Overlap and entanglement-witness measurements

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A feasible device for measurement of fidelity, overlap, purity, and Hilbert-Schmidt distance of two mixed states is proposed. In addition, this device realizes a decomposable entanglement witness-measurement for bipartite systems, corresponding to Werner criterion of entanglement. The measurement, based on interferometric setup and the control-phase gate, can be directly implemented in the cavity quantum electrodynamics, trapped ion, and electromagnetically induced transparency experiments.

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I. INTRODUCTION

A state of a quantum system is fully characterized by a density matrix, which contains all achievable information on the system. For simple systems, whose Hilbert space has a low dimension, quantum-state estimations are well mastered, but with increasing dimension of Hilbert space, the state reconstruction becomes a complicated experimental problem [1]. However, several important properties of the quantum state can be described by simple parameters, such as the fidelity $\langle\Psi|\rho|\Psi\rangle$ of density matrix ρ with a pure state $|\Psi\rangle$, overlap Tr $\rho^A\rho^B$ between two density matrices ρ^A and ρ^B [2], purity Tr ρ^2 of the state ρ, or Hilbert-Schmidt distance between two density matrices $d^2(\rho^A,\rho^B)=1/2\times\mathrm{Tr}(\rho^A-\rho^B)^2$. These quantities are widely employed in description of the quantum information protocols for both the discrete and continuous variables [3].

From the point of view of quantum information processing, detection of quantum entanglement is an important problem. A state is entangled if it cannot be written as a convex combination of product states. For any state of a two-qubit system and for two-mode Gaussian states of a continuous-variable system, there exist simple, necessary, and sufficient conditions for entanglement [4,5]. To detect entanglement, the concept of entanglement witness (EW) has been introduced [6,7]. Recently, a new area for the practical realization of such EW measurements has been opened. Typically, the experimentalists measure the particular correlations between the subsystems and consequently employ an entanglement criterion (Bell inequalities [8], Duan-Simon inequalities [5]) or directly reconstruct the density matrix and test the inseparability of the reconstructed state [9].

In this paper, we propose a direct method of fidelity, overlap, purity, and Hilbert-Schmidt distance measurement, which avoids the necessity of a complete state reconstruction. In addition, we suggest that the same device can be used to measure an entanglement witness related to the entanglement criterion that was first presented by Werner in his seminal paper [10]. We expect that both the suggested measurements can be very useful if the dimension of the Hilbert space is large and the number of available copies is small. For this case, our approach may be more efficient in extracting particular state characteristics, than the strategy based on complete state tomography and consequent calculation of the measured quantity. The paper is organized as follows. In Sec. II, we present theoretical analysis of generic measurement setup with possible experimental implementations, which is followed by discussion of overlap and entanglement-witness measurements in Sec. III and Sec. IV.

II. GENERIC SETUP

To begin, we describe the generic measurement setup depicted in Fig. 1. We assume two systems 0 and 1, representing the measured objects, whose Hilbert spaces may be infinite dimensional, in general. An auxiliary qubit A, with the basis states $|\uparrow\rangle$ and $|\downarrow\rangle$, will serve as a quantum meter. The device is composed of two interferometric setups: first, working with the main systems 0 and 1 and second, involving the auxiliary qubit A. To construct these interferometers, we need to implement the following unitary operations: for auxiliary qubit, unitary transformation U_H (Hadamard gate),

$$|\uparrow\rangle\to\frac{1}{\sqrt{2}}(|\uparrow\rangle+|\downarrow\rangle),$$

$$|\downarrow\rangle\to\frac{1}{\sqrt{2}}(|\downarrow\rangle-|\uparrow\rangle) \tag{1}$$

and phase-shift transformation U_{PS},

$$|\uparrow\rangle\to\exp(i\psi)|\uparrow\rangle,\quad |\downarrow\rangle\to|\downarrow\rangle. \tag{2}$$

