# Quantum control via encoded dynamical decoupling

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The ideas underlying dynamical decoupling methods are revisited within the framework of quantum-information processing, and their potential for direct implementations in terms of encoded rather than physical degrees of freedom is examined. The usefulness of encoded decoupling schemes as a tool for engineering both closed- and open-system encoded evolutions is investigated based on simple examples.

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## I. INTRODUCTION

Since the pioneering work on coherent averaging effects by Haeberlen and Waugh [1], the use of tailored pulse sequences for manipulating the effective Hamiltonian experienced by a target quantum system has developed a solid tradition in nuclear magnetic resonance (NMR) [2,3]. In particular, within the context of NMR quantum-information processing (QIP), decoupling and refocusing techniques provide the basic tools for enforcing Hamiltonian evolutions that correspond to quantum logic gates between selected spins [4]. The principles underlying these techniques, along with the powerful formalism offered by the average Hamiltonian theory (AHT) [1], have been recently extended beyond the NMR domain, and suggestive applications have resulted in various directions within QIP. On one hand, ideas from NMR decoupling motivated a "bang-bang" [5] control-theoretic framework for generic open quantum systems [6], which paved the way for the development of quantum symmetrization procedures and quantum error suppression strategies for QIP [6–10]. On the other hand, the application of active dynamical control in the bang-bang limit proved a valuable tool for engineering the evolution of coupled quantum subsystems [8], leading to various schemes for universal simulation of both closed-system [11] and open-system dynamics [12].

So far, in spite of the pervasive role played by quantum coding in QIP, the application of active refocusing and decoupling methods has been primarily thought of in terms of the basic *physical* degrees of freedom. Two exceptions are a proposal by Wu and Lidar for applying recoupling schemes on encoded qubits governed by exchange-type Hamiltonians [13], and an implementation by Fortunato *et al.* of encoded refocusing to experimentally demonstrate universal control on a decoherence-free qubit [14]. It is the purpose of this work to further comment on the significance of dynamical control methods as directly represented in terms of *encoded* degrees of freedom, by continuing the investigation of the interplay between quantum coding and decoupling techniques undertaken in [9], and by expanding the basic theoretical arguments sketched in [14]. While decoupling methods have been already shown to enable, in principle, to synthesize effective evolutions supporting noise-protected, encoded degrees of freedom [9,15], a different perspective is taken here—by imagining a preselected encoded structure and by looking at the evolutions that can be enforced through encoded dynamical control.

I envision two prospective types of applications of encoded dynamical decoupling methods within QIP and, more generally, quantum control. Similar to their unencoded counterparts, these include the simulation of both closed-system Hamiltonian evolutions and open-system nonunitary evolutions—all evolutions being, however, restricted to an underlying coding space that should be preserved throughout. After summarizing the basic facts about decoupling in Sec. II, encoded dynamical decoupling is introduced in Sec. III. I then discuss the two relevant areas of application based on simple representative examples in Secs. IV and V. A brief summary concludes in Sec. VI.

## II. BASICS OF DYNAMICAL DECOUPLING

A bang-bang (BB) quantum control problem is concerned with characterizing the effective evolutions that can be engineered by repeatedly interspersing, according to various possible schemes, the natural dynamics of a quantum system with full-power, instantaneous control operations (BB controls) [6–8,16]. Let $S$ denote the target control system, defined on a (finite-dimensional) state space $\mathcal{H}_S$, $\dim(\mathcal{H}_S)=N$, $N=2^n$ for qubit systems. In a general open-system setting, $S$ is coupled to an environment $E$ via an interaction Hamiltonian $H_{SE}$. The control problem can then be formulated in terms of the following data.

(1) $H$, the natural Hamiltonian of the composite system, $H=H_S\otimes \mathbb{1}_E+\mathbb{1}_S\otimes H_E+H_{SE}$, determining the free unitary evolution $U_0(t)=\exp(-iHt)$ on the joint state space $\mathcal{H}_S\otimes\mathcal{H}_E$. One may write $H_{SE}=\Sigma_\alpha S_\alpha\otimes E_\alpha$ for appropriate system ($S_\alpha$) and environment ($E_\alpha$) operators, respectively. The linear subspace $\mathcal{N}=\mathrm{span}\{S_\alpha\}$ of noise-inducing couplings is referred to as *error space*. Without loss of generality, both $H_S$ and the error generators in $\mathcal{N}$ can be assumed to be traceless. Detailed knowledge of $H_S$ or $\mathcal{N}$ may or may not be explicitly available from the start.

(2) $\mathcal{G}_{BB}$, the set of *all* BB operations that can be effected on $S$. As it is conceivable that $P^{-1}=P^\dagger\in\mathcal{G}_{BB}$ if $P\in\mathcal{G}_{BB}$, $\mathcal{G}$ can be associated with a subgroup of the group $\mathcal{U}(\mathcal{H}_S)$ of unitary transformations on the system alone.

(3) $\mathcal{G}$, the *discrete subset* of BB decoupling operations,

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$\mathcal{G}=\{U_k\}\subseteq\mathcal{G}_{BB}$, with $k\in\mathcal{K}$ for some finite set of indexes with order $|\mathcal{K}|\equiv|\mathcal{G}|$.

(4) $T_c$, the relevant control time scale (*cycle time*), associated with the duration of a single control cycle.

(5) $\{\tau_k\}$, the set of relative temporal separations between consecutive BB operations, $\tau_k=\Delta t_k/T_c>0$, in terms of the free evolution intervals $\Delta t_k$, and $\Sigma_{k\in\mathcal{K}}\tau_k=1$ (cyclicity condition).

AHT provides a general prescription for characterizing the controlled evolutions in terms of time-independent effective Hamiltonians that would result in the same unitary propagator if applied over the same evolution interval. Imagine that a single control period $T_c$ consists of a cyclic sequence of $|\mathcal{K}|$ BB pulses, $\mathcal{P}=\{P_k,\tau_k\}_{k=1}^{|\mathcal{K}|}$, with $\Pi_{k=1}^{|\mathcal{K}|}P_k=\mathbb{1}$. Then

$$\begin{aligned} U(T_c)&=\exp(-iH_{eff}T_c)\\ &=\prod_{m=0}^{M} U_m^{\dagger}U_0(\Delta t_m)U_m\\ &=\prod_{m=0}^{M}\exp(-iH_m\tau_m T_c), \end{aligned} \tag{1}$$

where the first equality *defines* the effective average Hamiltonian, and the "toggling-frame" transformed Hamiltonians $H_m$ are determined by the composite rotations $U_m=\Pi_{k=1}^{m}P_k$, $k=1,\dots,M-1$, $U_0=\mathbb{1}$. Here, $M=|\mathcal{K}|$ or $M=|\mathcal{K}|+1$ depending on whether the sequence is arranged so as to allow evolution in the $\mathbb{1}$ frame in a single or a pair of control subintervals—being, in any case, $U_{|\mathcal{K}|}=\mathbb{1}$ by cyclicity. Because $P_k\in\mathcal{G}_{BB}$, then $\mathcal{G}\subseteq\mathcal{G}_{BB}$ as anticipated. However, $\mathcal{G}$ need not itself be a subgroup, neither does $\mathcal{G}=\mathcal{G}_{BB}$, in general. For instance, allowing for $\mathcal{G}\neq\mathcal{G}_{BB}$ may be crucial for retaining universal control over decoupled dynamics [8]. Although the effective Hamiltonian (1) can be systematically calculated as a power series in the controllable parameter $T_c$ (Magnus series [3]), AHT is practically useful in the limit of *fast control* $T_c\to 0$ [2,3,6], where the lowest-order contributions in $T_c$ suffice for an accurate description. In this limit, $H_{eff}$ approaches

$$H_{eff}\mapsto\bar{H}=\sum_{k\in\mathcal{K}}\tau_k U_k^{\dagger}HU_k=\Lambda_{\mathcal{G}}(H), \tag{2}$$

with leading corrections accounted for by

$$\bar{H}^{(1)}=-\frac{i}{2T_c\hbar}\sum_{m>n}[H_m,H_n]\tau_m\tau_n. \tag{3}$$

Under the conditions ensuring convergence of the series defining $H_{eff}$, this correction is at least $O(T_c)$. The zeroth order approximation (2) is, of course, exact if the various transformed Hamiltonians commute—in which case any time scale constraint for the control implementation actually disappears.

