Optimal teleportation based on bell measurements

Sergio Albeverio,[1,*] Shao-Ming Fei,[1,2,†] and Wen-Li Yang[3,4,‡]
[1]*Institut für Angewandte Mathematik, Universität Bonn, D-53115, Germany*
[2]*Department of Mathematics, Capital Normal University, Beijing 100037, China*
[3]*Physikalisches Institut der Universität Bonn, 53115 Bonn, Germany*
[4]*Institute of Modern Physics, Northwest University, Xian 710069, China*
(Received 26 February 2002; published 15 July 2002)

We study optimal teleportation based on Bell measurements. An explicit expression for the quantum channel associated with the optimal teleportation with an arbitrary mixed state resource is presented. The optimal transmission fidelity of the corresponding quantum channel is calculated and shown to be related to the fully entangled fraction of the quantum resource, rather than the singlet fraction as in the standard teleportation protocol.

DOI: 10.1103/PhysRevA.66.012301 PACS number(s): 03.67.Hk, 89.70.+c, 03.65.−w

Quantum teleportation protocols play an important role in quantum information processing. In terms of a classical communication channel and a quantum resource (a nonlocal entangled state like an EPR pair of particles), the teleportation protocol gives ways to transmit an unknown quantum state from a sender traditionally named "Alice" to a receiver "Bob" who are spatially separated. These teleportation processes can be viewed as quantum channels. The nature of a quantum channel is determined by the particular protocol and the state used as a teleportation resource [1–3]. The standard teleportation protocol T_0 proposed in [1] uses Bell measurements and Pauli rotations. When the maximally entangled pure state $|\Phi\rangle = 1/\sqrt{n}\Sigma_{i=0}^{n-1}|ii\rangle$ is used as the quantum resource, it provides an ideal noiseless quantum channel $\Lambda_{T_0}^{(|\Phi\rangle\langle\Phi|)}(\rho)=\rho$. However, in a realistic situation, instead of the pure maximally entangled states, Alice and Bob usually share a mixed entangled state due to the decoherence. Teleportation using a mixed state as an entangled resource is, in general, equivalent to having a noisy quantum channel. Recently, an explicit expression for the output state of the quantum channel associated with the standard teleportation protocol T_0 with an arbitrary mixed state resource has been obtained [4,5].

In this paper we consider the following problem. Alice and Bob previously only share a pair of particles in an arbitrary mixed entangled state χ. In order to teleport an unknown state to Bob, Alice first performs a joint Bell measurement on her particles (particle 1 and particle 2) and gives her result to Bob by the classical communication channel. Then Bob, instead of the *Pauli* rotation like in the standard teleportation protocol [1], tries his best to choose a particular unitary transformation which depends on the quantum resource χ, so as to get the maximal transmission fidelity. We call our teleportation protocol the optimal teleportation based on the Bell measurement. We derive an explicit expression for the quantum channel associated with the optimal teleportation with an arbitrary mixed state resource. The transmission fidelity of the corresponding quantum channel is given in terms of the fully entangled fraction of the quantum resource.

Let $\{|i\rangle, i=0,\ldots,n-1\}$, $n<\infty$, be an orthogonal normalized basis of an n-dimensional Hilbert space $\mathcal{H}$. Any linear operator A: $\mathcal{H}\to\mathcal{H}$ can be represented by an $n\times n$ matrix as follows:

$$A(|i\rangle)=\sum_{j=0}^{n-1} A_{ij}|j\rangle, \quad A_{ij}\in\mathbb{C}.$$

We shall only consider the following three-tensor Hilbert space: $\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$ where Alice has the first and the second Hilbert space, and the third one belongs to Bob. Let h and g be $n\times n$ matrices such that $h|j\rangle=|(j+1) \bmod n\rangle$, $g|j\rangle=\omega^j|j\rangle$, with $\omega=\exp\{-2i\pi/n\}$. We can introduce n^2 linear-independent $n\times n$ matrices $U_{st}=h^t g^s$, which satisfy

$$U_{st}U_{s't'}=\omega^{st'-ts'}U_{s't'}U_{st}, \quad \mathrm{tr}(U_{st})=n\,\delta_{s0}\delta_{t0}. \tag{1}$$

