Time evolution of the Wigner function in the entangled-state representation

Hongyi Fan
CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, China and Department of Materials Science and Engineering, University of Science and Technology of China, Hefei, Anhui 230026, China
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For quantum-mechanical entangled states we introduce the entangled Wigner operator in the entangled-state representation. We derive the time evolution equation of the entangled Wigner operator . The trace product rule for entangled Wigner functions is also obtained.

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As is well known in quantum mechanics that the Wigner function $W(x,p)$ of a quantum state described by a density matrix ρ in one-dimensional case is defined as $(\hbar=1)$ [1–4]

$$W(x,p)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\langle x+\frac{v}{2}\right|\rho\left|x-\frac{v}{2}\right\rangle e^{-ipv}dv, \tag{1}$$

where $|x\rangle$ is the eigenvector of the coordinate operator $X_1=(a_1^\dagger+a_1)/\sqrt{2}$ (or named as the quadrature operator in quantum optics theory). $W(x,p)$ is a quasidistribution function in the phase space satisfying [2]

$$P(p)=\int W(x,p)dx, \quad P(x)=\frac{1}{2\pi}\int W(x,p)dp, \tag{2}$$

where $P(x)[P(p)]$ is proportional to the probability for finding the particle at x (at p in momentum space)—marginal distribution. Based on Eq. (1), physicists have developed tomography theory in quantum mechanics [5,6]. The single-mode Wigner operator is

$$\Delta(x,p)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|x-\frac{v}{2}\right\rangle\left\langle x+\frac{v}{2}\right|e^{-ipv}dv. \tag{3}$$

Using the technique of integration within an ordered product (IWOP) of operators [7,8] the integration in Eq. (3) can be performed and the result is an explicit operator [9]

$$\Delta(x,p)=\frac{1}{\pi}:e^{-(x-X_1)^2-(p-P_1)^2}:=\frac{1}{\pi}:e^{-2(a_1^\dagger-\alpha'^*)(a_1-\alpha')}:$$
$$\equiv\Delta(\alpha',\alpha'^*), \tag{4}$$

where $\alpha'=(1/\sqrt{2})(x+ip)$, : : denotes normal ordering. The Wigner operator also serves as an integral kernel of the Weyl rule [10] which is a quantization scheme connecting classical functions with their quantum correspondence operators. Supposing the operator Hamiltonian of a single particle is $H_1=P_1^2/2m+V(X_1)$, then one can derive the time evolution equation of the Wigner function governed by H_1 (after recovering $\hbar$) [3,4],

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}p_1\frac{\partial}{\partial x_1}-\frac{dV(x_1)}{dx_1}\frac{\partial}{\partial p_1}\right)W(x,p,t)$$
$$=\sum_{k=1}^{\infty}\left(\frac{\hbar}{2}\right)^{2k}\frac{(-1)^k}{(2k+1)!}\frac{d^{2k+1}V(X_1)}{dx_1^{2k+1}}\left(\frac{\partial}{\partial p_1}\right)^{2k+1}$$
$$\times W(x_1,p_1,t). \tag{5}$$

In the classical limit, letting $\hbar\to 0$, one obtains the Liouville-like equation

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}p_1\frac{\partial}{\partial x_1}-\frac{dV(x_1)}{dx_1}\frac{\partial}{\partial p_1}\right)W(x,p,t)=0.$$

In this work we shall deepen and generalize the Wigner function theory to two-mode entangled systems originally put forward by Einstein, Podolsky, and Rosen (EPR) [11]. In particular, we want to derive trace product rule for entangled Wigner functions as well as the time evolution equation of the Wigner function in the two-mode entangled-state representation. In EPR's treatment, when two systems are prepared in an entangled state, one of the two canonically conjugate variables is measured on one system and the entanglement is such that the value for a physical variable in the another system may be inferred with certainty. In two-mode systems there are a richer variety of quantum phenomena since there exists the possibility of quantum entanglements between the modes. For example, these entanglements may give rise to two-mode squeezing in a nondegenerate parametric amplifier [12]. Because entanglement is now widely used in quantum teleportation, quantum dense coding, the introduction of entangled-state representation is not only just for the convenience of some calculations, but also for revealing the intrinsic entanglement property inherent to some physical systems. For example, using the entangled-state representation for describing an electron moving in a uniform magnetic field [13], we have pointed out that EPR entanglement is also involved is such a system. As one can see shortly later from Eqs. (13) and (16) that since the marginal distributions of the Wigner function for entangled states are only meaningful in the entangled state $|\eta\rangle$ (or $|\xi\rangle$ state) representation, the equation of motion for the entangled Wigner function should also be expressed in the corresponding phase space.