In passing, it is worth noting that the most general transformation that AHT allows involves, as given in Eq. (2), a weighted (convex) mixture of unitary operators. Extended to the algebra of linear operators over $\mathcal{H}_S$, End$(\mathcal{H}_S)$, the action (2) defines a trace-preserving, unital, completely positive map $\Lambda_{\mathcal{G}}$. Interestingly, the fact that $\Lambda_{\mathcal{G}}$ is a unitary mixing operation implies that any Hamiltonian $\bar{H}$ reachable from $H$ via $\Lambda_{\mathcal{G}}$ is necessarily at least "as disordered" as $H$ in the sense of majorization [17]. Examples abound in the original NMR setting where decoupling is achieved through control actions of the general form (2). A relevant representative case is the so-called Waugh-Huber-Haeberlen sequence (WHH-4) used for homonuclear dipolar decoupling [2]. This corresponds to a sequence $\mathcal{P}=\{\tau_0,P_x,\tau_1,P_{-y},\tau_2,P_y,\tau_3,P_{-x},\tau_4\}$ of BB $\pi/2$ pulses, $P_a=\exp(-i\pi\sigma_a/4)$, and $\tau_0=\tau_1=\tau_3=\tau_4=1/6$, $\tau_2=1/3$—which effectively averages out two-spin interactions proportional to $3\sigma_z^i\sigma_z^j-\vec{\sigma}^i\cdot\vec{\sigma}^j$ over a cycle.

The simplest realization of Eq. (2) occurs under the additional assumptions that control operations are equally separated in time and the composite rotations in $\mathcal{G}$ close a group (*decoupling group* [6]). By letting $\tau_k=1/|\mathcal{G}|$, the quantum operation $\Lambda_{\mathcal{G}}$ of Eq. (2) reduces in this case to the projector $\Pi_{\mathcal{G}}$ on the centralizer $\mathcal{Z}(\mathcal{G})$ of $\mathcal{G}$ in End$(\mathcal{H}_S)$, $\mathcal{Z}(\mathcal{G})=\{X\in\text{End}(\mathcal{H}_S)|[X,U_k]=0\ \ \forall k\}$ [6–8]:

$$H_{eff}\mapsto\bar{H}=\frac{1}{|\mathcal{G}|}\sum_{k\in\mathcal{K}}U_k^{\dagger}HU_k=\Pi_{\mathcal{G}}(H). \tag{4}$$

The resulting effective Hamiltonian acquires now a direct symmetry characterization as $[\bar{H},\mathcal{G}]=0$–meaning that the controlled dynamics is symmetrized according to $\mathcal{G}$ [7], and the desired averaging action can be thought of as filtering out the dynamics that is not invariant under $\mathcal{G}$.

One of the most famous (and practically useful) examples of a decoupling sequence originating from a group is the Carr-Purcell sequence [18,2,3], which in its basic variant is used for suppressing undesired phase evolution due to $\sigma^z$ terms—representing, for instance, applied field inhomogeneity in NMR. The sequence consists of repeated BB $\pi_a$ pulses, $a=x$ or $y$, separated by $\Delta t$, corresponding to a decoupling group $\mathcal{G}_{CP_a}=\{\mathbb{1},\pi_a\}$, with $\pi_a=\exp(-i\pi\sigma_a/2)$. Within the cycle time $T_c=2\Delta t$, the pulses can be arranged so as either $\tau_1=\tau_2=1/2$ (with $M=|\mathcal{K}|=2$) or $\tau_1=\tau_3=1/4$, $\tau_2=1/2$ (with $M=|\mathcal{K}|+1=3$). Although the two sequences clearly lead to the same $\bar{H}$ as resulting from Eq. (4), the second option is actually superior in terms of the overall averaging accuracy —as it corresponds to a time-symmetric cycle for which all odd-order corrections $\bar{H}^{(2\ell+1)}$, $\ell=0,1,\dots$, vanish [3,6].

In a closed-system setting, $H_{SE}=0$ and the control problem consists in turning off (selectively or not) undesired contributions to $H_S$. Note that this amounts to simulating the AHT-reachable system's Hamiltonians by $H_S$. In particular, BB control may enable universal Hamiltonian simulation on $\mathcal{H}_S$ if an arbitrary effective evolution can be simulated by $H_S$. In an open-system setting, on the other hand, the focus is on averaging out the error generators $\{S_\alpha\}$, i.e., ensuring that $\Lambda_{\mathcal{G}}(\mathcal{N})=0$, in order to achieve decoherence suppression. The limit of fast control may be especially stringent to be met in this case, as it requires that $T_c\lesssim\tau_c$ for the shortest correlation time associated with the environmental noise [5,6,10]—which can be prohibitively small in Markovian or quasi-Markovian dynamics. In both situations, the attainable control goals are influenced by two main factors: the available knowledge of the interactions to be manipulated and/or averaged out, and the overall available control resources.

Some notable results about decoupling are worth mentioning.

(a) *Selective averaging* (or refocusing) requires knowledge of the transformation properties of the interactions to be turned off with respect to the achievable decoupling sets $\mathcal{G}$. For a decoupling group $\mathcal{G}$, a necessary condition for selectivity is that the centralizer is nontrivial, $\mathcal{Z}(\mathcal{G})\neq\{\lambda \mathbb{1}\}$ ($\lambda$ complex) [6,8]. A variety of *efficient* schemes (i.e., polynomial in $n$) exist for selective decoupling in systems with at most bilinear interactions [19]. These are applicable to NMR and NMR-like Hamiltonians in both the weak- and strong-coupling limit.

(b) For arbitrary interactions ($H_S$ or $\mathcal{N}$), *maximal averaging* (or complete decoupling, or annihilation) is possible, in principle, by letting $\mathcal{G}$ be a unitary error basis on $\mathcal{H}_S$ [6,11]. Although this choice can be shown to be optimal, the resulting scheme is inefficient as the number of required BB control operations is $|\mathcal{G}|=N^2=4^n$, which grows exponentially with $n$. Again, efficient schemes can be designed if no terms higher than bilinear ones are known to be relevant [19].