One can also check that $\{U_{st}\}$ satisfy the condition of *bases of the unitary operators* in the sense of [6], i.e.,

$$\begin{aligned}\mathrm{tr}(U_{st}U^\dagger_{s't'})&=n\,\delta_{tt'}\delta_{ss'},\\ U_{st}\ U^\dagger_{st}&=I_{n\times n},\end{aligned} \tag{2}$$

where $I_{n\times n}$ is the $n\times n$ identity matrix. $\{U_{st}\}$ form a complete basis of $n\times n$ matrices, namely, for any $n\times n$ matrix W, W can be expressed as

$$W=\frac{1}{n}\sum_{s,t}\ \mathrm{tr}(U^\dagger_{st}W)U_{st}. \tag{3}$$

From $\{U_{st}\}$ we can introduce the generalized Bell states,

$$|\Phi_{st}\rangle=(1\otimes U^*_{st})|\Phi\rangle=\frac{1}{\sqrt{n}}\sum_{i,j}\ (U_{st})^*_{ij}|ij\rangle,$$

$$\text{and}\quad |\Phi_{00}\rangle=|\Phi\rangle, \tag{4}$$

---

*Email address: albeverio@uni-bonn.de
†Email address: fei@uni-bonn.de
‡Email address: wlyang@th.physik.uni-bonn.de

$|\Phi_{st}\rangle$ are all maximally entangled states and form a complete orthogonal normalized basis of $\mathcal{H}\otimes\mathcal{H}$ shared by Alice and Bob. For any state χ shared by Alice and Bob, let us introduce the *singlet fraction* [2]: $F=\langle\Phi|\chi|\Phi\rangle$. In general, all the maximally entangled pure states are equivalent to $|\Phi\rangle$: $|\Psi_{\max}\rangle=1\otimes U|\Phi\rangle$, where U is a unitary transformation. One can define the *fully entangled fraction* [2] of a state χ by

$$\mathcal{F}(\chi)=\max\{\langle\Phi|(1\otimes U^\dagger)\ \ \chi\ \ (1\otimes U)|\Phi\rangle\},\qquad \text{for all}\ \ UU^\dagger=U^\dagger U=I_{n\times n}. \tag{5}$$

Since the group of unitary transformations in n dimensions is compact, there exists a unitary matrix W_χ such that

$$\mathcal{F}(\chi)=\langle\Phi|(1\otimes W_\chi^\dagger)\ \ \chi\ \ (1\otimes W_\chi)|\Phi\rangle. \tag{6}$$

Suppose now Alice and Bob previously shared a pair of particles in an arbitrary mixed entangled state χ. To transform an unknown state to Bob, Alice first performs a joint Bell measurement based on the generalized Bell states Eq. (4) on her parties. According to the measurement results of Alice, Bob chooses particular unitary transformations $\{T_{st}\}$ to act on his particle.

Theorem 1. The teleportation protocol defined by $\{T_{st}\}$, when used with an arbitrary mixed state with density matrix χ as a resource, acts as a quantum channel

$$\Lambda^{(\chi)}(\{T\})(\rho)=\frac{1}{n^2}\sum_{s,t}\sum_{s',t'}\langle\Phi_{st}|\chi|\Phi_{s't'}\rangle \times\left\{\sum_{\gamma\beta} T^\dagger_{\gamma\beta}U_{st}U_{\gamma\beta}\ \ \rho\ \ U^\dagger_{\gamma\beta}U^\dagger_{s't'}T_{\gamma\beta}\right\}. \tag{7}$$

Proof. The proof can be given in two steps:

Step 1: *Pure entangled state as a resource*. Each entangled pure state $|\Psi\rangle$ shared by Alice and Bob has the form

$$|\Psi\rangle=\sum_{i,j=0}^{n-1}a_{ij}|ij\rangle,\qquad \sum_{i,j=0}^{n-1}|a_{ij}|^2=1,\qquad a_{ij}\in\mathbb{C}. \tag{8}$$

Let A be the $n\times n$ matrix with elements $(A)_{ij}=a_{ij}$, $a_{ij}\in\mathbb{C}$. Suppose Alice wishes to teleport the unknown pure state $|\phi\rangle=\Sigma_{i=1}^n\alpha_i|i\rangle$. The initial state Alice and Bob have is then given by

$$|\phi\rangle\otimes|\Psi\rangle=\sum_{i,j,k=0}^{n-1}\alpha_i a_{jk}|ijk\rangle\ \ \in\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}. \tag{9}$$

