Continuous-variable Werner state: Separability, nonlocality, squeezing, and teleportation

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We investigate the separability, nonlocality, and squeezing of the continuous-variable analog of the Werner state: a mixture of a pure two-mode squeezed vacuum state with local thermal radiations. Utilizing this Werner state, coherent-state teleportation in the Braunstein-Kimble setup is discussed.

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I. INTRODUCTION

Quantum entanglement and nonlocality are fundamental resources for the quantum information processing, such as quantum teleportation [1], entanglement swapping [2], dense coding [3], quantum cryptography [4], and quantum computation [5]. The efficiency of quantum information processing significantly depends on the degree of entanglement or nonlocality of the quantum state shared by the parties involved in a given protocol. This dependence may be particularly vividly illustrated with the Werner states [6], which are formed by a mixture of the maximally entangled state and the separable maximally mixed state,

$$\rho = p|\Psi\rangle\langle\Psi| + \frac{(1-p)}{d^2} I_1 \otimes I_2, \quad 0 \leq p \leq 1, \tag{1}$$

where

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_1 |i\rangle_2 \tag{2}$$

is the maximally entangled state in d-dimensional Hilbert space and I denotes the identity operator.

The Werner state is characterized by a single parameter: the probability p of the maximally entangled state in the mixture and the Werner state being entangled iff $p > 1/(1+d)$. When the Werner state is used as a quantum channel for teleportation, then the average teleportation fidelity is given by $F = p + (1-p)/d$ [7,8]. This figure should be compared with the maximum fidelity achievable by means of classical communication and local operations $F_C = 2/(1+d)$ [9]. Since this boundary is reached exactly for $p = 1/(1+d)$, one concludes that all entangled Werner states are useful for the teleportation. Particularly interesting is the Werner state of two qubits, because for this system both the necessary and sufficient conditions for inseparability and nonlocality have been established by the Horodeckis [10–12]. An important feature of the two-qubit Werner states is a nonempty gap $1/3 < p \leq 1/\sqrt{2}$ between separable and nonlocal states.

In recent years, great attention has been paid to the quantum information processing with continuous variables (CVs). Most protocols developed originally for discrete quantum variables (qubits) have been extended to continuous variables, namely, teleportation [13], dense coding [14], entanglement swapping [15], and quantum cloning [16].

In this paper, we introduce a natural analog of the Werner state (1) for the CV systems: a mixture of a pure two-mode squeezed vacuum state and a mixed separable thermal state. We analyze in detail the separability, nonlocality and squeezing of the CV Werner state and we also discuss its usefulness in the teleportation of coherent states. Since there is no general method how to test the separability of a generic state in infinite-dimensional Hilbert space, one has to resort to some particular tests. We use the Peres-Horodecki (PH) criterion based on partial transposition [11]. Remarkably, nonpositive partial transpose is the necessary and sufficient condition for inseparability of two-mode bipartite Gaussian states [17,18]. However, the CV Werner state discussed in this paper is not Gaussian and hence in our case the PH criterion provides only a sufficient condition for the entanglement.

Testing of nonlocality for CV systems is based predominantly on the Banaszek-Wódkiewicz form of the Bell inequalities [19] that involves the Wigner function of a state. Here, we employ alternative Clauser-Horne-Shimony-Holt (CHSH) Bell-type inequalities for continuous quantum variables based on the single-mode realization of Pauli matrices [20]. By means of specific local transformations we map the two-mode CV Werner state into a state of two the qubits and then we employ the necessary and sufficient conditions for nonlocality of two-qubit system.

After discussing the separability and nonlocality of the Werner state we analyze its performance in quantum information processing. We consider the standard Braunstein-Kimble (BK) scheme for teleportation of CVs [13] where the Werner state serves as a quantum channel. Specifically, we focus on the teleportation of coherent states and we compare our findings with the results that have been obtained for qubit or qudit teleportation with Werner states [7,8].

The paper is organized as follows. In Sec. II, the CV analog of the Werner state is introduced. The mapping from infinite-dimensional Hilbert space to Hilbert space of two qubits is described in Sec. III. In Sec. IV, we will analyze the separability of the Werner state from two different points of view: after and before mapping into the two-qubit system. Sections V and VI are dedicated to the nonlocality and squeezing of the Werner state. In Sec. VII, the coherent-state teleportation with the Werner state is discussed. Finally, Sec. VIII contains conclusions.