(c) Universal simulation of an arbitrary effective Hamiltonian on $\mathcal{H}_S$ can be implemented in various ways, depending on the set of actual simulation requirements and control resources. Suppose that, for a given $H_S$, a decoupling set $\mathcal{G}$ exists, such that $[H_S,\Lambda_{\mathcal{G}}(H_S)]\neq 0$, and that periods of free evolution under $H_S$ can be alternated with periods of controlled evolution under $\Lambda_{\mathcal{G}}(H_S)$. Then, similarly to the twisted decoupler schemes discussed in [8], any Hamiltonian $L$ belonging to the Lie algebra generated by $iH_S$, $i\Lambda_{\mathcal{G}}(H_S)$ can be reached in principle—implying universal control in the generic case [20]. Even in the unfavorable situation where $H_S$ may consist of a single term (say $\sigma_z$ for a qubit), arbitrary Hamiltonians can still be engineered if, for instance, both a set of BB operations averaging $H_S$ and a slow application of a Hamiltonian $Z\in\mathcal{Z}(\mathcal{G})$ can be effected. Then one can again alternate evolutions under $H_S$ with controlled evolutions where $Z$ is applied in parallel with the decoupler $\mathcal{G}$ [8], obtaining as above universality in the generic case. (In the qubit example, with $H_S=\sigma_z$, one can choose $\mathcal{G}=\mathcal{G}_{CP_x}$ and apply $Z=\sigma_x$ via weak/slow control [8]). Of course, such programming procedures may require additional external capabilities beyond BB control and, in general, they will not ensure universality starting from an *arbitrary* (possibly unknown) $H_S$. An elegant approach applicable to this general situation has been recently presented in [11], where the possibility of arbitrary universal simulation is related to the identification of special decoupling groups called *transformer groups*. Starting from any Hamiltonian $A$, decoupling according to a transformer group is able to map $A$ into any desired effective Hamiltonian. For instance, a transformer group for a two-dimensional system (a single qubit) is generated under the natural representation by the four BB operations $\{i\sigma_x,i\sigma_y,i\sigma_z,R\}$, $R$ being the rotation by $2\pi/3$ about the axis $\hat{n}=(1,1,1)/\sqrt{3}$ that cyclically permutes the Pauli matrices.

## III. ENCODED DYNAMICAL DECOUPLING

So far, the full $N=2^n$-dimensional state space of $n$ physical qubits has been exploited. However, restricting to a $N_L$-dimensional *quantum code*, $N_L<N$, may prove extremely useful in QIP—the resulting benefits sometimes largely compensating the overheads and complications arising from dealing with a smaller number $n_L<n$ of encoded qubits. In particular, the two primary motivations for seeking appropriate encodings are either to ensure protection against noise—via active error-correcting codes [21] or passive noiseless codes on decoherence-free subspaces (DFSs, [22,23])/noiseless subsystems (NSs, [24,25,15])—or to allow for alternative routes to universality based on the physically available interactions on appropriately defined subsystem qubits—in the so-called approach of *encoded universality* [25,26].

A quantum code $\mathcal{H}_L$ can be generally thought of as a distinguished subsystem of the physical state space of $S$, determined by a correspondence of the form

$$\mathcal{H}_S\simeq\mathcal{H}_L\otimes\mathcal{H}_Z\oplus\mathcal{R}, \tag{5}$$

for some $\mathcal{H}_Z,\mathcal{R}$. $\mathcal{H}_L$ is the logical (or computational) factor, which reduces to a proper subspace $\mathcal{H}_L\subset\mathcal{H}_S$ when the "syndrome" cofactor is onedimensional, $\mathcal{H}_Z\simeq\mathbb{C}$. The summand $\mathcal{R}$ collects the noncomputational states in $\mathcal{H}_S$. A code can be algebraically characterized with respect to a suitable algebra $\mathcal{A}$ of operators on $\mathcal{H}_S$. The prototype example is a *noiseless code*, whereby the appropriate algebra $\mathcal{A}$ is the associative *interaction algebra* [24] containing the complex linear combinations of arbitrary products of $H_S$, all (or a subset of, see Sec. V) the error generators $S_\alpha$s, and the $\mathbb{1}$. Then, in general, $\mathcal{A}$ can be expressed, with respect to an appropriate basis in $\mathcal{H}_S$, as a direct sum of $d_J$-dimensional complex matrices, each appearing with a multiplicity $n_J$,

$$U\mathcal{A}U^\dagger=\oplus_{J\in\mathcal{J}}\mathbb{1}_{n_J}\otimes\mathrm{Mat}_{d_J}(\mathbb{C}), \tag{6}$$

where the change of basis $U$ in $\mathcal{H}_S$ is made explicit, and $\Sigma_{J\in\mathcal{J}}n_Jd_J=N$. With respect to the same basis, the algebra $\mathcal{A}'=\{X\in\mathrm{End}(\mathcal{H}_S)|[X,\mathcal{A}]=0\}$ [*commutant* of $\mathcal{A}$ in $\mathrm{End}(\mathcal{H}_S)$ [24]] is represented as

$$U\mathcal{A}'U^\dagger=\oplus_{J\in\mathcal{J}}\mathrm{Mat}_{n_J}(\mathbb{C})\otimes\mathbb{1}_{d_J}. \tag{7}$$

Thus, under the unitary map $U$, the state space $\mathcal{H}_S$ becomes isomorphic to

$$\mathcal{H}_S\simeq\oplus_{J\in\mathcal{J}}\mathcal{C}_J\otimes\mathcal{D}_J\simeq\oplus_{J\in\mathcal{J}}\mathbb{C}^{n_J}\otimes\mathbb{C}^{d_J}, \tag{8}$$

i.e., a direct sum of effectively bipartite subspaces. Each of the left (or right) factors in this decomposition can be associated with an $\mathcal{A}$ code (or $\mathcal{A}'$ code), respectively. Typically, to obtain a noiseless code (a DFS or a NS), one selects a fixed factor $\mathcal{C}_{J_*}=\mathcal{H}_L$ with dimension $N_L=n_{J_*}$—in which case, with respect to Eq. (5), $\mathcal{H}_Z=\mathcal{D}_{J_*}$ and $\mathcal{R}=\oplus_{J\neq J_*}\mathcal{C}_J\otimes\mathcal{D}_J$. Note that, because of Eq. (7), the code is an irreducible subspace of the commutant $\mathcal{A}'$ [24,9].

If $N_L\geqslant 2^{n_L}$, then $\mathcal{H}_L$ can protect the state space of $n_L$ logical qubits against noise in $\mathcal{A}$. While the *global* structure of $\mathcal{H}_L$ is sufficient for establishing storage or even existential controllability results over $\mathcal{H}_L$, an additional, crucial requirement on the *local* structure of $\mathcal{H}_L$ stems from the tensor product nature of QIP. In other words, an *encoded tensor product* structure on $\mathcal{H}_L$ is necessary for addressing notions of *efficient* simulation or *scalable* control over $\mathcal{H}_L$. This issue is especially evident in the encoded universality approach, where the primary algebraic structure one considers is the Lie algebra $\mathcal{L}$ generated under commutation by the set of accessible Hamiltonians [26]. (Formally, $\mathcal{L}$ plays the same role as $\mathcal{A}'$ in the noiseless coding approach.) Even if the code size $N_L$ is large enough to accommodate many qubits, without a precise mapping that defines encoded qubits, there is no way for assessing the potential of this set of interactions in terms of one- and two-qubit encoded gates useful for implementing a quantum circuit. In its essence, properly defining this encoded tensor product structure is equivalent to properly constructing qubits in a given physical system [25]. While no conclusive solution seems available to date, a practically motivated approach consists in identifying single-qubit encodings into *small* blocks of (two to four) physical qubits, and then inducing a tensor product structure by adjoining (or "conjoining") blocks [25,26,23]. To do so, the system $S$ is partitioned into clusters $\{c_\ell\}$ of physical qubits, i.e., $\mathcal{H}_S=\Pi_\ell\mathcal{H}^{(c_\ell)}$, and a mapping of the form (5) defines the state space $\mathcal{H}_{L_\ell}$ of the $\ell$th encoded qubit starting from $\mathcal{H}^{(c_\ell)}$. An overall structure as in Eq. (5) still emerges with

$$\mathcal{H}_L=\mathcal{H}_{L_1}\otimes\cdots\otimes\mathcal{H}_{L_{n_L}} \tag{9}$$

and $\mathcal{R}$ grouping all contributions involving $\mathcal{R}_\ell$ for at least one cluster. The code $\mathcal{H}_L$ can still be algebraically characterized as being, in general, embedded into one (or more) invariant subspaces of $\mathcal{A}'$—or $\mathcal{L}$, as appropriate [26,23]. For definiteness, I focus on the case where encoding is motivated by noise protection against $\mathcal{A}$ henceforth.