To transform the state $|\phi\rangle$ to Bob, Alice first performs a joint Bell measurement based on the generalized Bell states Eq. (4) on her party. After her measurement with outcoming in the state $|\Phi_{st}\rangle$, Bob's particle gets into a (unnormalized) state

$$|\phi\rangle\to\frac{1}{\sqrt{n}}AU_{st}|\phi\rangle.$$

Once Bob learns from Alice that she has obtained the result st, he performs on his previously entangled particle (particle 3) a unitary transformation T_{st}. Then the final state becomes $1/\sqrt{n}T^\dagger_{st}AU_{st}|\phi\rangle$. In terms of the density matrix, the teleportation based on the unitary matrices $\{T_{st}\}$, with the quantum resource being a pure state $|\Psi\rangle$, is a quantum channel with the output

$$\Lambda^{(|\Psi\rangle\langle\Psi|)}(\{T\})(\rho)=\frac{1}{n}\sum_{st} T^\dagger_{st}AU_{st}\ \ \rho\ \ U^\dagger_{st}AT_{st}.$$

Step 2: *An arbitrary mixed entangled state as a resource*. Let χ be a mixed state,

$$\chi=\sum_\alpha p_\alpha|\Psi_\alpha\rangle\langle\Psi_\alpha|,\qquad 0\leqslant p_\alpha\leqslant 1$$

and

$$\sum_\alpha p_\alpha=1,\qquad |\Psi_\alpha\rangle=\sum_{i,j}a^{(\alpha)}_{ij}|ij\rangle.$$

Applying the teleportation protocol T with a mixed state χ, the final state of Bob becomes

$$\Lambda^{(\chi)}(\{T\})(\rho)=\frac{1}{n}\sum_{s,t}\sum_\alpha p_\alpha T^\dagger_{st}A^{(\alpha)}U_{st}\ \ \rho\ \ U^\dagger_{st}(A^{(\alpha)})^\dagger T_{st}. \tag{10}$$

Since each matrix $A^{(\alpha)}$ can be decomposed in the basis of $\{U_{st}\}$ by $(A^{(\alpha)})_{ij}=\Sigma_{s,t}a^{(\alpha)}_{st}(U_{st})_{ij}$, Eq. (10) becomes

$$\Lambda^{(\chi)}(\{T\})(\rho)=\frac{1}{n}\sum_{s,t}\sum_{s',t'}\left(\sum_\alpha p_\alpha a^{(\alpha)}_{st}a^{(\alpha)*}_{s',t'}\right)\times\sum_{\gamma,\beta}T^\dagger_{\gamma\beta}U_{st}U_{\gamma\beta}\ \ \rho\ \ U^\dagger_{\gamma\beta}U^\dagger_{s't'}T_{\gamma\beta}.$$

Using the definition of generalized Bell states $\{|\Phi_{st}\rangle\}$ in Eq. (4), after a lengthy calculation, we arrive at

$$n\sum_\alpha p_\alpha A^{(\alpha)}_{st}A^{(\alpha)*}_{s',t'}=\langle\Phi_{st}|\chi|\Phi_{s't'}\rangle.$$

Substituting the above results into Eq. (10), one obtains Eq. (7). Using Eq. (2) and the identity

$$\sum_{s,t}U^\dagger_{st}\ \ AU_{st}=n\,\mathrm{tr}(A)I_{n\times n},\qquad \text{for any}\ \ n\times n\ \ \text{matrix}\ \ A,$$

the trace-preserving property of the quantum channel can be proved by

$$\begin{aligned}
&\mathrm{tr}[\Lambda^{(\chi)}(\{T\})(\rho)]\\
&=\frac{1}{n^2}\sum_{s,t}\sum_{s',t'}\langle\Phi_{st}|\chi|\Phi_{s't'}\rangle\\
&\quad\times\left\{\sum_{\gamma\beta}\mathrm{tr}(U^\dagger_{\gamma\beta}U^\dagger_{s't'}U_{st}U_{\gamma\beta}\rho)\right\}\\
&=\frac{1}{n}\sum_{s,t}\sum_{s',t'}\langle\Phi_{st}|\chi|\Phi_{s't'}\rangle\mathrm{tr}(U_{st}U^\dagger_{s't'})\mathrm{tr}(\rho)\\
&=\sum_{s,t}\langle\Phi_{st}|\chi|\Phi_{st}\rangle=\mathrm{tr}(\chi)=1. \qquad \blacksquare
\end{aligned}$$