In Ref. [14] we have successfully established the so-called entangled Wigner operator for correlated two-body systems, based on it, the corresponding Wigner function of two-body correlated system can be conveniently derived. This entangled Wigner operator is expressed in so-called entangled state $\langle\eta|$ representation as [14]

$$\Delta(\sigma,\gamma)=\int\frac{d^2\eta}{\pi^3}|\sigma-\eta\rangle\langle\sigma+\eta|\exp(\eta\gamma^*-\eta^*\gamma),\quad(6)$$

the corresponding Wigner function for a density matrix ρ is

$$W_\rho(\sigma,\gamma)=\int\frac{d^2\eta}{\pi^3}\langle\sigma+\eta|\rho|\sigma-\eta\rangle\exp(\eta\gamma^*-\eta^*\gamma),\quad(7)$$

where

$$|\eta\rangle=\exp\{-\tfrac{1}{2}|\eta|^2+\eta a_1^\dagger-\eta^*a_2^\dagger+a_1^\dagger a_2^\dagger\}|00\rangle,$$
$$\eta=\eta_1+i\eta_2\quad(8)$$

is the common eigenvector of X_1-X_2 and P_1+P_2 [15], which obeys the eigenvector equations

$$(X_1-X_2)|\eta\rangle=\sqrt{2}\eta_1|\eta\rangle,\quad(P_1+P_2)|\eta\rangle=\sqrt{2}\eta_2|\eta\rangle,\quad(9)$$

where $X_i=(1/\sqrt{2})(a_i+a_i^\dagger)$, $P_i=(1/\sqrt{2}i)(a_i-a_i^\dagger)$, the states $|\eta\rangle$ span a complete and orthonormal space

$$\int\frac{d^2\eta}{\pi}|\eta\rangle\langle\eta|=1,\quad\langle\eta|\eta'\rangle=\pi\delta(\eta-\eta')\delta(\eta^*-\eta'^*).\quad(10)$$

Using the IWOP technique we have performed the integration in Eq. (6) and obtained the explicit normally ordered form of $\Delta(\sigma,\gamma)$,

$$\Delta(\sigma,\gamma)=\pi^{-2}:\exp\{-|\sigma|^2-|\gamma|^2+\gamma(a_1^\dagger+a_2)+\gamma^*(a_2^\dagger+a_1)+\sigma(a_1^\dagger-a_2)+\sigma^*(a_1-a_2^\dagger)-2a_1^\dagger a_1-2a_2^\dagger a_2\}:,\quad(11)$$

which is just the product of two independent single-mode Wigner operators $\Delta(\sigma,\gamma)=\Delta(\alpha,\alpha^*)\Delta(\beta,\beta^*)$ provided we take

$$\gamma=\alpha+\beta^*,\quad\sigma=\alpha-\beta^*,\quad\alpha=(1/\sqrt{2})(x_1+ip_1),$$
$$\beta=(1/\sqrt{2})(x_2+ip_2).\quad(12)$$

We name $\Delta(\sigma,\gamma)$ in Eq. (6) as the entangled Wigner operator because performing the integration of $\Delta(\sigma,\gamma)$ over $d^2\gamma$ leads to the projection operator of the entangled state $|\eta\rangle$ and the marginal distribution in (η_1,η_2) phase space

$$\int d^2\gamma\Delta(\sigma,\gamma)=\frac{1}{\pi}|\eta\rangle\langle\eta||_{\eta=\sigma},$$
$$\left\langle\psi\left|\int d^2\gamma\Delta(\sigma,\gamma)\right|\psi\right\rangle=\frac{1}{\pi}|\psi(\eta)|^2|_{\eta=\sigma}.\quad(13)$$

Similarly, we can introduce the common eigenvector of X_1+X_2 and P_1-P_2 [16],

$$|\xi\rangle=\exp\{-\tfrac{1}{2}|\xi|^2+\xi a_1^\dagger+\xi^*a_2^\dagger-a_1^\dagger a_2^\dagger\}|00\rangle,$$
$$\xi=\xi_1+i\xi_2,\quad(14)$$

which obeys another pair of eigenvector equations

$$(X_1+X_2)|\xi\rangle=\sqrt{2}\xi_1|\xi\rangle,\quad(P_1-P_2)|\xi\rangle=\sqrt{2}\xi_2|\xi\rangle.\quad(15)$$