Let the set of encoded qubits be specified by encoded qubit observables,

$$\{\sigma_a^{L_\ell}\},\quad a=x,y,z;\quad \ell=1,\ldots,n_L, \tag{10}$$

satisfying Pauli-matrix commutation and anticommutation rules [25], and belonging to $\mathcal{A}'$. The relevant situation for introducing *encoded dynamical control* assumes that the natural system dynamics—specified by $H_S$ and possibly by some error generators in $\mathcal{N}$ whose effect is not eliminated by the encoding—are *expressible in terms of encoded qubit observables*. The goal is then to actively turn on/off selected encoded interactions without spoiling the benefits associated with the underlying encoding. Let $\mathcal{U}(\mathcal{H}_L)\simeq\mathcal{U}(2^{n_L})$ denote the group of unitary operators over $\mathcal{H}_L$. As in the unencoded case, the decoupling problem can be defined by a discrete set $\mathcal{G}^L=\{U_k^L\}$ of BB control operations, which represent encoded rotations over $\mathcal{H}_L$, $\mathcal{G}^L\subset\mathcal{U}(\mathcal{H}_L)$. In the simplest setting, $\mathcal{G}^L$ will itself form a group of encoded rotations (*encoded decoupling group*).

There are two minimal requirements for an operator $U_k^L$ to provide a legitimate unitary transformation over $\mathcal{H}_L$: (i) the gate should never draw the system outside the protected region, (ii) the qubit mapping should be preserved at the end of the gate [27]. Both conditions are satisfied if $U_k^L$ is generated by a Hamiltonian $A_k^L$ that is expressible in terms of encoded qubit observables—as it suffices for the present discussion. Accordingly, $U_k^L=\exp(i\eta\delta A_k^L)$, for effective strength and time parameters $\eta$, $\delta$, respectively—in such a way that the limit $\eta\to\infty$, $\delta\to 0$ with a finite BB action $\eta\delta$ can be achieved. While specifying *how* rotations are effected is irrelevant in the idealized BB limit of instantaneous control actions considered so far, it clearly becomes important in a realistic scenario where pulses have a finite duration and the evolution during the pulses should therefore be taken into account. Within AHT, compensation schemes have been developed for dealing with pulse-length corrections [2]. For application with encoded pulses, it is necessary to preliminarily make sure that the benefits of the encoding are not lost during the pulses. Suppose that unitary operations in $\mathcal{U}(\mathcal{H}_S)$ exist, whose action is *not* generated by Hamiltonians in $\mathcal{A}'$, but whose *net* effect matches, upon restriction to $\mathcal{H}_L$, that corresponding to some $U_k^L$. Then, besides the correction effects that also appear for $\exp(i\eta\delta A_k^L)$ for finite $\delta$ (and $A_k^L\in\mathcal{A}'$), additional errors are associated with the departure from $\mathcal{H}_L$ during $\delta$. While elimination of these effects motivates, in principle, the necessity of using encoded Hamiltonians, in practice different compensation techniques may be attempted, for instance, by resorting to robust control design [14].

With these definitions and caveats in mind, the applicability of decoupling methods directly carries over to the encoded case, once qubits and qubit operators are formally replaced with their encoded counterparts. Thus, if an operator $X=F[\{\sigma_a^j\}]$ has a given structure in terms of physical qubit observables, and a decoupling scheme according to $\mathcal{G}=\{U_k\}$ accomplishes a desired averaging effect via

$$\Lambda_{\mathcal{G}}(X)=\sum_{k\in\mathcal{K}}\tau_k U_k^\dagger X U_k, \tag{11}$$

then an equivalent averaging effect is obtained on an encoded operator $X^L$ with the same functional dependence $X^L=F[\{\sigma_a^{L_\ell}\}]$ on encoded qubit observables, via the encoded quantum operation

$$\Lambda_{\mathcal{G}^L}(X^L)=\sum_{k\in\mathcal{K}}\tau_k U_k^{L\dagger}X^L U_k^L. \tag{12}$$

## IV. ENGINEERING OF ENCODED CLOSED-SYSTEM DYNAMICS

Suppose that the noise generated by operators in $\mathcal{N}$ is fully taken care of by the chosen encoding, or that noise is not a concern to begin with, as in the encoded universality approach. Then encoded dynamical decoupling may provide a tool for encoded universal Hamiltonian simulation.

If, as assumed above, $H_S$ is expressible in terms of encoded observables, then the natural dynamics already implements a nontrivial logical transformation over the code. This provides the primary input to be exploited for engineering desired effective evolutions by encoded manipulations. As in the unencoded case, detailed knowledge on the structure of $H_S$ may or may not be a data of the problem. For *arbitrary* $H_S$, the results established in [11] imply that universal encoded simulation is achievable, in principle, if a finite encoded transformer can be constructed. The transformer groups so far identified [11] could be useful, in principle, for code size $N_L=2,3$—although, unfortunately, practical impact is limited by the large number of encoded rotations involved ($|\mathcal{G}^L|=24$ for the above-mentioned single-qubit transformer, and $|\mathcal{G}^L|\geqslant 168$ for dimension 3).

For the less ambitious target of generating a universal set of encoded evolutions starting from a given (known) $H_S$, simpler encoded programming strategies along the lines sketched in Sec. II may suffice. For instance, in the generic case, the Hamiltonian $H_S$, together with a noncommuting Hamiltonian $\Lambda_{\mathcal{G}^L}(H_S)$ obtained from $H_S$ via some encoded decoupling procedure, will fulfill the conditions for generating the whole Lie algebra $\mathrm{u}(\mathcal{H}_L)$ of anti-Hermitian encoded Hamiltonians—thereby implying universality over $\mathcal{H}_L$, at least at the existential level. The required group $\mathcal{G}^L$ of encoded BB rotations may be as simple as an encoded Carr-Purcell-type group. Let, for instance, $\pi^L$ denote an encoded $\pi$ rotation (acting on one or more qubits), and $\mathcal{G}^L_{CP}$ the associated encoded group. Then it is always possible to separate terms in $H_S$ which are symmetric ($s$) and antisymmetric ($a$) under $\mathcal{G}^L_{CP}$,

$$H_S=H_S^s+H_S^a, \tag{13}$$

with

$$\pi^L H_S^{(s)}\pi^L=H_S^s=\Pi_{\mathcal{G}^L}(H_S), \quad \pi^L H_S^{(a)}\pi^L=-H_S^a. \tag{14}$$

Thus, the above argument generally applies provided $[H_S,\Pi_{\mathcal{G}^L}(H_S)]=[H_S^s,H_S^a]\neq 0$, and the two Hamiltonians $H_S$, $\Pi_{\mathcal{G}_{CP}}(H_S)$ accordingly suffice for universal encoded control. An experimental demonstration using a DFS qubit encoded into the zero-quantum subspace of two nuclear spins is reported in [14].

