Quantum computation with two-level trapped cold ions beyond Lamb-Dicke limit

L. F. Wei, S. Y. Liu, and X. L. Lei
Department of Physics, Shanghai Jiao Tong University, 1954 Huashan Road, Shanghai 200030, China
(Received 31 July 2001; published 14 June 2002)

We propose a simple scheme for implementing quantum logic gates with a string of two-level trapped cold ions outside the Lamb-Dicke limit. Two internal states of each ion are used as one computational qubit (CQ) and the collective vibration of ions acts as the information bus, i.e., the bus qubit (BQ). Using the quantum dynamics for the laser-ion interaction as described by a generalized Jaynes-Cummings model, we show that quantum entanglement between any one CQ and the BQ can be coherently manipulated by applying classical laser beams. As a result, universal quantum gates, i.e., the one-qubit rotation and two-qubit controlled gates, can be implemented exactly. The required experimental parameters for the implementation, including the Lamb-Dicke (LD) parameter and the durations of the applied laser pulses, are derived. Neither the LD approximation for the laser-ion interaction nor the auxiliary atomic level is needed in the present scheme.

DOI: 10.1103/PhysRevA.65.062316 PACS number(s): 03.67.Lx, 89.70.+c, 32.80.Qk

I. INTRODUCTION

Since Shor's algorithm for efficiently factoring large numbers was proposed [1], many authors have addressed the problem of implementing quantum computation [2]. It has been shown that arbitrary rotations in the Hilbert space of an individual computational qubit (CQ), i.e., a one-qubit gate, and a controlled rotation, such as a controlled-NOT (C^N) or controlled-Z (C^Z), between two different CQs, i.e., a two-qubit controlled gate, are universal quantum gates. In other words, any unitary transformation on arbitrarily many CQs can be carried out by repeatedly performing these universal quantum gates [3]. Since one-qubit gates are generally easy to realize, implementing the two-qubit controlled gate operation is a central problem for constructing a quantum computer. Several kinds of physical systems have been proposed to implement quantum logic operations. Some of the most attractive schemes are based on nuclear magnetic resonance (NMR) [4], coupled quantum dots [5], cavity QED [6], and trapped cold ions [7], introduced first by Cirac and Zoller [8]. This system is based on the laser-ion interaction and possesses long qubit coherence times [9]. Information is stored in the spin states of an array of trapped cold ions and manipulated by using laser pulses. Two special conditions are required in the original Cirac-Zoller scheme [8]: (a) the ion must have three levels, and (b) the trap must operate in the Lamb-Dicke (LD) limit, i.e., the coupling between the internal spin and external vibrational degrees of freedom of the ion must be sufficiently weak. Laser pulses with different polarizations are used to coherently manipulate the quantum information of the system and five-step pulses are used to realize an exact C^N logic operation between two different cold ions. It is thus desirable to look for simpler schemes to construct an ion-trap quantum computer.

In the last few years, a number of modifications and extensions to the Cirac-Zoller idea have been proposed [10–13]. In particular, much attention has been paid to carrying out the universal quantum logic gates with a single trapped cooled ion, because imprisoning a single ion in a trap and cooling it to the vibrational ground state can easily be achieved with the present experimental technology. In 1995, Monroe *et al.* [14] demonstrated a simplified scheme for realizing the "reduced" two-qubit C^N logic operation between the internal and external degrees of freedom of the trapped cold ion by applying three laser pulses. Furthermore, by using a single resonant pulse, a two-qubit controlled operation with a single trapped two-level ion outside the LD limit was constructed [11]. However, the third auxiliary internal atomic state is still required in Ref. [14] and the operation reported in Ref. [11] is not the exact C^N gate, although it is equivalent to the "reduced" C^N logic gate [14] apart from phase factors. Subsequent operations must be carried out in order to eliminate the additional phase factors. Up to now, the problem of constructing a quantum computing network by connecting different ions in different traps, i.e., the scalability of the single trapped ion, has not been solved [15]. On the other hand, much attention has been paid to the problem of building a quantum network with cold ions confined in a single trap. Recent experiments showed that the collective motion of two ^{9}Be ions confined in a trap can be cooled to the ground state [16] and that quantum entanglement of up to four ions can be achieved [17], thus supporting the idea of realizing the quantum computation in a single trap. Theoretically, Jonathan *et al* [18] proposed an alternative scheme to realize the true two-qubit ion-ion gate by using the ac Stark shift induced by laser resonance with the ionic transition frequency. This scheme allows an increase in gate speed by at least an order of magnitude with respect to that of related single ion-trap experiments [14,19]. In addition, directly modifying the Cirac-Zoller proposal [8], Childs and Chuang [13] provided a general and accessible technique for performing a universal logic operation between ions with only two internal levels in a common trap. However, these schemes are based on the LD approximation, which requires that the spatial dimension of the ground state of the collective motion of these ions is much smaller than the effective wavelength of the laser wave. In fact, the quantum motion of the trapped ions is not limited to the LD regime [7]. It has been shown that utilizing the laser-ion interaction beyond the LD limit is helpful for reducing the noise in the trap and improving the cooling rate [20].

In this paper we propose an alternative scheme for realizing quantum computation with a set of two-level cold ions in a single trap driven by a series of laser pulses without using the LD approximation. The CQ is encoded by two internal states of the ion and the collective vibration of the trapped ions acts as the information bus, i.e., the bus qubit (BQ). We show how to realize universal quantum gates, including a simple rotation on any individual CQ and the exact two-qubit C^Z or C^N logic operation, by using suitable laser pulses. The paper is arranged as follows, In Sec. II the conditional quantum dynamics for the laser-ion interaction are derived and a method for coherently manipulating the entanglement between the CQ and BQ is proposed. Then, we present a simpler scheme to realize rotations on an individual CQ and the exact two-qubit controlled logic operations between the CQ and the BQ. We derive values of the required parameters, including the LD parameter and the durations of the applied laser pulses. Section III is devoted to the construction of logic gates between different CQs by making use of the elementary operations presented in Sec. II. We also discuss how to set up the experimental parameters for these realizations. Finally, we give some conclusions in Sec. IV.

II. QUANTUM DYNAMICS FOR THE LASER-ION INTERACTION BEYOND THE LD LIMIT AND EXACT QUANTUM GATES WITH A SINGLE TRAPPED COLD ION

The use of a BQ makes the physical construction of a quantum information processor much simpler. Most current ion-trap proposals use this idea. The BQ carries the quantum information in the computer. Instead of seeking a means to carry out the quantum logic operation between a pair of CQs directly, it is sufficient to implement the quantum operation between an arbitrary CQ and the BQ. In an ion-trap quantum computer a CQ is encoded by two internal spin states of the ion and the BQ is encoded by two Fock states of the external collective vibration of the trapped ions (center-of-mass vibrational degree of freedom). It has been assumed that a single laser beam can be directed at will to any chosen ion [8]. Thus, a single trapped cold ion driven by a laser beam can be used to describe a single CQ interacting with the BQ. In this section we show how to implement quantum gates with a single trapped cold ion driven by a traveling-wave laser field by considering the conditional quantum dynamics for the laser-ion interaction without taking the LD approximation.