On the other hand, for the main systems 0 and 1, we assume the linear coupling represented by unitary operation

$$U_R=\exp\left[\frac{\pi}{4}(a_0^\dagger a_1-a_1^\dagger a_0)\right]. \tag{3}$$

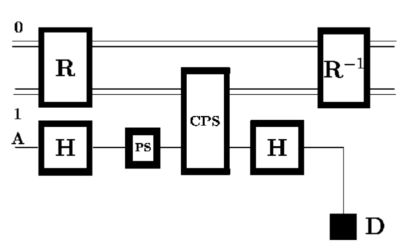

FIG. 1. Generic measuring device. H stands for Hadamard gate, CPS stands for controlled phase shift, PS stands for phase shift, R stands for coupling gate, and D stands for detector.

The linear coupling is generated by interaction Hamiltonian

$$H_{I1}=i\hbar\xi(a_0^\dagger a_1-\mathrm{H.c}),\tag{4}$$

where ξ is a linear coupling constant and a_0, a_1 are the annihilation operators of corresponding main systems. The time of coupling should be set to the value $\tau=\pi/4\xi$. The operations U_H, U_R, and U_{PS} can be performed by linear systems and their experimental implementations obviously do not represent a major problem.

A key point in the proposed setup is a realization of efficient controlled phase shift (CPS) operation

$$U_{CPS}=\exp(i\pi a_0^\dagger a_0|\uparrow\rangle\langle\uparrow|),\tag{5}$$

which couples the main system 0 and auxiliary qubit system *A*. This CPS operation is described by the following interaction Hamiltonian

$$H_{I2}=\hbar\kappa a_0^\dagger a_0|\uparrow\rangle\langle\uparrow|,\tag{6}$$

where κ is a real coupling constant and effective interaction time is set to be equal to π/κ. A kind of interaction (6) can be mediated by appropriate nonlinear interaction between systems 0 and *A*, which are frequently available in the cavity quantum electrodynamics, trapped ion experiments, and in the experiments employing electromagnetically induced transparency. If a state of auxiliary system is $|\uparrow\rangle$, the above described operation realizes phase shift about π in the system 0, whereas for state $|\downarrow\rangle$, no phase shift is induced. We compound the CPS gate with the coupling operations U_R, $U_R^\dagger$ to the following sequence of transformations $U_X=U_R^\dagger U_{CPS}U_R$ on the systems 0 and 1. In dependence on the qubit state, U_X effectively flips the states of systems 0 and 1,

$$\begin{aligned}U_X|\psi_n\rangle_0|\phi_m\rangle_1|\uparrow\rangle&=|\phi_m\rangle_0|\psi_n\rangle_1|\uparrow\rangle,\\ U_X|\psi_n\rangle_0|\phi_m\rangle_1|\downarrow\rangle&=|\psi_n\rangle_0|\phi_m\rangle_1|\downarrow\rangle,\end{aligned}\tag{7}$$

without any change of the basis states $|\uparrow\rangle$ and $|\downarrow\rangle$. If we are able to perform all the above-mentioned operations, then the generic measurement can be realized in the way depicted in Fig. 1. The measuring device consists of a main interferometer with the operations U_R, U_R^{-1} on systems 0,1 coupled by CPS gate to an auxiliary interferometer with the operations U_H, U_{PS} on the qubit system *A*. Finally, the auxiliary qubit is measured in the basis states $|\uparrow\rangle$ and $|\downarrow\rangle$ and corresponding probabilities $p_\uparrow$ and $p_\downarrow$ are obtained. From these probabilities, the overlap Tr $\rho^A\rho^B$ and entanglement witness can be inferred.