II. CONTINUOUS-VARIABLE WERNER STATE

A common resource of the quantum entanglement in CV information processing is the two-mode squeezed vacuum state generated by means of spontaneous parametric down-conversion in the nondegenerate optical parametric amplifier (NOPA),

$$\rho_{\rm NOPA}=(1-\lambda_1^2)\sum_{m,n=0}^{\infty}\lambda_1^{m+n}|m,m\rangle\langle n,n|. \tag{3}$$

Here $\lambda_1=\tanh r$, r is the squeezing parameter, and $|m,n\rangle=|m\rangle_A|n\rangle_B$ denotes the Fock state of two modes A and B. The NOPA state approaches the maximally entangled Einstein-Podolsky-Rosen (EPR) state [21] in the strong squeezing limit $r\to\infty$. In practice, the EPR state is well approximated by the NOPA state if $r>2$. Recently, squeezing as large as $r\approx 2$ has been achieved experimentally [22].

A natural extension of the NOPA state to the Werner state for CVs is based on the following observation: The factorized state $I_1\otimes I_2/d^2$ in the mixture (1) is a tensor product of density matrices I_1/d and I_2/d that can be identified with reduced density matrices of the subsystems 1 and 2 when the whole system is in the maximally entangled state $|\Psi\rangle$. Now, if the modes A and B are in the NOPA state, then each mode separately is in the thermal state. The thermal state of modes A and B can be expressed as follows:

$$\rho_T=(1-\lambda_2^2)^2\sum_{m,n=0}^{\infty}\lambda_2^{2(m+n)}|m\rangle\langle m|\otimes|n\rangle\langle n|, \tag{4}$$

where $\lambda_2=\tanh s$ and the mean number of thermal photons in each mode reads $\langle n\rangle_T=\sinh^2(s)$.

It is thus natural to define the CV analog of the Werner state [6] as a mixture of the NOPA state (3) and factorized thermal state (4),

$$\rho_W=p\rho_{\rm NOPA}+(1-p)\rho_T, \quad 0\leqslant p\leqslant 1. \tag{5}$$

The Werner states ρ_W form a three-parametric family of states. The simplest analog of the Werner state can be obtained assuming $r=s$. In this case the Werner state and the d-dimensional Werner state (1) become manifestly analogous in the strong squeezing limit when ρ_W approaches a mixture of a maximally entangled EPR state and a maximally mixed state in infinite-dimensional Hilbert space.

III. MAPPING INTO THE TWO-QUBIT SYSTEM

The simplest way in which one can study the separability and nonlocality properties of the two-mode state (5) is to map it by means of *local operations* into the two-qubit system for which separability and nonlocality conditions are well known [11,12]. In what follows the qubits corresponding to modes A and B are denoted as 1 and 2, respectively.

We introduce the Hermitian "spin one-half" operators S_j^{α}, $\alpha=A,B$,

$$S_1^{\alpha}+iS_2^{\alpha}=2\sum_{m=0}^{\infty}|2m\rangle_{\alpha\alpha}\langle 2m+1|,$$

$$S_3^{\alpha}=\sum_{m=0}^{\infty}(-1)^m|m\rangle_{\alpha\alpha}\langle m|, \tag{6}$$

which satisfy the Pauli matrix algebra

$$[S_i^{\alpha},S_j^{\beta}]=2i\varepsilon_{ijk}\delta_{\alpha\beta}S_k^{\alpha}, \quad (S_i^{\alpha})^2=1, \tag{7}$$

where ε_{ijk} is the totally antisymmetric tensor with $\varepsilon_{123}=+1$ and $\delta_{\alpha\beta}$ is the Kronecker symbol.

Let us for a moment restrict our attention to mode A and qubit 1. With the help of the operators (6) one can assign the following qubit density matrix ρ_1 to the density matrix ρ_A:

$$\rho_1=\frac{1}{2}(I_1+\mathbf{S}^A\cdot\boldsymbol{\sigma}), \tag{8}$$

where

$$\mathbf{S}^A\cdot\boldsymbol{\sigma}=\sum_{i=1}^{3}{\rm Tr}(\rho_A S_i^A)\sigma_i, \tag{9}$$

σ_i are standard Pauli matrices and I_1 is the identity operator on the Hilbert space of qubit 1.