For simplicity, we assume that a single ion is stored in a coaxial resonator rf-ion trap [21], which provides pseudopotential oscillation frequencies satisfying the condition $\omega_x \ll \omega_{y,z}$ along the principal axes of the trap. Only the quantized vibrational motion along the x direction is considered for the cooled ion [14]. Following Blockley *et al.* [22], the interaction between a single two-level trapped cold ion and a classical single-mode traveling light field of frequency ω_L can be described by the following Hamiltonian:

$$\hat{H}(t)=\hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)+\frac{1}{2}\hbar\omega_0\hat{\sigma}_z+\frac{\hbar\Omega}{2}\{\hat{\sigma}_+\exp[i\eta(\hat{a}+\hat{a}^\dagger)-i(\omega_L t+\phi)]+\text{H.c.}\}. \tag{1}$$

The first two terms correspond to the ion's external and internal degrees of freedom, respectively, and ω is the trap frequency. The final term gives the interaction between the ion and the light field with phase ϕ. Pauli operators $\hat{\sigma}_z$ and $\hat{\sigma}_\pm$ describe the internal degrees of freedom of the ion. $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators of the trap vibrational quanta. ω_0 is the atomic transition frequency, and Ω is the Rabi frequency. $\eta(<1)$ is the LD parameter. We consider the case in which the applied laser is resonant with the kth red-shifted vibrational sideband, i.e., the frequency of the laser field is chosen as $\omega_L=\omega_0-k\omega$, where k is a positive integer. Under the usual rotating-wave approximation, the Hamiltonian of the system reads

$$\hat{H}=\frac{\hbar\Omega}{2}\exp\left(-\frac{\eta^2}{2}-i\phi\right)\left\{\hat{\sigma}_+(i\eta)^k\left[\sum_{n=0}^{\infty}\frac{(i\eta)^{2n}\hat{a}^{\dagger n}\hat{a}^n}{n!(n+k)!}\right]\times\hat{a}^k+\text{H.c.}\right\}, \tag{2}$$

in the interaction picture defined by the unitary operator $\hat{U}_0(t)=\exp[-i\omega t(\hat{a}^\dagger\hat{a}+1/2)]\exp(-it\delta\hat{\sigma}_z/2)$, where $\delta=\omega_0-\omega_L$ is the detuning of the laser field from the ion. The above Hamiltonian is similar to that of the nonlinear coupled multiquantum Jaynes-Cummings model [23], which is exactly solvable. Therefore, it is easy to check the dynamical evolution of any two-qubit initial state by using the evolution operator $\hat{U}(t)=\exp[-(i/\hbar)\hat{H}t]$. Indeed, with the help of the relation [24]

$$\langle m-k|\langle e|\hat{H}|m\rangle|g\rangle=\begin{cases}0, & m<k,\\ \hbar i^k e^{-i\phi}\Omega_{m-k,m}, & m\geqslant k,\end{cases}$$

with

$$\Omega_{m-k,m}=\frac{\Omega\eta^k e^{-\eta^2/2}}{2}\sqrt{\frac{(m)!}{(m-k)!}}\sum_{n=0}^{m-k}\frac{(i\eta)^{2n}}{(k+n)!}C^n_{m-k},$$

the time evolution of the initial states $|m\rangle|e\rangle$ and $|m\rangle|g\rangle$ can be expressed as

$$|m\rangle|e\rangle\rightarrow\cos(\Omega_{m,m+k}t)|m\rangle|e\rangle-(-i)^{k-1}e^{i\phi}\sin(\Omega_{m,m+k}t)|m+k\rangle|g\rangle \tag{3}$$

and

$$|m\rangle|g\rangle\rightarrow\begin{cases}|m\rangle|g\rangle, & m<k,\\ \cos(\Omega_{m-k,m}t)|m\rangle|g\rangle+i^{k-1}e^{-i\phi}\times\sin(\Omega_{m-k,m}t)|m-k\rangle|e\rangle, & m\geqslant k,\end{cases} \tag{4}$$

respectively. The above treatment can also be modified directly to another laser excitation case, i.e., the kth blue sideband. In the present work only the red-sideband excitation is considered. It is seen from Eqs. (3) and (4) that entangled states are produced by the time evolution of the state $|m\rangle|e\rangle$ and the state $|m\rangle|g\rangle$ with $m\geqslant k$. From this conditional quantum dynamics we can define two kinds of elementary quantum operations: the one-qubit rotations

$$\hat{r}_c(m,\phi,t)=\{\cos(\Omega_{m,m}t)|g\rangle\langle g|-ie^{-i\phi}\sin(\Omega_{m,m}t)|e\rangle \times\langle g|-ie^{i\phi}\sin(\Omega_{m,m}t)|g\rangle\langle e|+\cos(\Omega_{m,m}t)|e\rangle\langle e|\}\otimes|m\rangle\langle m| \tag{5}$$

generated by applying a resonant pulse ($\omega_L=\omega_0$) to the chosen ion, and the two-qubit joint operation on the CQ and BQ

$$\hat{R}_{cb}(m,\phi,t)=\begin{cases} |m\rangle|g\rangle\langle m|\langle g|+[\cos(\Omega_{m,m+k}t)|m\rangle|e\rangle-(-i)^{k-1}e^{i\phi}\sin(\Omega_{m,m+k}t)|m+k\rangle|g\rangle]\langle m|\langle e|, & m<k,\\ [\cos(\Omega_{m-k,m}t)|m\rangle|g\rangle+i^{k-1}e^{-i\phi}\sin(\Omega_{m-k,m}t)|m-k\rangle|e\rangle]\langle m|\langle g| & \\ \quad+[\cos(\Omega_{m,m+k}t)|m\rangle|e\rangle-(-i)^{k-1}e^{i\phi}\sin(\Omega_{m,m+k}t)|m+k\rangle|g\rangle]\langle m|\langle e|, & m\geqslant k,\end{cases} \tag{6}$$

performed by using an off-resonant pulse. Here ϕ and t are the initial phase and duration of the applied pulse. By making use of these basic operations we now show how to realize exact quantum logic operations on a single trapped ion, once the LD parameter and the laser-ion interaction time are set up properly. We only need to consider the cases $k=0$ and $k=1$.