Note, that the measurement setup can be simplified if we reject the second $U_R^\dagger$ operation in Fig. 1. However, a nondemolition character of the measurement, which will be discussed in Sec. III, vanishes. The proposed measurements, based on the interferometric techniques and nonlinear coupling, could be experimentally implemented in the cavity quantum electrodynamics (QED) and trapped ion experiments. In both these areas of quantum optics, the interferometric experiments are well mastered and the nonlinear interaction (6) can be achieved with a sufficient strength. In cavity QED experiments, we would like to measure the overlap and entanglement witness for two light fields confined in the cavities. To accomplish this, we can directly employ an experimental setup, previously suggested in Ref. [11], based on Ramsey interferometer and the coupling between cavities. In trapped ion experiments, the generic setup can be used to measure the overlap and entanglement witness between two independent vibrational degrees of freedom of an ion confined in a linear Paul trap [12]. The construction of the CPS gate is similar to the commonly used QND measurement of vibrational energy in trapped ion experiments [13]. To measure the overlap and entanglement witness of traveling optical pulses, we can utilize Kerr coupling to design the CPS gate. Recently, an enhancement of the nonlinear Kerr coupling between two traveling pulses using electromagnetically induced transparency has been achieved. A change of light pulse phase about π by single photon pulse is expected [14]. To realize a large phase modulation on a single photon level, both the cavity and free-medium regimes have been considered [15].

III. OVERLAP MEASUREMENT

Now, we will discuss the overlap Tr $\rho^A\rho^B$ measurement, which is the building block for the measurement of purity and Hilbert-Schmidt distance. Let ρ^A and ρ^B denote the density matrices of the systems 0 and 1, which can be expressed in particular diagonal bases $|\psi_n\rangle$ and $|\phi_n\rangle$,

$$\rho_0^A=\sum_n p_n|\psi_n\rangle\langle\psi_n|,\quad \rho_1^B=\sum_m r_m|\phi_m\rangle\langle\phi_m|.\tag{8}$$

The scalar product $c_{nm}=\langle\psi_n|\phi_m\rangle$ characterizes the overlap between two basis states. We initially assume the qubit system in the state $|\uparrow\rangle_A$. After straightforward calculation, we can find that probability $p_\uparrow$ exhibits interference effect in dependence on the variable phase shift ψ. If we assume maximum and minimum of $p_\uparrow$ in dependence on ψ, we can simply calculate the visibility of interference fringes,

$$V=\frac{p_{max}-p_{min}}{p_{max}+p_{min}}=\sum_{n,m} p_n r_m|c_{nm}|^2=\mathrm{Tr}\ \rho^A\rho^B.\tag{9}$$

We find that if we consider the two independent states (8) at the input, then the visibility of the interference fringes is given exactly by the overlap $O=\mathrm{Tr}\ \rho^A\rho^B$. It is important that only a single parameter (the visibility) has to be estimated to measure the overlap *O*, irrespective of the complexity of the states ρ^A and ρ^B. In this way, we can define overlap observable *O* in the following form:

$$\begin{gathered}\mathcal{O}=\sum_{n',m'}|\psi_{n'}\rangle_{00}\langle\psi_{m'}|\otimes|\psi_{m'}\rangle_{11}\langle\psi_{n'}|,\\ O=\mathrm{Tr}\ \mathcal{O}\rho,\end{gathered}\tag{10}$$

where $\rho=\rho^A\otimes\rho^B$ is the input density matrix. Note, that operator $\mathcal{O}$ is exactly the flip operator [4,10] making transformation $\mathcal{O}(\psi\otimes\phi)=\phi\otimes\psi$. Particularly, the fidelity *F*

$=\langle\Psi|\rho|\Psi\rangle$ between the pure state and the density matrix can be measured. The overlap measurement can be also used to quantify other important state characteristics such as purity $P=\mathrm{Tr}\,\rho^2$ and Renyi entropy $S_R=-\ln P$, if two copies of the same state are available. In addition, the proposed measurement is overlap nondemolition measurement. It can be simply proved that after the overlap measurement, the output state

$$\rho_{\mathrm{out}}=\frac{\rho_1^A\otimes\rho_2^B+\rho_1^B\otimes\rho_2^A}{2} \tag{11}$$

is a balanced mix of the input density matrices. If we once more carry out the same overlap measurement, then identical visibility (9) is obtained. The nondemolition character can be utilized to measure the Hilbert-Schmidt distance

$$d^2(\rho^A,\rho^B)=\frac{P^A+P^B}{2}-O^{AB} \tag{12}$$

between two states ρ^A and ρ^B. To measure $d^2(\rho^A,\rho^B)$, we first perform the measurement of particular purities $P^A=\mathrm{Tr}(\rho^A)^2$ and $P^B=\mathrm{Tr}(\rho^B)^2$ and then use the same ensemble of the systems to measure the overlap $O^{AB}=\mathrm{Tr}\,\rho^A\rho^B$ between systems A and B. Then, we can calculate the Hilbert-Schmidt distance from the formula (12).