The fidelity of the teleportation is given by

$$f(\chi)=\overline{\langle\phi_{in}|\Lambda^{(\chi)}(\{T\})(|\phi_{in}\rangle\langle\phi_{in}|)|\phi_{in}\rangle}, \tag{11}$$

averaged over all pure input states ϕ_{in}.

In order to calculate the transmission fidelity Eq. (11), we need an irreducible n-dimensional representation of the unitary group $U(n)$, denoted by **G**. Let $U(g)$ be the unitary matrix representation of the element g of **G**. Recalling Schur's Lemma, one has the identity

$$\int_{\mathbf{G}}dg[U^\dagger(g)\otimes U^\dagger(g)]\ \ \sigma\ \ [U(g)\otimes U(g)]=\alpha_1 I\otimes I+\alpha_2 P, \tag{12}$$

$$\alpha_1=\frac{n^2\,\mathrm{tr}(\sigma)-n\,\mathrm{tr}(\sigma P)}{n^2(n^2-1)},\qquad \alpha_2=\frac{n^2\,\mathrm{tr}(\sigma P)-n\,\mathrm{tr}(\sigma)}{n^2(n^2-1)},$$

for any operator σ acting on the tensor space, where P is the flip operator such that $P|ij\rangle=|ji\rangle$. The invariant (Haar) measure dg on G is normalized by $\int_{\mathbf{G}}dg=1$.

Theorem 2. The transmission fidelity of the teleportation protocol defined by $\{T_{st}\}$ with arbitrary mixed state χ as a resource is given by

$$f(\chi)=\frac{1}{n(n+1)}\sum_{\gamma\beta}\langle\Phi|[1\otimes(T_{\gamma\beta}U^\dagger_{\gamma\beta})^\dagger]\times\chi(1\otimes T_{\gamma\beta}U^\dagger_{\gamma\beta})|\Phi\rangle\ +\frac{1}{n+1}. \tag{13}$$

Proof. From Theorem 1 and Eq. (12), one has

$$\begin{aligned}
f(\chi)&=\frac{1}{n^2}\sum_{s,t}\sum_{s',t'}\langle\Phi_{st}|\chi|\Phi_{s't'}\rangle\sum_{\gamma\beta}\overline{\langle\phi_{in}|T^\dagger_{\gamma\beta}U_{st}U_{\gamma\beta}|\phi_{in}\rangle\langle\phi_{in}|U^\dagger_{\gamma\beta}U^\dagger_{s't'}T_{\gamma\beta}|\phi_{in}\rangle}\\
&=\frac{1}{n^2}\sum_{s,t}\sum_{s',t'}\langle\Phi_{st}|\chi|\Phi_{s't'}\rangle\sum_{\gamma\beta}\overline{\langle\phi_{in}|\otimes\langle\phi_{in}|(T^\dagger_{\gamma\beta}U_{st}U_{\gamma\beta}\otimes U^\dagger_{\gamma\beta}U^\dagger_{s't'}T_{\gamma\beta})|\phi_{in}\rangle\otimes|\phi_{in}\rangle}\\
&=\frac{1}{n^2}\sum_{s,t}\sum_{s',t'}\langle\Phi_{st}|\chi|\Phi_{s't'}\rangle\sum_{\gamma\beta}\left\langle 00\left|\int_{\mathbf{G}}dg[U(g)^\dagger\otimes U(g)^\dagger](T^\dagger_{\gamma\beta}U_{st}U_{\gamma\beta}\otimes U^\dagger_{\gamma\beta}U^\dagger_{s't'}T_{\gamma\beta})[U(g)\otimes U(g)]\right|00\right\rangle\\
&=\frac{1}{n^3(n+1)}\sum_{s,t}\sum_{s',t'}\langle\Phi_{st}|\chi|\Phi_{s't'}\rangle\sum_{\gamma\beta}\{\mathrm{tr}(T^\dagger_{\gamma\beta}U_{st}U_{\gamma\beta})\mathrm{tr}(U^\dagger_{\gamma\beta}U^\dagger_{s't'}T_{\gamma\beta})+\mathrm{tr}(T^\dagger_{\gamma\beta}U_{st}U_{\gamma\beta}U^\dagger_{\gamma\beta}U^\dagger_{s't'}T_{\gamma\beta})\}\\
&=\frac{1}{n(n+1)}\sum_{\gamma\beta}\langle\Phi|(1\otimes(T_{\gamma\beta}U^\dagger_{\gamma\beta})^\dagger)\ \ \chi\ \ (1\otimes T_{\gamma\beta}U^\dagger_{\gamma\beta})|\Phi\rangle\ +\frac{1}{n+1},
\end{aligned}$$