A. Simple rotations of a single CQ

In general, one-qubit rotations are easy to implement in a physical system for quantum computation. In fact, it has been shown that, under the LD approximation, the simple rotation of a CQ can be realized directly by applying a resonant pulse ($\omega_L=\omega_0$) on the specifically chosen ion [8,13,25]. This operation does not depend on the state of the BQ. The reason is that, under the LD approximation, the applied resonant pulse does not result in a coupling between the internal and external degrees of freedom of the ion. Beyond the LD limit, however, such a coupling exists and thus the rotating angle of the one-qubit operation (5) depends on both the state of the BQ and the duration of the applied resonant pulse. In the space spanned by the basis states $\Gamma=\{|0\rangle,|1\rangle\}\otimes\{|g\rangle,|e\rangle\}$, the transformation $\hat{r}_c(m,\phi,t)$ takes the following matrix form:

$$\hat{r}_c(\phi,t)=\begin{pmatrix} \cos(\Omega_{0,0}t) & -ie^{i\phi}\sin(\Omega_{0,0}t) & 0 & 0\\ -ie^{-i\phi}\sin(\Omega_{0,0}t) & \cos(\Omega_{0,0}t) & 0 & 0\\ 0 & 0 & \cos(\Omega_{1,1}t) & -ie^{i\phi}\sin(\Omega_{1,1}t)\\ 0 & 0 & -ie^{-i\phi}\sin(\Omega_{1,1}t) & \cos(\Omega_{1,1}t)\end{pmatrix}. \tag{7}$$

Here $|0\rangle$ and $|1\rangle$ are the ground and first excited states of the external vibration of the trapped ion. They are usually used to encode the BQ. It is easily seen that the Walsh-Hadamard gate can also be implemented by this one-qubit rotation. Indeed, the resonant pulse with special initial phase ϕ and the duration $t=(\pi/4+2k\pi)/\Omega_{m,m}$ yields the following operation on the CQ:

$$\hat{r}_c(m,\phi,t)\rightarrow\begin{cases}\frac{1}{\sqrt{2}}|m\rangle\langle m|\otimes\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} & \text{for } \phi=\pi/2\pm2k\pi, k=0,1,2,\ldots,\\ \frac{1}{\sqrt{2}}|m\rangle\langle m|\otimes\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} & \text{for } \phi=3\pi/2\pm2k\pi,\end{cases} \tag{8}$$

which generates a uniform superposition of two encoded basis states of the CQ from one of them. The state of the BQ is unchanged during this operation. It is easily seen that, if the condition

$$\cos(\Omega_{0,0}t)=1, \sin(\Omega_{1,1}t)=1,$$

is satisfied, the operation (7) reduces to the controlled operation constructed in Refs. [11,26], which is equivalent to the C^N logic operation between the BQ and a CQ under local transformation.

B. Logic operation between the CQ and BQ: C^Z gate

It is easily seen that an off-resonant pulse is required for generating entanglement between the CQ and BQ from an unentangled two-qubit initial state. We assume in what follows that the cold ion is addressed by the first red-sideband laser pulse, i.e., the frequency of the applied laser field is $\omega_L=\omega_0-\omega$. The BQ is encoded by the states $|0\rangle$ and $|1\rangle$. As a consequence, the dynamical evolution equation (6) can be rewritten as

$$\hat{R}_{cb}(0,\phi,t)=|0\rangle|g\rangle\langle 0|\langle g|+[\cos(\Omega_{0,1}t)|0\rangle|e\rangle -e^{i\phi}\sin(\Omega_{0,1}t)|1\rangle|g\rangle]\langle 0|\langle e|$$

for the BQ's initial state $|0\rangle$, and

$$\hat{R}_{cb}(1,\phi,t)=[\cos(\Omega_{0,1}t)|1\rangle|g\rangle + e^{-i\phi}\sin(\Omega_{0,1}t)|0\rangle|e\rangle]\langle 1|\langle g| + [\cos(\Omega_{1,2}t)|1\rangle|e\rangle - e^{i\phi}\sin(\Omega_{1,2}t)|2\rangle|g\rangle]\langle 1|\langle e|$$

for the BQ's initial state $|1\rangle$. Obviously, if the applied off-resonant pulse satisfies the condition

$$\cos(\Omega_{0,1}t)=1, \quad \cos(\Omega_{1,2}t)=-1, \tag{9}$$

a universal two-qubit logic gate, the controlled-Z ($\hat{C}^Z_{cb}$) logic operation between the CQ and BQ in Γ space, i.e.,

$$\hat{C}^Z_{cb}=|0\rangle|g\rangle\langle 0\langle g|+|0\rangle|e\rangle\langle 0|\langle e| + |1\rangle|g\rangle\langle 1|\langle g|-|1\rangle|e\rangle\langle 1|\langle e| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{10}$$

can be realized directly. If the BQ is in the state $|0\rangle$, $\hat{C}^Z_{cb}$ has no effect, whereas if the BQ is in the state $|1\rangle$, $\hat{C}^Z_{cb}$ rotates the state of the CQ by the Pauli σ_z operator. Similarly, one can prove that if the duration of the applied pulse satisfies the condition

$$\cos(\Omega_{0,1}t')=1, \quad \sin(\Omega_{1,2}t')=1,$$

the $\hat{C}^Z_{cb}$ logic operation (10) can also be implemented by sequentially applying two red-sideband pulses with equal durations $t'=2p'\pi/\Omega_{0,1}, p'=1,2,3,\ldots$, i.e.,

$$|0\rangle|g\rangle\rightarrow|0\rangle|g\rangle\rightarrow|0\rangle|g\rangle, \quad |0\rangle|e\rangle\rightarrow|0\rangle|e\rangle\rightarrow|0\rangle|e\rangle,$$

$$|1\rangle|g\rangle\rightarrow|1\rangle|g\rangle\rightarrow|1\rangle|g\rangle, \quad |1\rangle|e\rangle\rightarrow-|2\rangle|g\rangle\rightarrow-|1\rangle|e\rangle. \tag{11}$$

However, in the following, only the direct way to implement the $\hat{C}^Z_{cb}$ gate (10) is considered. It is worth noting that there is no requirement on the initial phase of the applied pulse for realizing the operation $\hat{C}^Z_{cb}$.

C. Logic operation between CQ and BQ: C^N gate

The "reduced" $\hat{C}^N_{cb}$ logic gate [14] can be constructed exactly from the one-qubit operation $\hat{r}_c(m,\phi,t)$ and the two-qubit gate $\hat{C}^Z_{cb}$. Indeed, one can easily see that the operation $\hat{C}^Z_{cb}$, surrounded by two one-qubit operations $\hat{r}_c(\phi_1,t_1)$ and $\hat{r}_c(\phi_3,t_3)$, yields the following new operation between the BQ and a CQ:

$$\hat{r}_c(\phi_3,t_3)\hat{C}^Z_{cb}\hat{r}_c(\phi_1,t_1)=\begin{pmatrix} A_{0000} & A_{0001} & 0 & 0 \\ A_{0100} & A_{0101} & 0 & 0 \\ 0 & 0 & A_{1000} & A_{1011} \\ 0 & 0 & A_{1110} & A_{1111} \end{pmatrix}, \tag{12}$$

with

$$A_{0000}=\cos(\Omega_{0,0}t_3)\cos(\Omega_{0,0}t)_1 - e^{i(\phi_3-\phi_1)}\sin(\Omega_{0,0}t_3)\sin(\Omega_{0,0}t_1),$$