IV. ENTANGLEMENT-WITNESS MEASUREMENT

In the preceeding section, we had assumed the state of two systems 0 and 1 in the form $\rho=\rho_0^A\otimes\rho_1^B$ and demonstrated that the overlap $\mathrm{Tr}\,\rho^A\rho^B$ is directly measurable. Now we consider a general state of total system 0 and 1, written in the local basis $|\psi_n\rangle_0$ and $|\phi_m\rangle_0$,

$$\rho=\sum_{n,m,k,l}\rho_{nmkl}|\psi_n\rangle_{00}\langle\psi_k|\otimes|\phi_m\rangle_{11}\langle\phi_l|, \tag{13}$$

to illustrate the entanglement-witness measurement. We again employ the measurement setup depicted in Fig. 1, but now we focus on the difference of probabilities $\Delta=p_\uparrow-p_\downarrow$, instead of calculating the visibility V. The measurement procedure is the following. First, we use this setup without CPS gate and fix the phase ψ in such a way that $p_\uparrow=1$. Subsequently, after performing complete measurement with CPS gate, the difference of probabilities given by

$$\Delta=p_\uparrow-p_\downarrow=\sum_{n,m,k,l}\rho_{nmkl}\langle\phi_l|\psi_n\rangle\langle\psi_k|\phi_m\rangle \tag{14}$$

can be rewritten by partial transposition operation in this interesting form

$$\Delta=\langle\Lambda|\rho^{T_1}|\Lambda\rangle. \tag{15}$$

Here, the partial transposition is defined as follows:

$$\rho^{T_1}_{mn,m'n'}=\langle m|_0\langle n|_1\rho^{T_1}|n'\rangle_1|m'\rangle_0=\rho_{mn',m'n} \tag{16}$$

and $|\Lambda\rangle$ is an unnormalized maximally entangled state,

$$|\Lambda\rangle=\sum_j|\phi_j\rangle_0|\phi_j\rangle_1. \tag{17}$$

Measured variable Δ is given by partial transposition of the density matrix, which is closely related to the entanglement witness operator $\mathcal{W}$,

$$\begin{gathered}\Delta=\mathrm{Tr}\,\rho\mathcal{W},\quad \mathcal{W}=\Pi_+-\Pi_-,\\ \Pi_+=\sum_{n>m}|+,m,n\rangle\langle+,m,n|+\sum_n|n,n\rangle\langle n,n|,\\ \Pi_-=\sum_{n>m}|-,m,n\rangle\langle-,m,n|,\\ |+,m,n\rangle=\frac{1}{\sqrt{2}}(|\psi_n\rangle_0|\psi_m\rangle_1+|\psi_m\rangle_0|\psi_n\rangle_1),\\ |-,m,n\rangle=\frac{1}{\sqrt{2}}(|\psi_n\rangle_0|\psi_m\rangle_1-|\psi_m\rangle_0|\psi_n\rangle_1),\\ |n,n\rangle=|\psi_n\rangle_0|\psi_n\rangle_1.\end{gathered} \tag{18}$$