Performing the integration of $\Delta(\sigma,\gamma)$ over $d^2\sigma$ yields

$$\int d^2\sigma\Delta(\sigma,\gamma)=\frac{1}{\pi}|\xi\rangle\langle\xi||_{\xi=\gamma},$$
$$\left\langle\psi\left|\int d^2\sigma\Delta(\sigma,\gamma)\right|\psi\right\rangle=\frac{1}{\pi}|\psi(\xi)|^2|_{\xi=\gamma},\quad(16)$$

which is the marginal distribution in (ξ_1,ξ_2) phase space. $|\psi(\xi)|^2|_{\xi=\gamma}$ $[|\psi(\eta)|^2|_{\eta=\sigma}]$ is proportional to the probability for finding the two particles that possess relative momentum $\sqrt{2}\xi_2$ (total momentum $\sqrt{2}\eta_2$) and simultaneously center-of-mass position $\xi_1/\sqrt{2}$ (relative position $\sqrt{2}\eta_1$). The introduction of the entangled Wigner operator also brings much convenience for calculating the Wigner function of some entangled states. For example, using Eqs. (6) and (10) the Wigner function of the entangled state $|\eta\rangle$ itself can be immediately derived, i.e.,

$$\langle\eta|\Delta(\sigma,\gamma)|\eta\rangle=\int\frac{d^2\eta'}{\pi^3}\langle\eta|\sigma-\eta'\rangle\langle\sigma+\eta'|\eta\rangle\exp(\eta'\gamma^*-\eta'^*\gamma)=(2\pi)^{-1}\delta(\sqrt{2}\eta_1-(x_1-x_2))\delta(\sqrt{2}\eta_2-(p_1+p_2)).\quad(17)$$

For the two-mode squeezed vacuum state $S|00\rangle$, S is the two-mode squeezing operator that has a neat form in the $\langle\eta|$ representation [17],

$$S=\exp[-\lambda(a_1^\dagger a_2^\dagger-a_1a_2)]=g\int\frac{d^2\eta}{\pi}|g\eta\rangle\langle\eta|,\quad g=e^\lambda,\quad(18)$$

using Eqs. (6) and (12) we immediately get the Wigner function

$$W_\Psi(\sigma,\gamma)\equiv\langle00|S^\dagger\Delta(\sigma,\gamma)S|00\rangle=\frac{g^2}{\pi^2}\exp[-|\sigma|^2/g^2-g^2|\gamma|^2].\quad(19)$$

We now examine the following expression composed by integrating two Wigner functions

$$4\pi^2\int d^2\sigma d^2\gamma W_{\rho_1}(\sigma,\gamma)W_{\rho_2}(\sigma,\gamma).\quad(20)$$

Substituting Eq. (7) into Eq. (20) and using Eq. (10) we have

$$4\pi^2\int d^2\sigma d^2\gamma W_{\rho_1}(\sigma,\gamma)W_{\rho_2}(\sigma,\gamma)$$
$$=4\int d^2\sigma d^2\gamma\int\frac{d^2\eta}{\pi}\langle\sigma+\eta|\rho_1|\sigma-\eta\rangle e^{\eta\gamma^*-\eta^*\gamma}$$
$$\times\int\frac{d^2\eta'}{\pi^3}\langle\sigma+\eta'|\rho_2|\sigma-\eta'\rangle e^{\eta'\gamma^*-\eta'^*\gamma}$$
$$=4\int d^2\sigma\int\frac{d^2\eta}{\pi^2}\langle\sigma+\eta|\rho_1|\sigma-\eta\rangle\langle\sigma-\eta|\rho_2|\sigma+\eta\rangle. \tag{21}$$