$$A_{0001}=-ie^{i\phi_1}\cos(\Omega_{0,0}t_3)\sin(\Omega_{0,0}t_1) - ie^{i\phi_3}\sin(\Omega_{0,0}t_3)\cos(\Omega_{0,0}t_1),$$

$$A_{0100}=-ie^{-i\phi_3}\sin(\Omega_{0,0}t_3)\cos(\Omega_{0,0}t_1) - ie^{-i\phi_1}\cos(\Omega_{0,0}t_3)\sin(\Omega_{0,0}t_1),$$

$$A_{0101}=\cos(\Omega_{0,0}t_3)\cos(\Omega_{0,0}t_1) - e^{-i(\phi_3-\phi_1)}\sin(\Omega_{0,0}t_3)\sin(\Omega_{0,0}t_1);$$

$$A_{1010}=\cos(\Omega_{1,1}t_3)\cos(\Omega_{1,1}t_1) + e^{i(\phi_3-\phi_1)}\sin(\Omega_{1,1}t_3)\sin(\Omega_{1,1}t_1),$$

$$A_{1011}=-ie^{i\phi_1}\cos(\Omega_{1,1}t_3)\sin(\Omega_{1,1}t_1) + ie^{i\phi_3}\sin(\Omega_{1,1}t_3)\cos(\Omega_{1,1}t_1),$$

$$A_{1110}=ie^{-i\phi_1}\cos(\Omega_{1,1}t_3)\sin(\Omega_{1,1}t_1) - ie^{-i\phi_3}\sin(\Omega_{1,1}t_3)\cos(\Omega_{1,1}t_1),$$

$$A_{1111}=-\cos(\Omega_{1,1}t_3)\cos(\Omega_{1,1}t_1) - e^{-i(\phi_3-\phi_1)}\sin(\Omega_{1,1}t_3)\sin(\Omega_{1,1}t_1),$$

where t_1,t_3 and ϕ_1,ϕ_3 are the durations and initial phases of the first and third applied resonant laser pulses, respectively. They should be set up properly for realizing the specific quantum operation between the BQ and CQ. It is easily seen that, if the two initial phases ϕ_1,ϕ_3 satisfy the relation

$$\phi_3-\phi_1=\pm 2k\pi,,k=0,1,2,\ldots, \tag{13}$$

the matrix elements on the right side of Eq. (12) become

$$A_{0000}=A_{0101}=\cos[\Omega_{0,0}(t_3+t_1)],$$

$$A_{0001}=-ie^{i\phi_1}\sin[\Omega_{0,0}(t_3+t_1)],$$

$$A_{0100}=-ie^{-i\phi_1}\sin[\Omega_{0,0}(t_3+t_1)],$$

$$A_{0101}=-A_{1111}=\cos[\Omega_{0,0}(t_3-t_1)],$$

$$A_{1011}=ie^{i\phi_1}\sin[\Omega_{1,1}(t_3-t_1)],$$

$$A_{1110}=-ie^{-i\phi_1}\sin[\Omega_{1,1}(t_3-t_1].$$

Furthermore, if the matching conditions

$$\cos[\Omega_{0,0}(t_3+t_1)]=1,$$

$$\sin[\Omega_{1,1}(t_3-t_1)] = \begin{cases} 1 & \text{for } \phi_1=3\pi/2\pm 2k'\pi, \quad k'=0,1,2,\ldots, \\ -1 & \text{for } \phi_1=\pi/2\pm 2k\pi, \quad k=0,1,2,\ldots \end{cases} \tag{14}$$

are satisfied, we have

$$A_{0000}=A_{0101}=A_{1011}=A_{1110}=1,$$

$$A_{0001}=A_{0100}=A_{1000}=A_{1111}=0.$$

This means that, under the conditions (13) and (14), the operation (12) is nothing but the exact reduced C^N logic gate [14], i.e.,

$$\hat{r}_c(\phi_3,t_3)\hat{C}^Z_{cb}\hat{r}_c(\phi_1,t_1)\to\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}=\hat{C}^N_{cb}. \tag{15}$$

In this logic operation, if the BQ (control qubit) is in the state $|0\rangle$, the operation has no effect, whereas if the BQ is in the state $|1\rangle$, a NOT operation is applied to the CQ. The BQ remains in its initial state after the operation.

We now show how to set up the experimental parameters to realize the $\hat{C}^N_{cb}$ gate (15). These parameters include the LD parameter and the durations of the applied laser pulses. First, the requisite LD parameter and the duration of the off-resonant red-sideband pulse are determined by Eq. (9), which implies

$$\frac{\Omega_{1,2}}{\Omega_{0,1}}=\frac{2-\eta^2}{\sqrt{2}}=\frac{q-0.5}{p}, \quad p,q=1,2,3,\ldots,$$

$$t_2=\frac{2p\pi}{\Omega_{0,1}}=\frac{4p\pi e^{\eta^2/2}}{\Omega\eta}. \tag{16}$$

Second, using the LD parameter determined from the above equation, the durations of two resonant pulses surrounding the first red-sideband pulse are further determined by solving Eq. (14) for different initial phases; e.g.,

$$t_1=\pi\left(\frac{p'+1}{\Omega_{0,0}}+\frac{q'+0.25}{\Omega_{1,1}}\right)=T_1,$$

$$t_3=\pi\left(\frac{p'+1}{\Omega_{0,0}}-\frac{q'+0.25}{\Omega_{1,1}}\right)=T_3 \quad \text{for } \phi_1=\phi_3=\pi/2;$$

$$t_1=T_3,\ t_3=T_1 \quad \text{for } \phi_1=\phi_3=3\pi/2;$$

$$p',q'=1,2,3,\ldots. \tag{17}$$

So far we have shown that, once the coupling parameter η is set up appropriately, the "reduced" two-qubit C^N logic operation between the CQ and BQ can be implemented exactly by sequentially applying three pulses (a red-sideband pulse surrounded by two resonant ones) with controllable durations. As in Refs. [11,26], neither the weak coupling limit (i.e. the LD approxiamtion) nor the auxiliary atomic level is required in the present scheme. Therefore, pulses having different polarizations are not needed and the coupling parameter between the CQ and BQ may be large (e.g., 0.9064, 0.9692, etc.). We also note that this logic operation does not depend on the initial phase of the applied off-resonant pulse, while the initial phases of two resonant pulses should be set up accurately to satisfy the conditions (13), (14). The main points of our approach can be summarized as follows:

(a) Use a resonant pulse with initial phase $\pi/2$(or $3\pi/2$) and duration $T_1(T_3)$ to rotate the chosen CQ,

(b) Use an off-resonant red-sideband pulse to complete the C^Z logic operation between the chosen CQ and BQ and

(c) Use another resonant pulse with initial phase $\pi/2$ (or $3\pi/2$) and duration T_3 (T_1) on the chosen CQ to complete the gate operation C^N between the chosen CQ and BQ.