where the identity $\mathrm{tr}_{12}[(A\otimes B)P]=\mathrm{tr}(AB)$, Eqs. (2) and (3) have been used. $\blacksquare$

Obviously when the term $\langle\Phi|(1\otimes(T_{\gamma\beta}U^\dagger_{\gamma\beta})^\dagger)\chi(1\otimes T_{\gamma\beta}U^\dagger_{\gamma\beta})|\Phi\rangle$ is maximized, i.e., $T_{\gamma\beta}U^\dagger_{\gamma\beta}=W_\chi$, one gets the maximal fidelity. Recalling the definition of the *fully entangled fraction* Eqs. (5) and (6), we arrive at our main result:

Theorem 3. The optimal teleportation based on the Bell measurements, when used with an arbitrary mixed state with density matrix χ as a resource, acts as a general trace-preserving quantum channel

$$\Lambda_O^{(\chi)}(\rho)=\frac{1}{n^2}\sum_{s,t}\sum_{s',t'}\langle\Phi_{st}|\chi|\Phi_{s't'}\rangle\times\left\{\sum_{\gamma\beta}U^\dagger_{\gamma\beta}W^\dagger_\chi U_{st}U_{\gamma\beta}\ \ \rho\ \ U^\dagger_{\gamma\beta}U^\dagger_{s't'}W_\chi U_{\gamma\beta}\right\}. \tag{14}$$

The corresponding transmission fidelity is given by

$$f_{\max}(\chi) = \frac{n\mathcal{F}(\chi)}{n+1} + \frac{1}{n+1}, \tag{15}$$

where $\mathcal{F}(\chi)$ is the fully entangled fraction Eq. (5) and W_χ is the unitary matrix which fulfills such a fully entangled fraction Eq. (6).

Our results show that the maximum transmission fidelity of the teleportation based on the Bell measurement depends on the *fully entangled fraction* only, whereas that of a standard teleportation depends on the singlet fraction [5]. Our result also agrees with the fidelity formula of the general optimal teleportation given by the Horodecki family [7].

Summarizing, we obtain the explicit expression of the output state of the optimal teleportation, with arbitrary mixed entangled state as a resource, in terms of some noisy quantum channel. This allows us to calculate the transmission fidelity of the quantum channel. It is shown that the transmission fidelity depends only on the *fully entangled fraction* of the quantum resource shared by the sender and the receiver. The fidelity in our optimal teleportation protocol is in general greater than the one in standard teleportation protocol [1,4,5].

W.-L.Y. would like to thank Professor von Gehlen for his continuous encouragement. He has been supported by the Alexander von Humboldt Foundation.

[1] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W.K. Wooters, Phys. Rev. Lett. **70**, 1895 (1993).

[2] M. Horodecki, P. Horodecki, and R. Horodecki, e-print quant-ph/0109124.

[3] G.M. D'Ariano, P. Lo Presti, and M.F. Sacchi, Phys. Lett. A **272**, 32 (2000).

[4] G. Bowen and S. Bose Phys. Rev. Lett. **87**, 267901 (2001).

[5] S. Albeverio, S.-M. Fei, and W.-L. Yang (unpublished).

[6] R.F. Werner, e-print quant-ph/0003070.

[7] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **60**, 1888 (1999).