bar{\gamma}\sin\bar{x}\sin\bar{\theta},$$

$$\dot{\bar{\beta}}=-\frac{\Delta}{\bar{\Omega}_+^2}\dot{\bar{x}}\bar{\gamma}\sin\bar{x}\cos\bar{\theta}, \tag{10}$$

$$\dot{\bar{\gamma}}=-\frac{\Delta}{\bar{\Omega}_+^2}\dot{\bar{x}}\sin\bar{x}(\bar{\alpha}\sin\bar{\theta}-\bar{\beta}\cos\bar{\theta}),$$

$$\ddot{\bar{x}}=-\frac{\varepsilon^2\Delta}{\bar{\Omega}_+}\sin\bar{x}(\bar{\alpha}\sin\bar{\theta}-\bar{\beta}\cos\bar{\theta}),$$

where $\bar{\Omega}_+=\sqrt{\Delta^2+\cos^2\bar{x}}$. The secular terms ($\bar{\alpha}$, $\bar{\beta}$, . . .) are considered to be constants. In this approximation θ can be treated as $\bar{\theta}=\bar{\Omega}_+t$ and Eqs. (10) can be integrated easily. Next we obtain the equations describing the secular motion. We substitute the solution in the form (9) with fast oscillating terms given by the solution of Eq. (10) into the system (8). Leaving the terms of order ε^2 we can derive the following equations for regular motion (overbars are omitted):

$$\dot{\alpha}=\frac{\Delta^2}{2\Omega_+^5}\dot{x}^2\sin^2x\,\beta+f_1(\varepsilon^2),$$

$$\dot{\beta}=-\frac{\Delta^2}{2\Omega_+^5}\dot{x}^2\sin^2x\,\alpha+f_2(\varepsilon^2), \tag{11}$$

$$\dot{\gamma}=0+f_3(\varepsilon^2),$$

$$\ddot{x}=-\frac{\varepsilon^2\Delta}{2\Omega_+}\sin 2x\,\gamma+f_4(\varepsilon^2),$$

where $f_i(\varepsilon^2)$ denotes fast oscillating terms of the order of ε^2. These terms do not affect the slow regular motion and can be neglected (or averaged) if we consider the motion over time intervals greater than the characteristic period of the fast oscillations.

The system (11) is integrable and decomposes into two parts describing the external and internal dynamics separately. The center-of-mass motion depends on the internal state of the atom via the initial condition only and can be described as oscillations in an effective potential of the form

$$U(x)=-\varepsilon^2\gamma\sqrt{\Delta^2+\cos^2x}. \tag{12}$$

The maximum and minimum of the potential $U(x)$ are located at nodes or antinodes of the standing wave. If γ is positive (negative) the nodes of the standing wave correspond to maxima (minima) of $U(x)$.

The system (5) was integrated numerically for $\Delta=1$, $\varepsilon=0.01$, and the fixed value of $S_2=\varepsilon^2$. In Fig. 1 we plot the projection of the phase trajectory on the unit sphere $S_1=u^2+v^2+w^2=1$, which was parametrized in the following way:

$$u=\sin\theta\cos\varphi,\quad v=\cos\theta,\quad w=\sin\theta\sin\varphi. \tag{13}$$

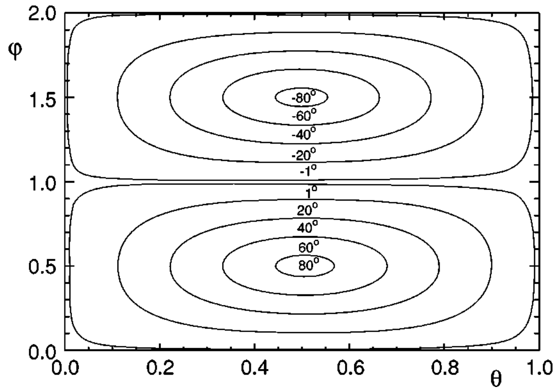

FIG. 1. Poincaré section of the system (5) for $\Delta=1$, $\varepsilon=0.01$. Every curve has an index indicating the value of θ_0 in degrees. Angles are scaled on π.

The values of u, v, and w were sampled at moments when the atom had coordinate $x=\pi/2$ and positive speed $\dot{x}>0$. The corresponding angles θ and φ were then calculated. The initial conditions for the Bloch-vector components were taken in the form (13) with $\varphi(0)=\pi/2$ and different values of $\theta(0)=\theta_0$. We also choose the initial coordinate $x(0)=\pi/2$ and initial velocity $\dot{x}(0)$ so as to conserve the value of S_2. In this case, it follows that $\gamma(0)=\sin\theta_0$ from Eqs. (7). According to our analysis $\gamma(t)=\gamma(0)$ is an integral of motion and we can use the angle θ_0 to parametrize the curves in Fig. 1.