Some values of these experimental parameters are given in Table I. The durations of the applied pulses are given by the quantities $\Omega t_j/\pi, j=1,2,3$ in the table. It is seen from the table that the switching speed of the C^N gate depends on the Rabi frequency Ω and the LD parameter η. It can be estimated numerically once the experimental parameters are defined. For example, in the case of a recent experiment [14] a single $^9Be^+$ ion confined in a coaxial-resonator radio frequency ion trap and cooled to its quantum ground state by Raman cooling, the target qubit consists of two $^2S_{1/2}$ hyperfine ground states of $^9Be^+$: $^2S_{1/2}|F=2,m_F=2\rangle$ and

TABLE I. Some experimental parameters for realizing the exact "reduced" C^N gate by three-step sequential pulses with $\phi_1=\phi_3=\pi/2$.

p	q	η	$\Omega t_2/\pi$	p'	q'	$\Omega t_1/\pi$	$\Omega t_3/\pi$
2	2	0.9692	13.2024	5	1	29.1785	2.8108
10	8		33.0061	8	1	38.7753	12.4076
				10	1	45.1732	18.8055
2	3	0.4819	18.6448	1	1	2.9777	1.5148
6	8		55.9343	2	2	8.1496	0.8354
				3	1	7.4702	6.0073
3	3	0.9064	19.9648	2	1	10.2554	1.8081
9	8		59.8943	3	1	13.2713	4.8239
				8	2	45.2453	3.0088
4	6	0.2355	69.8532	1	1	2.6005	1.5119
				2	1	4.6567	3.5682
				2	2	6.8337	1.3913
14	20	0.1738	16.3571	1	1	2.5538	1.5071
				2	1	4.5843	3.5374
				2	2	6.6779	1.4438
3	4	0.5919	24.1611	1	1	3.2991	1.4661
				2	1	5.6817	3.8487
				2	2	9.3477	0.1827

$^2S_{1/2}|F=2,m_F=1\rangle$, separated by $\omega_0/2\pi=1.25$ GHz. If the Rabi frequencies are chosen as $\Omega=2\pi\times140$ kHz for resonant excitations and $\eta\Omega=2\pi\times30$ kHz for off-resonant excitations, one can easily show that the shortest duration of the applied pulses for the realization of the exact C^N gate is about 10^{-4}.

In summary, we have suggested a method for implementing quantum logic operations between the BQ and CQ beyond the LD limit. In the controlled operations, the BQ acts as the control qubit. It is worthwhile to point out that the BQ is really not an additional qubit in the quantum computer because one cannot perform any single-qubit operation on the BQ. However, the BQ can serve as an intermediary to perform quantum logic operations between different CQs. This will be the subject of the next section.

III. UNIVERSAL GATES FOR QUANTUM COMPUTATION WITH TWO-LEVEL TRAPPED COLD IONS BEYOND THE LD LIMIT

The ion-trap quantum computer consists of a string of ions stored in a linear radio-frequency trap and cooled sufficiently. The motion of the ions, which are coupled together due to the Coulomb force between them, is quantum mechanical in nature. Each qubit is formed by two internal levels of an ion. The ions are sufficiently separated to be addressed by different laser beams [7], i.e., each ion can be illuminated individually. The communication and logic operations between qubits are performed by using laser pulses sequentially to excite or deexcite quanta of the collective vibration (i.e., the shared phonon) modes, which act as the BQ. Following Monroe *et al.* [11], only the lowest vibration mode of the ion string, i.e., the harmonic motion of the center of mass, is considered. In the interaction picture defined by the unitary operator

$$\hat{U}_0^N(t)=\exp[-i\omega t(\hat{a}^\dagger\hat{a}+1/2)]\prod_{j=1}^{N}\exp(-it\delta_j\hat{\sigma}_{j,z}/2),$$

the Hamiltonian of the system takes the form

$$\hat{H}=\frac{\hbar}{2}\sum_{j=1}^{N}\Omega_j\{\hat{\sigma}_{j,+}\exp[i\eta_j(\hat{a}+\hat{a}^\dagger)]+\text{H.c.}\},\qquad(18)$$

with $\Omega_j\equiv\Omega$, $\eta_j\equiv\eta$. We have shown in the above section that simple rotations on a single CQ and controlled operations between a CQ and the BQ can be realized exactly by applying suitable laser pulses. In the following we show how to implement universal quantum gates involving many CQs assisted by their common BQ.

A. Operation of multiple CQs

Operations on different CQs can be performed by applying different laser beams to different ions. As a consequence,rotation operations on different CQs can be performed individually and synchronously. For example, for a quantum register with N trapped cold ions, it is easily proven that the uniform superposition state of the N-CQ register

$$|\Psi_0\rangle=\frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}c_i|i\rangle,\quad c_i=\pm1,$$

which is the computational initial state for almost all quantum algorithms, can easily be prepared by applying N resonant pulses with equal initial phases $\pi/2$ (or $3\pi/2$) and durations $\Omega_{m,m}t=\pi/4$ to N trapped ions synchronously. We note that the CQs in such an operator are not correlated with each other except by sharing a common phonon mode. For realizing quantum computation with the N-ion quantum register, we shall construct the universal two-qubit controlled gates between different CQs.

B. C^Z logic gate between different trapped ions

The C^Z logic operation between the ith ion and jth ion,

$$\hat{C}^Z_{c_ic_j}=|g_i\rangle|g_j\rangle\langle g_i|\langle g_j|+|g_i\rangle|e_j\rangle\langle g_i|\langle e_j|+|e_i\rangle|g_j\rangle\times\langle e_i|\langle g_j|-|e_i\rangle|e_j\rangle\langle e_i|\langle e_j|,\qquad(19)$$

means that if the first CQ (control qubit) is in the state $|g_i\rangle$, the operation has no effect, whereas if the control qubit is in the state $|e_i\rangle$, the state of the second CQ (target qubit) is rotated by the Pauli operator $\hat{\sigma}_z$. To realize this operation we consider the quantum evolution of a three-qubit system (i.e., two CQs and the BQ initially in the state $|0\rangle$) driven by an off-resonant laser pulse. Using three-step red-sideband pulses with the same frequency $\omega_L=\omega_0-\omega$ to the ith, jth, and again the ith ions sequentially, we have the following dynamical evolutions:

$$|0\rangle|g_i\rangle|g_j\rangle\rightarrow|0\rangle|g_i\rangle|g_j\rangle,$$

$$|0\rangle|g_i\rangle|e_j\rangle\rightarrow B_1|0\rangle|g_i\rangle|e_j\rangle+B_2|0\rangle|e_i\rangle|g_j\rangle+B_3|1\rangle|g_i\rangle|g_j\rangle,$$

$$|0\rangle|e_i\rangle|g_j\rangle\rightarrow C_1|0\rangle|g_i\rangle|e_j\rangle+C_2|0\rangle|e_i\rangle|g_j\rangle+C_3|1\rangle|g_i\rangle|g_j\rangle,$$