### A. Example: A single NS-encoded qubit

A similar procedure could be relevant for obtaining universal encoded control over a NS-encoded qubit of three 1/2 spins under general collective noise [24,25,28]. Suppose that the noise generators $S_\alpha$ are the global, permutation-invariant Pauli operators $S_a=\Sigma_{j=1}^3\sigma_a^j$, $a=x,y,z$, and that the natural system Hamiltonian has the isotropically coupled form

$$H_S=\Omega S_z+J_{12}s_{12}+J_{23}s_{23}+J_{31}s_{31}, \tag{15}$$

where the Heisenberg exchange coupling $s_{jk}=\vec{\sigma}^j\cdot\vec{\sigma}^k$ and the parameters $\Omega, J_{jk}$ are real. Then $\mathcal{A}$ is the algebra of completely symmetric operators over $\mathcal{H}_S\simeq(\mathbb{C}^2)^{\otimes 3}$, and $\mathcal{A}'$ can be identified with the group algebra of the permutation group $\mathcal{S}_3$ (under the natural representation in $\mathcal{H}_S$). A NS code under $\mathcal{A}$ is identified by a correspondence of the form (5) – with $\mathcal{H}_Z\simeq\mathbb{C}^2$ carrying the irreducible representation of su(2) corresponding to total angular momentum $J=1/2$, $\mathcal{H}_L\simeq\mathbb{C}^2$ carrying the two-dimensional irreducible representation of $\mathcal{S}_3$, and $\mathcal{R}\simeq\mathbb{C}^4$ being the invariant subspace of symmetric states with total angular momentum $J=3/2$ [24,25,29]. Explicit expressions for encoded qubit observables (and the associated logical states) are given in [25]. In particular,

$$\sigma_x^L=_L\frac{1}{2}(\mathbb{1}+s_{12}), \quad \sigma_y^L=_L\frac{\sqrt{3}}{6}(s_{23}-s_{31}), \tag{16}$$

where the notation $=_L$ denotes identity upon restriction to the logical subsystem. This allows to rewrite the Hamiltonian (15), up to irrelevant contributions which are constant over $\mathcal{H}_L$, as

$$H_S=_L(2J_{12}-J_{23}-J_{31})\sigma_x^L+\sqrt{3}(J_{23}-J_{31})\sigma_y^L. \tag{17}$$

Note that the vanishing of $H_S$ for a fully symmetric coupling network, $J_{jk}=J\ \ \forall j,k$, correctly verifies the identity action of permutation-invariant operators over $\mathcal{H}_L$ ($H_S\in\mathcal{A}\cap\mathcal{A}'$ in this case). Given $H_S$ in the form (17), the above-mentioned universality scheme based on encoded Carr-Purcell sequences becomes applicable, in principle, provided one has to ability to enact rapid, encoded $\pi^L$ pulses. Because, with respect to the chosen code, the action of the $\sigma_x^L$ operator is effectively identical to swapping the physical qubits 1 and 2, this is, for instance, achievable if the exchange Hamiltonian $s_{12}$ can be switched on for the appropriate time. As a result, one generates an effective encoded Hamiltonian $\Pi_{\mathcal{G}^L_{CP_x}}(H_S)=(2J_{12}-J_{23}-J_{31})\sigma_x^L$, which, together with $H_S$, allows for universality.

Although not directly applicable to the weakly coupled molecule used in [28] to experimentally realize the above three-spin NS, and generally demanding for NMR implementations as the $J$-coupling parameters are not controllable, these ideas could prove viable for a wide class of QIP devices supporting, in principle, fully tunable Heisenberg interactions [30]. In particular, explicit universality constructions relevant to solid-state quantum computing architectures based on exchange interactions are provided in [13].

### B. Example: A pair of DFS-encoded qubits

As a further example, motivated from NMR, imagine two pairs of spin-1/2 nuclei, corresponding to different species (hydrogen and carbon, for instance), subjected to block-collective dephasing noise. Let the two clusters be associated with the pairs (1,2) and (3,4), respectively. Then the two error generators are $S_z^{(c_1)}=\sigma_z^1+\sigma_z^2$, $S_z^{(c_2)}=\sigma_z^3+\sigma_z^4$, and $\mathcal{A}=\mathcal{A}_z^{(c_1)}\otimes\mathcal{A}_z^{(c_2)}$ in terms of the interaction algebras $\mathcal{A}_z^{(c_\ell)}$ generated by $S_z^{(c_\ell)}$ for dephasing on two qubits [14]. Protection against errors in $\mathcal{A}$ can be accomplished by replacing each physical pair with a DFS-encoded qubit supported by the states having $S_z^{(c_1)}=0$ and $S_z^{(c_2)}=0$, i.e.,

$$|i_{L_1} j_{L_2}\rangle = |i_{L_1}\rangle \otimes |j_{L_2}\rangle, \quad |0_{L_\ell}\rangle = |01\rangle^{(c_\ell)},$$
$$|1_{L_\ell}\rangle = |10\rangle^{(c_\ell)}, \quad \ell = 1,2. \tag{18}$$

In this case, with reference to the general structure (5), each DFS encoding has the form $\mathcal{H}^{(c_\ell)} \simeq \mathcal{H}_{L_\ell} \oplus \mathcal{R}_\ell$, with $\mathcal{H}_{L_\ell} = \mathrm{span}\{|0_{L_\ell}\rangle, |1_{L_\ell}\rangle\} \simeq \mathbb{C}^2$, $\mathcal{R}_\ell = \mathrm{span}\{|00\rangle_\ell, |11\rangle_\ell\} \simeq \mathbb{C}^2$ ($\mathcal{H}_{Z_\ell} \simeq \mathbb{C}$ is irrelevant for subspace encodings). Correspondingly, for four spins, $\mathcal{H}_L \simeq \mathbb{C}^2 \otimes \mathbb{C}^2$ is a four-dimensional subspace of the six-dimensional zero-quantum subspace corresponding to total $z$ angular momentum $S_z = \Sigma_j \sigma_z^j = 0$, $j=1,\ldots,4$, and $\mathcal{R}$ collects all contributions where $S_z^{(c_\ell)} \neq 0$ for at least one pair. Encoded observables for the above qubits are provided, for example, by the choice [14]

$$\sigma_z^{L_1} =_L \frac{1}{2}(\sigma_z^1 - \sigma_z^2), \quad \sigma_x^{L_1} =_L \frac{1}{2}(\sigma_x^1 \sigma_x^2 + \sigma_y^1 \sigma_y^2), \tag{19}$$

and similarly for qubit $L_2$.

In general, the physical spin system will exhibit a strongly coupled spectrum described by an internal Hamiltonian composed of both Zeeman spin-field and indirect spin-spin interactions [3,14],

$$H_S = \sum_{j=1,\ldots,4} \pi \nu_j \sigma_z^j + \sum_{j<j'=1,\ldots,4} \frac{\pi}{2} J_{jj'} \vec{\sigma}^j \cdot \vec{\sigma}^{j'}$$
$$= \sum_j H_{Z_j} + \sum_{j<j'} H_{J_{jj'}}, \tag{20}$$

where the chemical shifts and $J$-coupling parameters are understood in frequency units. For typical values of the static Zeeman field, the contribution to the total energy of a given pair of spins $(j,j')$ due to the coupling $H_{J_{jj'}}$ can be treated as a perturbation with respect to $H_{Z_{j(j')}}$—the diagonal $\sigma_z^j \sigma_z^{j'}$ and off-diagonal $\sigma_x^j \sigma_x^{j'} + \sigma_y^j \sigma_y^{j'}$ terms leading to first- and second-order correction effects in $J_{jj'}$, respectively. Because the differences in the chemical shifts $|\nu_j - \nu_{j'}|$ are larger when different nuclear species are involved, the approximation of neglecting off-diagonal couplings (weak-coupling limit) is well justified for heteronuclear interactions. Thus, Eq. (20) can be effectively replaced by

$$H_S = \sum_{j=1,\ldots,4} \pi \nu_j \sigma_z^j + \frac{\pi}{2}(J_{12} \vec{\sigma}^1 \cdot \vec{\sigma}^2 + J_{34} \vec{\sigma}^3 \cdot \vec{\sigma}^4)$$
$$+ \sum_{j=1,2; j'=3,4} \frac{\pi}{2} J_{jj'} \sigma_z^j \sigma_z^j. \tag{21}$$