IV. LOSS OF INTEGRABILITY

In the previous section we have obtained integrals for motion of the system (5). The following conditions were considered to be satisfied: (i) $\dot{x}(0)\ll 1$, (ii) $\Delta\gg\varepsilon$, and (iii) $\varepsilon\ll 1$, each being essential for the validity of our approximation.

Let us consider system (5) with $\dot{x}(0)\sim 1$. In this case the center-of-mass motion cannot be treated as slow with respect to the internal dynamics and therefore the analysis developed in the previous section is no longer applicable.

It follows from integral S_2 that $\dot{x}\approx 2S_2-\varepsilon^2(\Delta w-\cos xu)+\cdots$. This means that we can seek the solution of Eq. (5) in the form of an infinite series with ε^2 being a small parameter of the asymptotic procedure:

$$u=u_0+\varepsilon^2u_2+O(\varepsilon^4),$$

$$v=v_0+\varepsilon^2v_2+O(\varepsilon^4), \tag{14}$$

$$w=w_0+\varepsilon^2w_2+O(\varepsilon^4),$$

$$x=x_0+\varepsilon^2x_2+O(\varepsilon^4).$$

After substitution of Eq. (14) into Eq. (5) we derive the zeroth order approximation equations

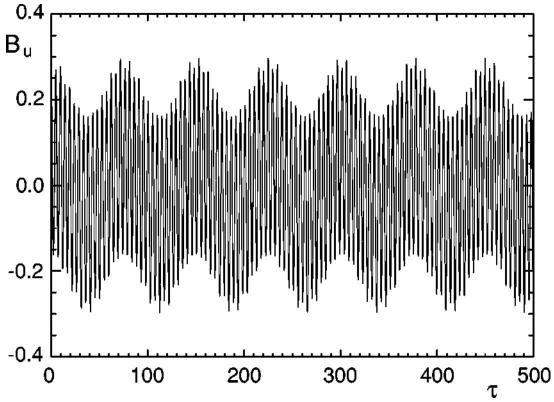

FIG. 2. Autocorrelation function $B_u(\tau)$ for $\Delta=1$, $\varepsilon=0.05$, and $\dot{x}(0)=0.7$.

$$\dot{u}_0=-\Delta v_0,$$
$$\dot{v}_0=\Delta u_0+w_0\cos x_0, \tag{15}$$
$$\dot{w}_0=-v_0\cos x_0,$$
$$\ddot{x}_0=0.$$

The last equation is trivial and implies immediately that $x_0=\dot{x}(0)t$. The internal dynamics are governed by Bloch-type equations with harmonic perturbation and, according to the Floquet theorem, quasiperiodic behavior is expected for u_0, v_0, and w_0.

Collecting terms proportional to ε^2, it is easy to obtain the second order approximation equations

$$\dot{u}_2=-\Delta v_2,$$
$$\dot{v}_2=\Delta u_2+w_2\cos x_0-x_2w_0\sin x_0, \tag{16}$$
$$\dot{w}_2=-v_2\cos x_0+x_2v_0\sin x_0,$$
$$\ddot{x}_2=u_0\sin x_0.$$

The equation for x_2 of system (16) is expressed in terms of the zeroth order variables only and can be integrated analytically. After substituting this solution into the first three equations of system (16) we obtain quasiperiodically perturbed Bloch-type equations. Continuing this procedure, we can obtain the solution with any desired accuracy.

In Fig. 2 the autocorrelation function

$$B_u(\tau)=\overline{u(t+\tau)u(t)-u(t)^2} \tag{17}$$

of $u(t)$ is shown for $\varphi_0=\theta_0=\pi/3$, $x(0)=0$, and a large value of the initial velocity $\dot{x}(0)=0.7$. The autocorrelation function does not decay, suggesting that the motion is regular.

Other ways of losing the applicability of the solution obtained in Sec. III include decreasing the detuning Δ to the order of the small parameter ε or increasing ε to the order of 1. In both these cases the dynamics becomes chaotic. In Fig. 3 we plotted the Poincaré section for system (5) with the angles θ and φ defined as in Fig. 1. We took parameter values $\Delta=0.01$, $\varepsilon=0.01$, and initial conditions $\varphi_0=\theta_0=\pi/3$, $x(0)=0.4\pi$, $\dot{x}(0)=0$. The values of θ and φ were sampled at moments when the atom had coordinate $x=0$ and positive speed $\dot{x}>0$. We also calculated the Lyapunov exponent, which was found to be $\lambda=0.01\pm0.002$.