$$|0\rangle|e_i\rangle|e_j\rangle\rightarrow D_1|1\rangle|g_i\rangle|e_j\rangle+D_2|1\rangle|e_i\rangle|g_j\rangle+D_3|2\rangle|g_i\rangle|g_j\rangle+D_4|0\rangle|e_i\rangle|e_j\rangle,\qquad(20)$$

with

$$B_1=\cos(\Omega_{0,1}t_2'),\quad B_2=-e^{i(\phi_2'-\phi_3')}\sin(\Omega_{0,1}t_2')\sin(\Omega_{0,1}t_3'),$$

$$B_3=-e^{i\phi_2'}\sin(\Omega_{0,1}t_2')\cos(\Omega_{0,1}t_3');$$

$$C_1=-e^{i(\phi_1'-\phi_2')}\sin(\Omega_{0,1}t_1')\sin(\Omega_{0,1}t_2'),$$

$$C_2=\cos(\Omega_{0,1}t_1')\cos(\Omega_{0,1}t_3')-e^{i(\phi_1'-\phi_3')}\sin(\Omega_{0,1}t_1')\times\cos(\Omega_{0,1}t_2'\sin(\Omega_{0,1}t_3'),$$

$$C_3=-e^{i\phi_1'}\sin(\Omega_{0,1}t_1')\cos(\Omega_{0,1}t_2')\cos(\Omega_{0,1}t_3') - e^{i\phi_3'}\cos(\Omega_{0,1}t_1')\sin(\Omega_{0,1}t_3');$$

$$D_1=-[e^{i\phi_1'}\sin(\Omega_{0,1}t_1')\cos(\Omega_{1,2}t_2')\cos(\Omega_{0,1}t_3') + e^{i\phi_3'}\cos(\Omega_{0,1}t_1')\cos(\Omega_{0,1}t_2')\sin(\Omega_{0,1}t_3')],$$

$$D_2=-e^{i\phi_2'}[\cos(\Omega_{0,1}t_1')\sin(\Omega_{0,1}t_2')\cos(\Omega_{1,2}t_3') + e^{i(\phi_1'-\phi_3')}\sin(\Omega_{0,1}t_1')\sin(\Omega_{1,2}t_2')\sin(\Omega_{1,2}t_3')],$$

$$D_3=e^{i\phi_2'}[e^{i\phi_1'}\sin(\Omega_{0,1}t_1')\sin(\Omega_{1,2}t_2')\cos(\Omega_{1,2}t_3') + e^{i\phi_3'}\cos(\Omega_{0,1}t_1')\sin(\Omega_{0,1}t_2')\sin(\Omega_{1,2}t_3')],$$

$$D_4=\cos(\Omega_{0,1}t_1')\cos(\Omega_{0,1}t_2')\cos(\Omega_{0,0}t_3')-e^{i(\phi_1'-\phi_3')} \times\sin(\Omega_{0,1}t_1')\cos(\Omega_{1,2}t_2')\sin(\Omega_{0,1}t_3').$$

Here ϕ_i' and t_i' $(i=1,2,3)$ are the phase and duration of the first, second, and third off-resonant pulses. They should be set up properly for realizing the operation $\hat{C}^Z_{c_ic_j}$. Obviously, if the LD parameter η and the duration of the second off-resonant pulse are set up according to Eq. (9) to implement the logic operation $\hat{C}^Z_{c_jb}$ between the BQ and the jth CQ, the coefficients in Eq. (20) reduce to

$$B_1=1,\quad B_2=B_3=0;$$

$$C_1=0,$$

$$C_2=\cos(\Omega_{0,1}t_1')\cos(\Omega_{0,1}t_3') - e^{i(\phi_1'-\phi_3')}\sin(\Omega_{0,1}t_1')\sin(\Omega_{0,1}t_3'),$$

$$C_3=-e^{i\phi_1'}\sin(\Omega_{0,1}t_1')\cos(\Omega_{0,1}t_3') - e^{i\phi_3'}\cos(\Omega_{0,1}t_1')\sin(\Omega_{0,1}t_3');$$

$$D_1=e^{i\phi_1'}\sin(\Omega_{0,1}t_1')\cos(\Omega_{0,1}t_3') - e^{i\phi_3'}\cos(\Omega_{0,1}t_1')\sin(\Omega_{0,1}t_3'),$$

$$D_2=D_3=0,$$

$$D_4=\cos(\Omega_{0,1}t_1')\cos(\Omega_{0,1}t_3') + e^{i(\phi_1'-\phi_3')}\sin(\Omega_{0,1}t_1')\sin(\Omega_{0,1}t_3').$$

These coefficients further become

$$B_2=B_3=0,\quad B_1=1,\quad C_1=C_3=0,\quad C_2=1,$$

$$D_1=D_2=D_3=0,\quad D_4=-1,$$

if ϕ_1',ϕ_3' and t_1',t_3' are further set up to satisfy the conditions

$$\phi_1'=\phi_3'\pm 2k\pi,\quad k=0,1,2,\ldots,$$

$$\cos[\Omega_{0,1}(t_1'+t_3')]=1, \cos[\Omega_{0,1}(t_1'-t_3')]=-1. \qquad (21)$$

This means that, once the LD parameter is set up properly, a $\hat{C}^Z_{c_jb}$ gate operation, surrounded by two off-resonant pulses addressing the ith CQ, yields the exact C^Z logic operation between the ith and jth ions, i.e.,

$$\hat{R}_{c_ib}(\phi_3',t_3')\hat{C}^Z_{c_jb}\hat{R}_{c_ib}(\phi_1',t_1')\rightarrow\hat{C}^Z_{c_ic_j}\otimes|0\rangle\langle 0|. \qquad (22)$$

The BQ remains in its initial state after the operation. Equation (21) implies that

$$\Omega_{0,1}t_1'=(k+k')\pi-\frac{\pi}{2}, \Omega_{0,1}t_3'=(k-k')\pi+\frac{\pi}{2},$$

$$k,k'=1,2,3,\ldots. \qquad (23)$$

This shows that, once the LD parameter η is set up properly, the C^Z logic operation (19) between the ith and the jth CQs can be realized exactly by sequentially applying three-step red-sideband pulses with adjustable durations. The procedure for realizing this gate can be summarized as follows: (a) A red-sideband pulse with $\Omega_{0,1}t_1'=3\pi/2$ is applied to the ith ion; (b) a red-sideband pulse with $\Omega_{0,1}t_2'=2\pi$ is applied to the jth ion; and (c) a red-sideband pulse with $\Omega_{0,1}t_3'=\pi/2$, whose phase equals that of the first one, is applied to the ith ion again.

Comparing to Cirac-Zoller's proposal [8], we note that the same number of pulses for implementing the C^Z logic operation between different ions are required. An obvious advantage of the present method is that no auxiliary atomic level is needed. Thus laser pulses with different polarizations are not required.