Let $\sigma+\eta=\tau,\ \ \sigma-\eta=\lambda$, then Eq. (21) becomes

$$[\text{Eq. (21)}]=\int d^2\tau\int\frac{d^2\lambda}{\pi^2}\langle\tau|\rho_1|\lambda\rangle\langle\lambda|\rho_2|\tau\rangle$$
$$=\int\frac{d^2\tau}{\pi}\langle\tau|\rho_1\rho_2|\tau\rangle=\mathrm{Tr}(\rho_1\rho_2), \tag{22}$$

where both $|\lambda\rangle$ and $|\tau\rangle$ are $|\eta\rangle$-*type* entangled states. Hence we have

$$\mathrm{Tr}(\rho_1\rho_2)=4\pi^2\int d^2\sigma d^2\gamma W_{\rho_1}(\sigma,\gamma)W_{\rho_2}(\sigma,\gamma), \tag{23}$$

this is the trace product rule for entangled Wigner function. Especially, when $\rho_1=|\psi\rangle\langle\psi|$ and $\rho_2=|\phi\rangle\langle\phi|$ are both two-mode correlated pure states, we have (after recovering $\hbar$)

$$\mathrm{Tr}(\rho_1\rho_2)=|\langle\psi|\phi\rangle|^2=\left|\int\frac{d^2\eta}{\pi}\psi^*(\eta)\phi(\eta)\right|^2$$
$$=4\pi^2\hbar^2\int d^2\sigma d^2\gamma W_{|\psi\rangle}(\sigma,\gamma)W_{|\phi\rangle}(\sigma,\gamma), \tag{24}$$

which means the transition amplitude $\langle\psi|\phi\rangle$ can be expressed by the product of two states' Wigner function integrated over the phase space. O'Connell and Wigner first expressed the inner product of two quantum states as the overlap between two ordinary Wigner functions in ordinary phase space [18]. Further, in Eq. (23) when $\rho_1=\rho_2\equiv\rho$, due to pure state's $\mathrm{Tr}(\rho^2)\leqslant 1$, we obtain the relation for entangled Wigner function

$$4\pi^2\hbar^2\leqslant 1/\int d^2\sigma d^2\gamma W_\rho^2(\sigma,\gamma). \tag{25}$$

We now turn to the following question. Corresponding to the entangled Wigner operator (Wigner function) what is its time evolution equation when the Hamiltonian of two-body correlated system is

$$H=P_1^2/2m_1+P_2^2/2m_2+U(X_1-X_2), \tag{26}$$

where the potential just depends on the relative distance of the two particles. By introducing the reduced mass and total mass $M=m_1+m_2$, $\mu=(m_1m_2)/M$, and the mass-weight relative momentum $P_r=\mu_2P_1-\mu_1P_2$, and the relative coordinate $x_r=X_1-X_2$, where $\mu_i=m_i/M$, we can convert Eq. (26) to

$$H=P^2/2M+P_r^2/2\mu+U(x_r),\quad P=P_1+P_2. \tag{27}$$

From the Heisenberg equation $(\partial/\partial t)\rho=-i[H,\rho]$ $(\hbar=1)$, we see

$$\frac{\partial}{\partial t}\langle\sigma+\eta|\rho|\sigma-\eta\rangle$$
$$=-i\langle\sigma+\eta|(P^2/2M+P_r^2/2\mu+U(x_r))\rho|\sigma-\eta\rangle$$
$$+i\langle\sigma+\eta|\rho(P^2/2M+P_r^2/2\mu+U(x_r))|\sigma-\eta\rangle. \tag{28}$$

To simplify Eq. (28) we must know $\langle\eta|P_r$, for this purpose we appeal to the Schmidt decomposition of $|\eta\rangle$ [19] in the momentum eigenstate space

$$|\eta\rangle=e^{-i\eta_1\eta_2}\int_{-\infty}^{\infty}dp|p+\sqrt{2}\eta_2\rangle_1\otimes|-p\rangle_2e^{-i\sqrt{2}\eta_1p}. \tag{29}$$

It then follows from Eq. (29) and $\mu_2+\mu_1=1$ that

$$P_r|\eta\rangle=e^{-i\eta_1\eta_2}\int_{-\infty}^{\infty}dp[\mu_2(p+\sqrt{2}\eta_2)+\mu_1p]|p+\sqrt{2}\eta_2\rangle_1$$
$$\otimes|-p\rangle_2e^{-i\sqrt{2}\eta_1p}$$
$$=\left[i\frac{\partial}{\partial(\sqrt{2}\eta_1)}-\frac{1}{\sqrt{2}}(\mu_1-\mu_2)\eta_2\right]|\eta\rangle. \tag{30}$$