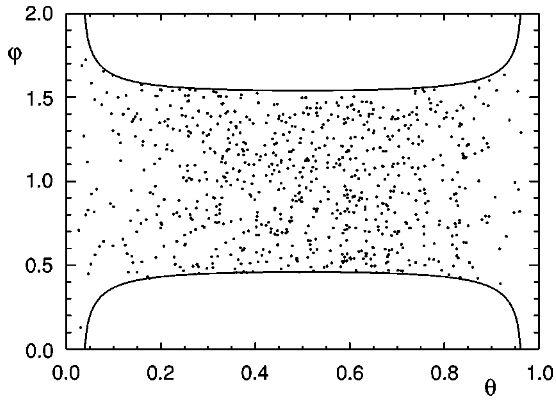

FIG. 3. Poincaré section of the system (5) for $\Delta=0.01$, $\varepsilon=0.01$. Angles are scaled on π. The solid curve shows the boundary of the region of the phase space accessible to the system (5) and related to the conservation of energy S_2.

In Fig. 4 the Poincaré section is plotted for $\Delta=1$ and $\varepsilon=1$. The initial conditions were taken to be the same as for the previous case. The Lyapunov exponent in this case was $\lambda=0.22\pm0.05$.

In both figures the trajectories belong to a chaotic component covering most of the phase space accessible to the system.

V. CONCLUSION

In this paper we have studied the motion of a neutral atom in the field of a plane standing electromagnetic wave. The dynamics of the atom were investigated in a semiclassical approximation: the motion of the atom was described classically and the internal dynamics were considered quantum mechanically. The estimation of the scaled Planck constant

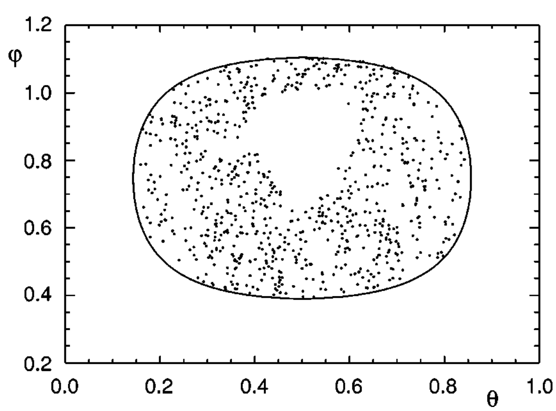

FIG. 4. Poincaré section of the system (5) for $\Delta=1$, $\varepsilon=1$. Angles are scaled on π. The solid curve has the same definition as in Fig. 3.

indicates that such an approach is feasible.

We show, both analytically and numerically, that the motion is regular and can be characterized by the absence of energy exchange between the internal and external motion of the atom if the following conditions are satisfied: (i) the initial velocity of atom $\dot{x}(0) \ll c\Omega/\omega$; (ii) the detuning between the atomic transition frequency and field frequency $\Delta \gg \varepsilon\Omega$; and (iii) the scaled Planck constant $\varepsilon \ll 1$.

We show that if *any* of these conditions is violated, the analytical solution is no longer applicable. For example, if the first condition (i) is violated the motion of the atom can be represented in the form of a series with ε^2 being a small parameter. To first order, the motion of the atom becomes quasiperiodic. However, if we consider the next order the internal dynamics are described by quasiperiodically perturbed Bloch-type equations. A series of papers was devoted to the investigation of this system [12–15]. It was shown to exhibit behavior that possesses quasicontinuous power spectra of dynamical variables for appropriately chosen frequencies of perturbation. Although the dynamics of the atom are regular in this case, we can expect them to be quite complicated.

It was shown that if either condition (ii) or (iii) is violated then the motion becomes chaotic and can be characterized by positive Lyapunov exponents. However, it is important to note that if (iii) is violated the spatial motion of the atom cannot be treated classically and our model fails. In this regard it is interesting to investigate the behavior of the corresponding quantum mechanical model describing the motion of a neutral atom for values of $\varepsilon \geq 1$.

The estimation of the parameters suggests that a laser field can be used as a source of the electromagnetic wave and we believe that the chaotic motion of neutral atoms can be detected experimentally.

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