C. C^N logic operation between different trapped ions

As another universal two-qubit gate for an ion-trap quantum computer, the C^N logic operation between a pair of CQs (e.g., the ith and jth ions),

$$\hat{C}^N_{c_ic_j}=|g_i\rangle|g_j\rangle\langle g_i|\langle g_j|+|g_i\rangle|e_j\rangle\langle g_i|\langle e_j|+|e_i\rangle|g_j\rangle \times\langle e_i|\langle e_j|+|e_i\rangle|e_j\rangle\langle e_i|\langle g_j|, \qquad (24)$$

means that if the first CQ (control qubit) is in the state $|g_i\rangle$, the operation has no effect, whereas if the control qubit is in the state $|e_i\rangle$, the state of the second CQ (target qubit) undergoes a NOT operation. In Ref. [11] Monroe *et al.* showed that if the C^N gate (15) between the target qubit j and the BQ is surrounded by two extra operations, which map and reset the state of the control qubit i onto the state of the BQ, the C^N gate operation (24) may be carried out. We now give an alternative method to realize the C^N logic operation (24) by making use of the two-qubit operation $\hat{C}^Z_{c_ic_j}$ and the single-qubit operation $\hat{r}_c(m,\phi,t)$. When the BQ is in the state $|0\rangle$, we know from Sec. II that the rotation on the jth CQ

$$\hat{r}_{c_j}(0,\phi,t)=\cos(\Omega_{0,0}t)|g_j\rangle\langle g_j|-ie^{i\phi}\sin(\Omega_{0,0}t)|g_j\rangle\langle e_j|-ie^{-i\phi}\sin(\Omega_{0,0}t)|e_j\rangle\langle g_j|+\cos(\Omega_{0,0}t)|e_j\rangle\langle e_j| \quad (25)$$

can be carried out by applying a single resonant laser pulse to the jth ion. The state of the BQ is not changed during this operation. If the $\hat{C}^Z_{c_ic_j}$ gate is surrounded by the operations $\hat{r}_{c_j}(0,\phi_1'',t_1'')$ and $\hat{r}_{c_j}(0,\phi_3'',t_3'')$, we have

$$\begin{aligned}\hat{r}_{c_j}(0,\phi_3'',t_3'')\hat{C}^Z_{c_ic_j}\hat{r}_{c_j}(0,\phi_1'',t_1'')=&B_{0000}|g_i\rangle|g_j\rangle\langle g_i|\langle g_j|+B_{0001}|g_i\rangle|g_j\rangle\langle g_i|\langle e_j|+B_{0010}|g_i\rangle|e_j\rangle\langle g_i|\langle g_j|+B_{0011}|g_i\rangle|e_j\rangle\\&\times\langle g_i|\langle e_j|+B_{1100}|e_i\rangle|g_j\rangle\langle e_i|\langle g_j|+B_{1101}|e_i\rangle|g_j\rangle\langle e_i|\langle e_j|+B_{1110}|e_i\rangle|e_j\rangle\langle e_i|\langle g_j|\\&+B_{1111}|e_i\rangle|e_j\rangle\langle e_i|\langle e_j|,\end{aligned} \quad (26)$$

with

$$B_{0000}=\cos(\Omega_{0,0}t_3'')\cos(\Omega_{0,0}t_1'')-e^{i(\phi_3''-\phi_1'')}\sin(\Omega_{0,0}t_3'')\sin(\Omega_{0,0}t_1''),$$

$$B_{0001}=-i[e^{i\phi_1''}\cos(\Omega_{0,0}t_3'')\sin(\Omega_{0,0}t_1'')+e^{i\phi_3''}\sin(\Omega_{0,0}t_3'')\cos(\Omega_{0,0}t_1'')],$$

$$B_{0010}=-i[e^{-i\phi_3''}\sin(\Omega_{0,0}t_3'')\cos(\Omega_{0,0}t_1'')+e^{-i\phi_1''}\cos(\Omega_{0,0}t_3'')\sin(\Omega_{0,0}t_1'')],$$

$$B_{0011}=\cos(\Omega_{0,0}t_3'')\cos(\Omega_{0,0}t_1'')-e^{-i(\phi_3''-\phi_1'')}\sin(\Omega_{0,0}t_3'')\sin(\Omega_{0,0}t_1''),$$

$$B_{1100}=\cos(\Omega_{0,0}t_3'')\cos(\Omega_{0,0}t_1'')+e^{i(\phi_3''-\phi_1'')}\sin(\Omega_{0,0}t_3'')\sin(\Omega_{0,0}t_1''),$$

$$B_{1101}=-i[e^{i\phi_1''}\cos(\Omega_{0,0}t_3'')\sin(\Omega_{0,0}t_1'')-e^{i\phi_3''}\sin(\Omega_{0,0}t_3'')\cos(\Omega_{0,0}t_1'')],$$

$$B_{1110}=-i[e^{-i\phi_3''}\sin(\Omega_{0,0}t_3'')\cos(\Omega_{0,0}t_1'')-e^{-i\phi_1''}\cos(\Omega_{0,0}t_3'')\sin(\Omega_{0,0}t_1'')],$$

$$B_{1111}=-\cos(\Omega_{0,0}t_3'')\cos(\Omega_{0,0}t_1'')-e^{-i(\phi_3''-\phi_1'')}\sin(\Omega_{0,0}t_3'')\sin(\Omega_{0,0}t_1'').$$

Here, ϕ_1'',t_1'' and ϕ_3'',t_3'' are the phases and durations of the resonant pulses applied to perform two rotations on the target CQ, i.e., the jth ion. Similarly, one can easily prove that, if ϕ_1'',ϕ_3'' and t_1'',t_3'' satisfy

$$\phi_3''=\phi_1''\pm 2k\pi,\quad k=0,1,2,\ldots,$$

$$\Omega_{0,0}t_1''=(p+p'-3/4)\pi,\ \Omega_{0,0}t_3''=(p-p'+3/4)\pi$$

$$\text{for}\quad \phi_1=\pi/2\pm 2k\pi, p,p'=1,2,3,\ldots; \quad (27)$$

$$\Omega_{0,0}t_1''=(p+p'-1/4)\pi,\ \Omega_{0,0}t_3''=(p-p'+1/4)\pi$$

$$\text{for}\quad \phi_1=3\pi/2\pm 2k\pi,$$

the coefficients in Eq. (26) become

$$B_{0000}=B_{0011}=B_{1101}=B_{1110}=1,$$

$$B_{0001}=B_{0010}=B_{1100}=B_{1111}=0.$$

This means that an exact $\hat{C}^Z_{c_ic_j}$ surrounded by two resonant pulses applied to the jth CQ can give rise to an exact C^N logic operation between the ith and jth CQs, i.e.,

$$\hat{r}_{c_j}(0,\phi_3'',t_3'')\hat{C}^Z_{c_ic_j}\hat{r}_{c_j}(0,\phi_1'',t_1'')\to\hat{C}^N_{c_ic_j}. \quad (28)$$

The durations of two applied resonant pulses are determined again by Eq. (18). We summarize the process for realizing the logic operation $\hat{C}^N_{c_ic_j}$ as follows: (a) A resonant pulse with initial phase $\pi/2$ ($3\pi/2$) and $\Omega_{0,0}t_1''=5\pi/4(7\pi/4)$ is applied to the target CQ (the jth ion); (b) three-step red-sideband pulses are sequentially used to implement C^Z gate (19) between the ith and jth CQs; and (c) another resonant pulse with initial phase $\pi/2$ ($3\pi/2$) and $\Omega_{0,0}t_3''=3\pi/4$ ($\pi/4$) is applied to the target jth CQ.