In terms of the encoded qubit observables given in Eq. (19), the chemical shift terms can be immediately rewritten as combinations of $S_z^{(c_\ell)}$ operators (constant over $\mathcal{H}_L$) and logical $\sigma_z^{L_\ell}$ operators, whereas the homonuclear spin-spin interactions contribute with logical $\sigma_x^{L_\ell}$ operators (and additional constant terms proportional to $\sigma_z^j \sigma_z^j$). The remaining heteronuclear bilinear couplings in Eq. (21) give

$$A S_z^{(c_1)} S_z^{(c_2)} + B S_z^{(c_1)} \sigma_z^{L_2} + C S_z^{(c_2)} \sigma_z^{L_1} + D \sigma_z^{L_1} \sigma_z^{L_2}, \tag{22}$$

with coefficients

$$A = \frac{1}{8}(J_{13} + J_{14} + J_{23} + J_{24}), \quad B = \frac{1}{4}(J_{13} - J_{14} + J_{23} - J_{24}),$$
$$C = \frac{1}{4}(J_{13} + J_{14} - J_{23} - J_{24}), \quad D = \frac{1}{4}(J_{13} - J_{14} - J_{23} + J_{24}). \tag{23}$$

Again, an overall identity action over $\mathcal{H}_L$ is obtained for a fully symmetric coupling network—as well as for other special coupling patterns in the system. Assuming that none of these nongeneric circumstances is met, the action of the overall Hamiltonian $H_S$ over the code is finally

$$H_S =_L \pi(\Delta\nu_{12} \sigma_z^{L_1} + \Delta\nu_{34} \sigma_z^{L_2} + J_{12} \sigma_x^{L_1} + J_{34} \sigma_x^{L_2} + D \sigma_z^{L_1} \sigma_z^{L_2}), \tag{24}$$

with $\Delta\nu_{12} = \nu_1 - \nu_2$, $\Delta\nu_{34} = \nu_3 - \nu_4$, respectively.

Remarkably, the structure of the Hamiltonian (24) for the two encoded qubits is very similar to that of the Hamiltonian describing two weakly coupled physical spins—except that, in the ordinary NMR setting, control along the transverse $\sigma_x$ (or $\sigma_y$) directions is directly supplied by external radio-frequency fields. While the potential of Eq. (24) for universal encoded control should be expected by analogy with the unencoded case, various schemes may be specified depending on the actual control capabilities. At the existential level, universality follows from the ability of rapidly effecting a single encoded rotation, say a "hard" (nonselective) encoded $\pi^L$ pulse, $\pi_x^L = \pi_x^{L_1} \pi_x^{L_2}$ about the encoded $x$ axis. Because the associated averaging produces the encoded effective Hamiltonian $\Pi_{\mathcal{G}^L}(H_S) = J_{12} \sigma_x^{L_1} + J_{34} \sigma_x^{L_2} + D \sigma_z^{L_1} \sigma_z^{L_2}$, by appropriately alternating evolution periods under $H_S$ and $\Pi_{\mathcal{G}^L}(H_S)$, arbitrary encoded Hamiltonians can be enacted, in principle, through repeated commutation [20].

Constructive prescriptions become possible as soon as a wider range of control options is accessible. In particular, schemes for effecting a universal set of encoded gates—including (i) all single-qubit rotations and (ii) a two-qubit phase coupling—can be provided. Assume that, in formal analogy with the standard (unencoded) NMR setting, the parameters $J_{12}$ and $J_{34}$ are independently tunable at will. This is the scenario also analyzed in [13]. To selectively rotate each encoded qubit about the $x$ axis, turn on the appropriate $\sigma_x^{L_\ell}$ term in Eq. (24) and simultaneously refocus $\sigma_z^{L_\ell}$ evolutions by applying encoded $\pi_x^{L_1} \pi_x^{L_2}$ pulses as above. Note that although this action preserves the $\sigma_z^{L_1} \sigma_z^{L_2}$ coupling, the corresponding evolution can be neglected under the usual assumption that single-qubit rotations can be effected rapidly enough. In fact, the whole evolution induced by Eq. (24) can be neglected during the $x$ rotation if the strength of the appropriate $J_{12}$ or $J_{34}$ parameter can be arbitrarily controlled as assumed. Also, note that the same sequence of hard $\pi_x^{L_1}\pi_x^{L_2}$ pulses allows to selectively leave this phase coupling alone if both $J_{12}$ and $J_{34}$ are kept to zero—except during the (infinitesimal) durations of the refocusing pulses. Selective rotations about the appropriate encoded $z$ axis are only slightly more demanding, as they require the hard logical $\pi_x^L$ pulses to be replaced by "soft" encoded $\pi_x^{L_\ell}$ on the intended spin alone. The relevance of these and related universality constructions is further highlighted in [13].

However, the above tunability requirements are too stringent, in general, for implementations where the natural Hamiltonian is *always* active. This is, for instance, the relevant situation in NMR, where interactions can be only effectively set to zero via appropriate refocusing, and additional constraints also arise for typical implementation parameters. In particular, one may expect that the encoded chemical shift evolutions will dominate in Eq. (24), i.e., $\Delta\nu_{12},\Delta\nu_{34}\gg J_{12},J_{34}\gg D$. Nevertheless, universal control can still be gained by having access to a small set of encoded operations. In the scheme I now outline, *no ability of arbitrarily tuning encoded Hamiltonians is assumed*, apart from the *fixed* combinations of strength and times corresponding to logical $\pi^L$ pulses. Unlike in the previous case, however, encoded $\pi^L$ rotations about *two* noncommuting axes will be generally needed. The encoded decoupling schemes for implementing the required universal set of coupling are as follows. Selective rotations about the appropriate $z$ axis are easiest, as one can take advantage of the natural averaging of the $\sigma_x^{L_\ell}$ and $\sigma_z^{L_1}\sigma_z^{L_2}$ that is enforced by the above hierarchy of energy scales. One need only refocus the undesired encoded phase evolution (say, $\sigma_z^{L_2}$) by applying sequences of encoded selective $\pi_x^{L_\ell}$ pulses (say, $\pi_x^{L_2}$). It is easily seen that decoupling purely based on simultaneous $\pi_x^{L_1}\pi_x^{L_2}$ pulses is not sufficient, without the ability to separately control the coefficients in the resulting effective Hamiltonian, to implement selective $x$ rotations or the two-qubit phase coupling. However, a selective implementation of, say, $\sigma_x^{L_1}$ can be engineered, for instance, by the encoded pulse sequence

$$\pi_x^{L_1}\pi_x^{L_2}-\Delta t-\pi_x^{L_1}\pi_z^{L_2}-\Delta t-\pi_x^{L_1}\pi_x^{L_2}-\Delta t-\pi_x^{L_1}\pi_z^{L_2},$$

$$\Delta t=T_c/4, \tag{25}$$

which corresponds to subjecting qubit 2 to maximal encoded averaging according to $\mathcal{G}^L_{max}=\{\mathbb{1}^L,\sigma_x^L,\sigma_y^L,\sigma_z^L\}$, and cycling twice qubit 1 through the encoded Carr-Purcell sequence refocusing $\sigma_z^{L_1}$. A similar sequence works for rotating the second encoded qubit, by interchanging 1 and 2 in Eq. (25). If all single-qubit interactions are refocused instead, by applying the above $\mathcal{G}^L_{max}$ to both qubits,

$$\pi_x^{L_1}\pi_x^{L_2}-\Delta t-\pi_z^{L_1}\pi_z^{L_2}-\Delta t-\pi_x^{L_1}\pi_x^{L_2}-\Delta t-\pi_z^{L_1}\pi_z^{L_2},$$

$$\Delta t=T_c/4, \tag{26}$$

the $\sigma_z^{L_1}\sigma_z^{L_2}$ evolution is selectively extracted from Eq. (24).