Note that due to relation $\mathrm{Tr}\,A^{T_1}B=\mathrm{Tr}\,AB^{T_1}$, the flip operator V can be also expressed in a form $\mathcal{W}=(|\Lambda\rangle\langle\Lambda|)^{T_1}$. A value of $\Delta=\langle\Lambda|\rho^{T_1}|\Lambda\rangle$ can be both negative or positive. Due to Peres-Horodecki criterion [4] of separability, if Δ is negative we can be sure that the total state is entangled. Unfortunately, for positive Δ, we are not able to decide whether the state is entangled or separable. Note that measurement of the entanglement witness requires the interaction between systems 0 and 1. It is still an open question, whether a modification of the proposed measurement utilizing only local measurements and classical communication can be found. Analogically to the overlap measurement, we need to estimate only a single parameter to determine the entanglement witness, irrespective of the complexity of the total Hilbert space. The particular positive operator valued measures (POVMs) Π_+ and Π_- represent projectors onto symmetric (antisymmetric) subspaces of the total space of systems 0 and 1. Thus, if we are able to distinguish between the symmetrical and antisymmetrical states by a direct measurement, we can straightforwardly implement the overlap and entanglement-witness measurements. Hence $\mathcal{W}$ is a dichotomic variable with eigenvalues ±1. This dichotomic variable was first introduced by Werner in his famous paper [10], where an entanglement criterion corresponding to this entanglement-witness measurement was suggested. This is not as strong as the one established later by Peres and Horodecki's family, but it is very efficient for the Werner states, for example. We present here the simplest illustrative example: one can simply find that for the two-qubit Werner state $\rho_W=p|\Psi\rangle\langle\Psi|+[(1-p)/4]1\otimes1$, where $|\Psi\rangle=1/\sqrt{2}(|01\rangle-|10\rangle)$ in systems 0 and 1, the witness parameter $\Delta=1/2(1-3p)$ says that Werner state is entangled only if $p>1/3$, which is in exact coincidence with the result of Peres-Horodecki criterion [4]. More generally, the Werner state in d-dimensional Hilbert space [16]

$$\rho_W = \frac{2p}{d^2-d}\Pi_- + \frac{2(1-p)}{d^2+d}\Pi_+ \tag{19}$$

is entangled if and only if Δ is negative [10]. Assuming the unitary operations on the system 0 and 1 and relation $(A \otimes B\rho C \otimes D)^{T_1} = A \otimes D^T \rho^{T_1} C \otimes B^T$, where T denotes the transposition, we are able to extend the entanglement-witness measurement to a generalized form $\tilde{\mathcal{W}} = |\Omega\rangle\langle\Omega|^{T_1}$, where $|\Omega\rangle$ is any maximally entangled state on the total Hilbert space. This entanglement witness belongs to the class of *decomposable* entanglement witnesses that have a general form $\mathcal{W} = aP + (1-a)Q^{T_1}$, where $a \geqslant 0$ and Q,P are positive operators with unit trace [7]. Decomposable entanglement witnesses are only able to detect the entangled states with nonpositive partial transposition. On the other hand, entangled states with positive partial transposition connected with so-called bound entanglement cannot be detected in this way.

V. CONCLUSION

We propose overlap Tr $\rho^A\rho^B$ and an entanglement-witness $\mathcal{W}$ measurements employing interferometric techniques and specific nonlinear interaction (to realize CPS operation). Experimental feasibility of this CPS operation is the most important requirement in the proposed setup. We can directly implement the suggested measurement in contemporary laboratories: in the cavity QED experiments, trapped ion experiments, and for traveling light pulses utilizing electromagnetically induced transparency. These overlap and entanglement-witness measurements can be especially useful for the state in Hilbert spaces with large dimension when direct state reconstruction could be too complicated. In these cases, it seems to be more efficient to use the device proposed in this paper, than directly reconstruct an unknown state and, consequently, calculate the particular state characteristics.

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[1] For an extensive review, see D.-G. Welsch, W. Vogel, and T. Opatrný, *Homodyne Detection and Quantum-State Reconstruction*, Progress in Optics XXXIX (Elsevier, New York, 1999), p. 63.

[2] Tr $\rho^A\rho^B$ is not an exact generalization of overlap $|\langle\Psi|\Phi\rangle|^2$ between two pure states, in fact. However, we will use a notion of overlap for Tr $\rho^A\rho^B$ in the following text.

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