In summary, we have given a method to implement arbitary one-qubit rotations on any single CQ and the C^N or C^Z gate between any pair of CQs with trapped cold ions. These operations form a universal set. Any unitary operations on an arbitrary number of CQs can be expressed as a composition of the elements in the set. We note that the initial state of the BQ in the above discussion is always assumed to be its ground state $|0\rangle$. One can prove that, if the BQ is initially in an excited state, e.g., $|1\rangle$, $|2\rangle$, etc., universal two-qubit gates between different ions cannot be realized by a one-quantum excitation process. Implementing quantum computation by making use of the multiquantum (multiphonon) interaction between the CQ and BQ will be discussed in a future publication.

IV. CONCLUSIONS

We have demonstrated the possibility of performing ion-trap quantum computation with two-level ions beyond the LD limit. The method for realizing the exact universal quantum gates, i.e., single-qubit rotations and two-qubit controlled logic operations, has been discussed in detail. The present scheme is not limited to small values of the LD parameter and does not require an extra atomic level. Therefore, laser pulses with different polarizations are not needed. There are two requirements for our scheme. First, the LD parameter η must be set up accurately for realizing the exact C^Z logic operation between the internal and external degrees of freedom. Second, the durations of all applied laser pulses must be adjusted accurately. We have shown that the C^Z logic operation between the external and internal states can be realized by using a single off-resonant pulse. This C^Z gate $\hat{C}^Z_{cb}$ surrounded by two resonant pulses with suitable durations gives rise to the exact corresponding C^N logic gate. We have further shown that the C^Z logic operation between different ions in a trap can be implemented by using a three-step sequence of off-resonant pulses. The second off-resonant pulse is used to realize the C^Z logic operation between the internal and external states of the target ion, and the first and third ones are applied to the control ion. This C^Z gate $\hat{C}^Z_{c_ic_j}$ surrounded by two resonant pulses with suitable durations applied to the target ion yields the corresponding C^N logic operation between two ions.

Compared to other methods for realizing the exact C^N logic operation on an ion-trap quantum register, the present scheme has some advantages. Although the same number of pulses is needed for realizing the C^N logic gate as that reported in Refs. [8], [13] and [14], the auxiliary atomic level and LD approximation are not required in the present scheme. Our approach to realizing the C^N gate is also different from that in Refs. [11] and [26], where a controlled operation between the BQ and CQ, which is equivalent to the exact C^N gate only under local transformation, was realized by applying a single resonant pulse. In the present scheme the exact C^N logic gate is implemented by an off-resonant pulse surrounded by two resonant ones. We hope that the present scheme will be useful for future experiments.

ACKNOWLEDGMENTS

This work was supported by the Natural Science Foundation of China, the Special Funds for Major State Basic Research Project (Grant No. 200683), the Shanghai Municipal Commission of Science and Technology, and the Shanghai Foundation for Research and Development of Applied Materials. One of us (L.F.W.) is grateful to Professor J. Q. Liang for helpful discussions.

[1] P. W. Shor, in *Proceedings of the 35th Annual Symposium on Foundations of Computer Science*, edited by S. Goldwasser (IEEE Computer Society Press, Los Alamitos, CA, 1994), p. 124.
[2] A. Ekert and R. Jozsa, Rev. Mod. Phys. **68**, 733 (1995).
[3] A. Barenco *et al.*, Phys. Rev. A **52**, 3457 (1995).
[4] N.A. Gershenfeld and I.L. Chuang, Science **275**, 350 (1997); D.G. Cory, A.F. Fahmy, and T.F. Havel, Proc. Natl. Acad. Sci. U.S.A. **94**, 1634 (1997).
[5] D. Loss and D.P. Divincenzo, Phys. Rev. A **57**, 120 (1998); T. Tanamoto, *ibid.* **61**, 022305 (2000).
[6] Q.A. Turchette *et al.*, Phys. Rev. Lett. **75**, 4710 (1995).
[7] A. Steane, Appl. Phys. B: Lasers Opt. **64**, 623 (1997); D.F.V. James, *ibid.* **66**, 181 (1998).
[8] J.I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).
[9] D. Kielpinski *et al.*, Science **291**, 1013 (2001); F. Schmidt-Kaler *et al.*, J. Mod. Opt. **47**, 2573 (2000).
[10] Q.A. Turchette *et al.*, Phys. Rev. Lett. **81**, 1525 (1998).
[11] C. Monroe, D. Leibfried, B.E. King, D.M. Meekhof, W.M. Itano, and D.J. Wineland, Phys. Rev. A **55**, R2489 (1997).
[12] D. Leibfried, Phys. Rev. A **60**, R3335 (1999); E. Solano, R.L. de Matos Filho, and N. Zagury, *ibid.* **59**, R2539 (1999).
[13] A.M. Childs and I.L. Chuang, Phys. Rev. A **63**, 012306 (2001).
[14] C. Monroe, D.M. Meekhof, B.E. King, W.M. Itano, and D.J. Wineland, Phys. Rev. Lett. **75**, 4714 (1995).
[15] J.I. Cirac and P. Zoller, Nature (London) **404**, 579 (2000).
[16] B.E. King *et al.*, Phys. Rev. Lett. **81**, 1525 (1998).
[17] C.A. Sackett *et al.*, Nature (London) **404**, 25 (2000).
[18] D. Jonathan, M.B. Plenio, and P.L. Knight, Phys. Rev. A **62**, 042307 (2000).
[19] Ch. Roos *et al.*, Phys. Rev. Lett. **83**, 4713 (1999).
[20] D. Stevens, J. Brochard, and A.M. Steane, Phys. Rev. A **58**, 2750 (1998).
[21] S.R. Jefferts, C. Monroe, E.W. Bell, and D.J. Wineland, Phys. Rev. A **51**, 3112 (1995).
[22] C.A. Blockley, D.F. Walls, and H. Kisken, Europhys. Lett. **17**, 509 (1992).
[23] W. Vogel and R.L. de Matos Filho, Phys. Rev. A **52**, 4214 (1995); W. Vogel and D.-G. Welsch, *ibid.* **40**, 7113 (1989).
[24] A. Steane *et al.*, Phys. Rev. A **62**, 042305 (2000).
[25] M. Sasura and V. Buzek, Phys. Rev. A **64**, 012305 (2001).
[26] L.X. Li and G.C. Guo, J. Opt. B: Quantum Semiclassical Opt. **2**, 62 (2000).