The transformation (8) is physical because it can be, at least in principle, performed in the laboratory. Let us assume that the qubit is represented by a two-level atom resonantly interacting with a single mode of electromagnetic field. Suppose that the interaction is governed by the following Hamiltonian:

$$H_{\rm int}=i\hbar\Omega(|1\rangle\langle 0|a\sqrt{n}-|0\rangle\langle 1|\sqrt{n}a^{\dagger}), \tag{10}$$

where $a(a^{\dagger})$ is the annihilation (creation) operator of mode A and $n=a^{\dagger}a$. The Hamiltonian (10) can be considered as a kind of nonlinear Jaynes-Cummings model. The specific feature of $H_{\rm int}$ is that its eigenvalues (the Rabi frequencies) are linearly proportional to the number of photons n. If the two-level atom is initially in its ground state $|0\rangle$ and if the interaction time t is adjusted in such a way that $\Omega t=\pi/2$, then the output state of the atom is exactly given by Eq. (8). Although the Hamiltonian (10) may be hard to implement in practice, it provides a clear physical picture behind the mathematical transformation (8).

Formally, the transformation (8) is a trace-preserving completely positive (CP) map. Making use of the correspondence between CP maps and positive-semidefinite operators [23] we can express the transformation (8) as follows:

$$\rho_1={\rm Tr}_A(\chi_{A1}\rho_A^T\otimes I_1), \tag{11}$$

where

$$\chi_{A1}=\sum_{m=0}^{\infty}\sum_{k,l=0}^{1}|2m+k\rangle_{AA}\langle 2m+l|\otimes|k\rangle_{11}\langle l| \tag{12}$$

is a positive-semidefinite operator acting on the direct product of the Hilbert spaces of mode A and of qubit 1; $|1\rangle_1$ and $|0\rangle_1$ are basis states of qubit 1; and T stands for the transposition. Mapping now the two-mode density matrix (5) into the two-qubit density matrix

$$\rho'_W = \mathrm{Tr}_{AB}(\chi_{A1}\chi_{B2}\rho_W^T \otimes I_1 \otimes I_2), \tag{13}$$

where the χ_{B2} is obtained from Eq. (12) by replacements $A \to B$ and $1 \to 2$, one gets

$$\rho'_W = \frac{1}{4}\left(I_1 \otimes I_2 + \mathbf{S}^A \cdot \boldsymbol{\sigma} \otimes I_2 + I_1 \otimes \mathbf{S}^B \cdot \boldsymbol{\sigma} + \sum_{i,j=1}^{3} t_{ij}\sigma_i \otimes \sigma_j \right). \tag{14}$$

The elements $t_{ij} = \mathrm{Tr}\ (\rho_W S_i^A S_j^B)$ of the correlation tensor $\mathcal{T}$ explicitly read

$$t_{11} = -t_{22} = \frac{2\lambda_1 p}{1+\lambda_1^2},$$
$$t_{33} = p + (1-p)\left(\frac{1-\lambda_2^2}{1+\lambda_2^2}\right)^2,$$
$$t_{ij} = 0, \quad i \neq j. \tag{15}$$

By calculating the matrices $\mathbf{S}^A \cdot \boldsymbol{\sigma}$ and $\mathbf{S}^B \cdot \boldsymbol{\sigma}$ and taking into account the expressions (15), one obtains after some algebra

$$\rho'_W = \begin{pmatrix} \frac{p}{1+\lambda_1^2} + \frac{1-p}{(1+\lambda_2^2)^2} & 0 & 0 & \frac{\lambda_1 p}{1+\lambda_1^2} \\ 0 & \frac{\lambda_2^2(1-p)}{(1+\lambda_2^2)^2} & 0 & 0 \\ 0 & 0 & \frac{\lambda_2^2(1-p)}{(1+\lambda_2^2)^2} & 0 \\ \frac{\lambda_1 p}{1+\lambda_1^2} & 0 & 0 & \frac{\lambda_1^2 p}{1+\lambda_1^2} + \frac{\lambda_2^4(1-p)}{(1+\lambda_2^2)^2} \end{pmatrix}. \tag{16}$$

Thus we have mapped the state ρ_W into this two-qubit state. Note that the transformation (13) is local; it is carried out separately on each subsystem (A,1) and (B,2). The essential feature of local unconditional transformations is that they cannot increase the amount of entanglement or nonlocality present in any bipartite state. This ensures that the properties of the state ρ'_W reflect the properties of the original Werner state ρ_W. If we find that the state ρ'_W is entangled or nonlocal, then the same holds true for the original state ρ_W.