The problem of how to actually effect the required logical $\pi^L$ rotations may remain challenging in reality, as the evolutions induced by the external controlling fields will not, in general, correspond to encoded Hamiltonians. Again, this is the case in NMR for control with available radio-frequency magnetic fields [14]. Thus, even though it is always possible to enforce a unitary propagator whose net action is the same as that of the required encoded rotation, the corresponding sequence of physical gates may be complicated and, as noted above, will cause departure from the code during the finite pulsing time. A possible way out is explored in [14], based on the idea of compensating for the resulting exposure to noise by optimizing the length of the relevant control sequences and by incorporating intrinsic robustness features using composite pulse techniques [31]. The situation is relatively straightforward for the special case of $\pi^L$ pulses, as the encoded rotations they effect have a simple translation in terms of realizable physical spin propagators. The simplest instance is the action of a $\pi_x^{L_1}\pi_x^{L_2}$, which is identical to the action induced over $\mathcal{H}_L$ by a hard $\pi_x$ pulse on all four spins. In fact, the following correspondence for the logical $\pi^L$ pulses involved in the above encoded sequences holds (up to irrelevant phase factors):

$$\pi_x^{L_1}\leftrightarrow\sigma_x^1\sigma_x^2,$$

$$\pi_x^{L_2}\leftrightarrow\sigma_x^3\sigma_x^4,$$

$$\pi_x^{L_1}\pi_x^{L_2}\leftrightarrow\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4,$$

$$\pi_x^{L_1}\pi_z^{L_2}\leftrightarrow\sigma_x^1\sigma_x^2\sigma_z^4,$$

$$\pi_z^{L_1}\pi_x^{L_2}\leftrightarrow\sigma_z^2\sigma_x^3\sigma_x^4,$$

$$\pi_z^{L_1}\pi_z^{L_2}\leftrightarrow\sigma_z^2\sigma_z^4.$$

Accordingly, universal manipulation of encoded evolutions with reduced error rate is still achievable if robust ways for effecting the above physical operations can be devised [14,32].

## V. ENGINEERING OF ENCODED OPEN-SYSTEM DYNAMICS

A further direction for application is suggested by looking at encoded dynamical decoupling as a tool for encoded quantum error suppression. Similar to the idea of concatenating passive noise protection via DFSs/NSs with quantum error correction—by operating error-correcting codes directly on DFS/NS-encoded qubits [33], the idea is now to concatenate passive noise protection with active dynamical control—by effecting error suppression directly on encoded qubits.

### A. Example: A decohering encoded qubit

By analogy to the prototype example of a single decohering qubit analyzed in [5], the prototype situation is provided by a single decohering *encoded* qubit. Imagine a system $S$ composed of two physical qubits, which are subjected to a purely dephasing interaction as a result of the coupling to two bosonic environments $B_1$, $B_2$, i.e., the overall system is described by a Hamiltonian of the form

$$H = H_S + H_{B_1} + H_{B_2} + H_{SB_1} + H_{SB_2} \tag{27}$$

for the uncoupled subsystem's Hamiltonians

$$H_S = \sum_{j=1,2} \frac{\omega_k}{2} \sigma_z^j,$$

$$H_{B_1} + H_{B_2} = \sum_k \omega_k b_k^{(1)\dagger} b_k^{(1)} + \sum_\ell \omega_\ell b_\ell^{(2)\dagger} b_\ell^{(2)}, \tag{28}$$

and a total interaction Hamiltonian

$$H_{SB_1} + H_{SB_2} = \sum_{j=1,2} \sigma_z^j \otimes \Bigg\{ \sum_k g_{kj}^{(1)} (b_k^{(1)} + b_k^{(1)\dagger}) + \sum_\ell g_{\ell j}^{(2)} (b_\ell^{(2)} + b_\ell^{(2)\dagger}) \Bigg\}. \tag{29}$$

In the above equations, $\omega_j$, $j=1,2$ are the natural frequencies of single-qubit evolutions, $b_k^{(1)}, b_k^{(1)\dagger}$ are bosonic operators for the environment $B_1$ (similarly for $B_2$), and the parameters $g_{kj}^{(1)}$ ($g_{\ell j}^{(2)}$) determine the coupling strength of qubit $j$ to mode $k$ of bath 1 (or mode $\ell$ of bath 2, respectively). Identity operators on the appropriate subsystems are also understood as needed. Clearly, implementing decoupling according to the group $\mathcal{G}_{CP_x} = \{\mathbb{1}, \sigma_x^1 \sigma_x^2\}$ would dynamically suppress both error generators—thereby allowing to reduce the dephasing noise on both qubits to a level, in principle, as low as desired. However, as recalled earlier, this is only effective in practice in the limit of fast control, where [5,10]

$$T_c \lesssim \min_i \{\tau_c^{(i)}\}, \tag{30}$$

$\tau_c^{(i)}$ denoting the (shortest) correlation time of the $i$th environment. Suppose now that $B_2$ is sufficiently "slow," but $B_1$ is "fast," so that decoupling at the rate determined by $\tau_c^{(1)}$ cannot be afforded. Although implementing decoupling at the slower rate set by $B_2$ reduces the noise effects originating from $B_2$, there is no actual guarantee that the overall error rate is suppressed due to the possibility of decoherence acceleration from modes at frequency higher than $\sim 1/\tau_c^{(2)}$ in $B_1$ [5]. In any event, both qubits would remain effectively exposed to noise. Are there other noise control options worth being considered?

For arbitrary dephasing and just two qubits, quantum error correction does not help, neither does passive noise protection—unless some symmetries can be identified in the noise. Suppose that, in the above interaction Hamiltonian (29), $g_{k1}^{(1)} = g_{k2}^{(1)} = g_k^{(1)}$ to a good accuracy, meaning that the environment $B_1$ couples collectively to the qubits. Then the overall interaction can be rewritten as

$$H_{SB_1} + H_{SB_2} = S_z \otimes \mathcal{B}_z^{(1)} + \sigma_z^1 \otimes \mathcal{B}_1^{(2)} + \sigma_z^2 \otimes \mathcal{B}_2^{(2)}, \tag{31}$$

for $S_z = \sigma_z^1 + \sigma_z^2$ as earlier defined, and appropriate environment operators $\mathcal{B}_z^{(1)} = \Sigma_k g_k^{(1)} (b_k^{(1)} + b_k^{(1)\dagger})$, $\mathcal{B}_j^{(2)} = \Sigma_\ell g_{\ell j}^{(2)} (b_\ell^{(2)} + b_\ell^{(2)\dagger})$, $j=1,2$. Physically, this situation corresponds to a noise process consisting of both fast collective dephasing, due to $B_1$, and slow independent dephasing, due to $B_2$. Whenever a symmetry in the open-system dynamics is present, major gains should be expected by seeking for an appropriate encoding into a DFS or a NS. In our case, the relevant DFS is the one already introduced in Sec. IV B, i.e., the one spanned by states corresponding to total zero angular momentum along $z$, $S_z = 0$. Thus, a logical DFS qubit is defined by the encoding given in Eq. (18) (for just one qubit), with

$$\mathcal{H}_L = \text{span}\{|0_L\rangle, |1_L\rangle\} = \text{span}\{|01\rangle, |10\rangle\}. \tag{32}$$