As a result of Eqs. (30) and (9) we have

$$\langle\sigma+\eta|(P^2/2M)=(\sigma_2+\eta_2)^2/M\langle\sigma+\eta|,$$
$$\langle\sigma+\eta|(P_r^2/2\mu)=(1/4\mu)[i\partial/\partial(\sigma_1+\eta_1)$$
$$+(\mu_1-\mu_2)(\sigma_2+\eta_2)]^2\langle\sigma+\eta|. \tag{31}$$

On the other hand, using $\sigma+\eta=\tau$, $\sigma-\eta=\lambda$, we have $\partial/\partial(\sigma_1\pm\eta_1)=(1/2)(\partial/\partial\sigma_1\pm\partial/\partial\eta_1)$, using this and Eqs. (10), (30), and (31) we rewrite Eq. (28) as

$$i\frac{\partial}{\partial t}\langle\sigma+\eta|\rho|\sigma-\eta\rangle$$
$$=\left\{\frac{4\sigma_2\eta_2}{M}+\frac{\sigma_2\eta_2}{\mu}(\mu_1-\mu_2)^2-\frac{1}{4\mu}\frac{\partial}{\partial\sigma_1}\frac{\partial}{\partial\eta_1}\right.$$
$$+\frac{i(\mu_1-\mu_2)}{2\mu}\left(\sigma_2\frac{\partial}{\partial\sigma_1}+\eta_2\frac{\partial}{\partial\eta_1}\right)+U[\sqrt{2}(\sigma_1+\eta_1)]$$
$$\left.-U[\sqrt{2}(\sigma_1-\eta_1)]\right\}\langle\sigma+\eta|\rho|\sigma-\eta\rangle, \tag{32}$$

where $4/M+(1/\mu)(\mu_1-\mu_2)^2=1/\mu$ and $U(\sqrt{2}(\sigma_1+\eta_1))-U(\sqrt{2}(\sigma_1-\eta_1))$

$$=2\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\frac{\partial^{2k+1}U[\sqrt{2}\sigma_1]}{\partial(\sqrt{2}\sigma_1)^{2k+1}}(\sqrt{2}\eta_1)^{2k+1}\equiv A. \tag{33}$$

Substituting Eqs. (32) and (33) into

$$\frac{\partial}{\partial t}W_\rho(\sigma,\gamma,t)=\frac{\partial}{\partial t}\int\frac{d^2\eta}{\pi^3}\langle\sigma+\eta|\rho|\sigma-\eta\rangle\exp(\eta\gamma^*-\eta^*\gamma), \tag{34}$$

and performing the integration by parts we obtain

$$i\frac{\partial}{\partial t}W_\rho(\sigma,\gamma)=\int\frac{d^2\eta}{\pi^3}e^{2i(\eta_2\gamma_1-\eta_1\gamma_2)}\left\{\frac{\sigma_2\eta_2}{\mu}-\frac{1}{4\mu}\frac{\partial}{\partial\sigma_1}\frac{\partial}{\partial\eta_1}+\frac{i(\mu_1-\mu_2)}{2\mu}\left(\sigma_2\frac{\partial}{\partial\sigma_1}+\eta_2\frac{\partial}{\partial\eta_1}\right)+A\right\}\langle\sigma+\eta|\rho|\sigma-\eta\rangle$$
$$=\left\{-\frac{i}{2\mu}\sigma_2\frac{\partial}{\partial\gamma_1}-\frac{i\gamma_2}{2\mu}\frac{\partial}{\partial\sigma_1}+\frac{i(\mu_1-\mu_2)}{2\mu}\left(\sigma_2\frac{\partial}{\partial\sigma_1}+\gamma_2\frac{\partial}{\partial\gamma_1}\right)\right.$$
$$\left.+2\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\frac{\partial^{2k+1}U[\sqrt{2}\sigma_1]}{\partial(\sqrt{2}\sigma_1)^{2k+1}}\left(\frac{i\partial}{\partial(\sqrt{2}\gamma_2)}\right)^{2k+1}\right\}\int\frac{d^2\eta}{\pi^3}e^{2i(\eta_2\gamma_1-\eta_1\gamma_2)}\langle\sigma+\eta|\rho|\sigma-\eta\rangle. \quad (35)$$