By construction, this encoded qubit is perfectly protected (with infinite distance) against the noise due to $B_1$. However, mixing with the degrees of freedom of $B_2$ is still induced through the error generators $\sigma_z^j$, $j=1,2$ in Eq. (31). The key observation is that this residual noise preserves the coding space, corresponding to a purely decohering coupling between the encoded qubit and $B_2$. Because, by using Eq. (19), the error generators are simply expressed in terms of encoded observables

$$\sigma_z^1 =_L \frac{S_z}{2} + \sigma_z^L, \quad \sigma_z^2 =_L \frac{S_z}{2} - \sigma_z^L, \tag{33}$$

the action of the total Hamiltonian $H$, Eq. (27) on $\mathcal{H}_L$ finally rewrites as

$$H =_L \Delta\omega \sigma_z^L + H_B + \sigma_z^L \otimes \mathcal{B}_z, \tag{34}$$

where $\Delta\omega = (\omega_1 - \omega_2)/2$, $H_B = H_{B_1} + H_{B_2}$, and $\mathcal{B}_z = (\mathcal{B}_1^{(2)} - \mathcal{B}_2^{(2)})/2$. This form makes it clear that the action of $H$ on the encoded qubit is formally identical to the action of the diagonal spin-boson Hamiltonian on the physical qubit considered in [5,34].

By the same argument valid for suppressing decoherence on a single physical qubit, implementing a sequence of equally spaced, encoded $\pi_x^L$ pulses sufficiently fast with respect to the *slower* rate determined by $1/\tau_c^{(2)}$ is now guaranteed to suppress the encoded error rate by a factor of (at least) $O[(T_c/\tau_c^{(2)})^2]$ [6]. This corresponds to encoded dynamical decoupling according to $\mathcal{G}_{CP_x}^L = \{\mathbb{1}^L, \sigma_x^L\}$, where $\sigma_x^L$ is again given in Eq. (19). Thus, if the required encoded $\pi_x^L$ rotation can be effected, suppression of the slow dephasing from $B_2$ can be accomplished without reintroducing exposure of the encoded qubit to the fast dephasing from $B_1$.

### B. Generalizations

Some generalizations of the above example are worth pointing out. First, the same encoded decoupling scheme is effective at suppressing encoded phase errors in a situation where the internal two-qubit Hamiltonian is governed by a generally anisotropic exchange Hamiltonian

$$H_S=\frac{\omega_1}{2}\sigma_z^1+\frac{\omega_1}{2}\sigma_z^2+J_{12}^x\sigma_x^1\sigma_x^2+J_{12}^y\sigma_y^1\sigma_y^2+J_{12}^z\sigma_z^1\sigma_z^2, \tag{35}$$

with $J_{12}^x=J_{12}^y=J$ and either $J_{12}^z$ arbitrary, $J_{12}^z=0$, or $J_{12}^z=J$ (the so-called $XXZ$, $XY$, and isotropic models, respectively [13]). In the isotropic case, in particular, which was also examined in Sec. IV and is directly relevant to NMR and solid-state devices, the encoded open-system Hamiltonian (34) is modified to an encoded spin-boson Hamiltonian [34] as

$$H'={}_L\Delta\omega\sigma_z^L+J\sigma_x^L+H_B+\sigma_z^L\otimes\mathcal{B}_z. \tag{36}$$

Once encoded suppression of the $\sigma_z^L$ error generator is accomplished as above, control according to the $\sigma_z^L$ Hamiltonian can be re-introduced, if desired, by implementing one of the programming schemes described in [8]—implying the possibility of retaining universal noise-suppressed encoded control.

Situations involving hybrid dephasing processes with highly correlated components and slow residual noise in multiqubit systems may be also relevant. If, for instance, in a four-qubit system, dephasing from $B_1$ is pairwise correlated on qubits (1,2) and (3,4) as considered in Sec. IVB, then encoding into the tensor product of the two DFSs described there ensures protection against $B_1$. Once this is done, applying sequences of encoded $\pi_x^{L_1}\pi_x^{L_2}$ pulses at the appropriate rate causes a suppression of any (independent or correlated) residual dephasing due to $B_2$.

Finally, similar ideas may be more generally applicable to situations where the encoded error generators for the residual noise are expressible through encoded observables. If, for instance, starting from the above two-qubit Hamiltonian (27), the action over $\mathcal{H}_L$ may be cast into the form

$$H={}_L\Delta\omega\sigma_z^L+H_B+\sigma_z^L\otimes\mathcal{B}_z+\sigma_x^L\otimes\mathcal{B}_x \tag{37}$$

for appropriate operators $\mathcal{B}_z,\mathcal{B}_x$ on environment $B_2$, then the encoded qubit suffers from both decoherence and dissipation effects. The resulting error rate can be suppressed, in principle, by using the encoded annihilator $\mathcal{G}_{max}^L=\{\mathbb{1}^L,\sigma_x^L,\sigma_y^L,\sigma_z^L\}$ introduced above. Note that, in terms of coupling to the physical degrees of freedom, the interaction (37) involves *multiple-qubit* excitations as error generators. Interestingly, system-bath couplings allowing for similar multiqubit processes to the lowest order are the only class of interactions, in addition to those exhibiting spatial symmetry, for which DFSs are known to exist [35].

To summarize, if noise has both slow and fast components, so that achieving full decoupling of the physical degrees of freedom becomes unfeasible, then encoded dynamical decoupling may be useful in situations where a dominant symmetry in the fast noise can be exploited to obtain encoded qubits, and the generators for the residual slow noise are expressible in terms of encoded observables. In this event, using encoded rather than physical degrees of freedom allows the system to benefit already from enacting decoupling operations at the slower rate. However, this looser constraint on control time scales competes, as already emphasized, with tighter symmetry constraints on the useful control Hamiltonians—as encoded Hamiltonians are demanded, and they are not always easily available in physical systems.

## VI. CONCLUSIONS

I have analyzed the relevance and potential usefulness of active dynamical control, as inspired by NMR spectroscopy, in the light of the QIP-motivated notion of encoded degrees of freedom. Ultimately, the resulting approach of concatenating active control with encoded qubits naturally stems from the program of regarding the information-carrying subsystems as the primary degrees of freedom for realizing QIP. Applications of encoded dynamical decoupling to control both Hamiltonian and non-Hamiltonian encoded evolutions have been suggested by illustration through specific examples. In spite of the important role played by liquid-state NMR QIP as a motivating experimental setting accessible with present-day technology, it is worth stressing that the principles of encoded dynamical decoupling as outlined here are potentially useful, as their unencoded counterpart, in full generality. In particular, applications to solid-state proposals based on coupled quantum dots [36], silicon-based devices [37], or Josephson-junction circuits [38] appear especially promising, thanks to the natural availability of controllable exchange-type interactions. The usefulness of decoupling techniques for control of neutral atoms in optical lattices [39] would also be worth exploring in view of rapid experimental progress in the field [40]. While assessing the actual viability of the proposed schemes for a specific realization is only possible upon careful consideration of both the natural internal dynamics and the available control resources, I hope that these ideas will motivate further investigation and serve as guiding principles to expand our capabilities to manipulate quantum systems and quantum information.

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