Note that if one wants to discuss the classical limit of the time evolution equation of the entangled Wigner function, one should write Eq. (33) as

$$A=2\sum_{k=0}^{\infty}\frac{\hbar^{2k+1}}{(2k+1)!}\frac{\partial^{2k+1}U(\sqrt{2}\sigma_1)}{\partial(\sqrt{2}\sigma_1)^{2k+1}}\left(\frac{\sqrt{2}\eta_1}{\hbar}\right)^{2k+1}, \quad (36)$$

and at the same time change $e^{2i(\eta_2\gamma_1-\eta_1\gamma_2)}$ in Eq. (35) to $e^{2i(\eta_2\gamma_1-\eta_1\gamma_2)/\hbar}$). Therefore the equation of motion for the entangled Wigner function is

$$\left\{\frac{\partial}{\partial t}+\frac{1}{2\mu}[\sigma_2-(\mu_1-\mu_2)\gamma_2]\frac{\partial}{\partial\gamma_1}+\frac{1}{2\mu}[\gamma_2-(\mu_1-\mu_2)\sigma_2]\frac{\partial}{\partial\sigma_1}-2\frac{\partial U[\sqrt{2}\sigma_1]}{\partial(\sqrt{2}\sigma_1)}\left(\frac{\partial}{\partial(\sqrt{2}\gamma_2)}\right)\right\}W_\rho(\sigma,\gamma,t)$$
$$=\sum_{k=1}^{\infty}\frac{2(-1)^k}{(2k+1)!}\frac{\partial^{2k+1}U[\sqrt{2}\sigma_1]}{\partial(\sqrt{2}\sigma_1)^{2k+1}}\left(\frac{\partial}{\partial(\sqrt{2}\gamma_2)}\right)^{2k+1}\times W_\rho(\sigma,\gamma,t). \quad (37)$$

In order to see the physical meaning of Eq. (37) more clearly, from Eq. (12) we notice $\gamma=\alpha+\beta^*=\gamma_1+i\gamma_2$, $\gamma_1=(1/\sqrt{2})(x_1+x_2)$, $\gamma_2=(1/\sqrt{2})(p_1-p_2)$, $\sigma=\alpha-\beta^*=\sigma_1+i\sigma_2$, $\sigma_1=(1/\sqrt{2})(x_1-x_2)$, $\sigma_2=(1/\sqrt{2})(p_1+p_2)$, thus

$$(1/2\mu)[\sigma_2-(\mu_1-\mu_2)\gamma_2]=\frac{1}{\sqrt{2}}\left(\frac{p_1}{m_1}+\frac{p_2}{m_2}\right),$$
$$(1/2\mu)[\gamma_2-(\mu_1-\mu_2)\sigma_2]=(1/\sqrt{2})(p_1/m_1-p_2/m_2). \quad (38)$$

Hence Eq. (37) is equal to

$$\left\{\frac{\partial}{\partial t}+\frac{1}{\sqrt{2}}\left(\frac{p_1}{m_1}+\frac{p_2}{m_2}\right)\frac{\partial}{\partial\gamma_1}+\frac{1}{\sqrt{2}}\left(\frac{p_1}{m_1}-\frac{p_2}{m_2}\right)\frac{\partial}{\partial\sigma_1}-2\frac{\partial U[\sqrt{2}\sigma_1]}{\partial(\sqrt{2}\sigma_1)}\left(\frac{\partial}{\partial(\sqrt{2}\gamma_2)}\right)\right\}W_\rho(\sigma,\gamma,t)$$
$$=\sum_{k=1}^{\infty}\frac{2(-1)^k}{(2k+1)!}\frac{\partial^{2k+1}U[\sqrt{2}\sigma_1]}{\partial(\sqrt{2}\sigma_1)^{2k+1}}\left(\frac{\partial}{\partial(\sqrt{2}\gamma_2)}\right)^{2k+1}\times W_\rho(\sigma,\gamma,t), \quad (39)$$

which is comparable with Eq. (5). From Eq. (39) we see that this equation is expressed with $(1/\sqrt{2})(x_1+x_2)$,$(1/\sqrt{2})(x_1-x_2)$, and $(1/\sqrt{2})(p_1-p_2)$, corresponding to center-of-mass coordinate, relative coordinate, and relative momentum, respectively, which is as expected.

In summary, we have derived the equation of motion for the Wigner operator in the entangled-state representation. The trace product rule for entangled Wigner functions is also obtained.

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