IV. SEPARABILITY

According to the PH partial transposition criterion [11,12] the state (16) is entangled iff its partial transpose

$$(\rho'_W)^{T_1}_{m\mu,n\nu} \equiv (\rho'_W)_{n\mu,m\nu} \tag{17}$$

has some negative eigenvalue. Due to the specific structure of the matrix (16), it is easy to see that its partial transposition has a negative eigenvalue if the off-diagonal elements of ρ'_W are larger than the central diagonal elements,

$$\frac{\lambda_1 p}{1+\lambda_1^2} > \frac{\lambda_2^2(1-p)}{(1+\lambda_2^2)^2}. \tag{18}$$

It is instructive to rewrite this condition as an inequality for the probability p. After some algebra, one finds that the state (14) is entangled iff

$$p > \frac{1}{1 + 2\frac{\tanh(2r)}{\tanh^2(2s)}}, \tag{19}$$

where we have used the relations $\lambda_1 = \tanh r$ and $\lambda_2 = \tanh s$. For the direct analog of the Werner state with $r = s$, and in the strong squeezing limit, the state (16) and hence also the state (5) are entangled if $p > \frac{1}{3}$ as in the case of the two-qubit Werner state [24].

Surprisingly, the PH criterion can also be applied directly to the two-mode state (5) for which it is the only sufficient condition for entanglement. The partially transposed matrix $\rho_W^{T_A}$ has a block-diagonal form with 1×1 blocks in one-dimensional subspaces spanned by vectors $\{|m,m\rangle\}$, m

$=0,1,\ldots$ and 2×2 blocks in two-dimensional subspaces spanned by vectors $\{|m,n\rangle,|n,m\rangle,m\neq n\},m,n=0,1,\ldots$. Consequently, the eigenvalues of the partially transposed matrix $\rho_W^{T_A}$ can easily be calculated as roots of quadratic equations and read

$$x^{(l)}=p(1-\lambda_1^2)\lambda_1^{2l}+(1-p)(1-\lambda_2^2)^2\lambda_2^{4l},$$

$$x_{1,2}^{(mn)}=(1-p)(1-\lambda_2^2)^2\lambda_2^{2(m+n)}\pm p(1-\lambda_1^2)\lambda_1^{m+n}, \tag{20}$$

where $l=0,1,\ldots$ and $m\neq n=0,1,\ldots$. According to the above-mentioned separability criterion [11], the state (5) is entangled if there are such m,n for which $x_2^{(mn)}<0$. If $\lambda_2=0$, then $x_2^{(mn)}<0$ for all $p>0$ and the state (5) is always entangled. If $\lambda_2\neq0$, then the inseparability condition $x_2^{(mn)}<0$ is equivalent with the inequality

$$p>\frac{(1-\lambda_2^2)^2}{(1-\lambda_2^2)^2+(1-\lambda_1^2)\left(\frac{\lambda_1}{\lambda_2^2}\right)^{m+n}}\equiv p_{m+n}. \tag{21}$$

Three different cases must be considered to be dependent on the value of the ratio $q=\lambda_1/\lambda_2^2$.

(i) If $q>1$, then the factor q^{m+n} in the denominator of Eq. (21) increases with increasing $m+n$ and consequently the right-hand side (RHS) of Eq. (21) decreases attaining zero value in the limit $m+n\to\infty$. Hence, the state (5) is entangled for any $p>0$. In particular, a direct analog of the Werner state $(r=s)$ is entangled for every $p>0$.

(ii) If $q=1$, then also $q^{m+n}=1$. From that it follows that the inequality (21) is independent of m and n and the state (5) is entangled if

$$p>\frac{1-\lambda_1}{2}=\frac{1-\tanh r}{2}. \tag{22}$$

(iii) If $q<1$, then the RHS of inequality (21) increases with growing $m+n$. Since the RHS attains its minimum value for $m+n=1$, the state (5) is entangled if

$$p>\frac{(1-\lambda_2^2)^2}{(1-\lambda_2^2)^2+(1-\lambda_1^2)\frac{\lambda_1}{\lambda_2^2}}=p_1, \tag{23}$$

or equivalently,

$$p>\frac{1}{1+\frac{\cosh^4(s)}{\cosh^2(r)}\frac{\tanh r}{\tanh^2(s)}}. \tag{24}$$

The partial transposition criterion applied to the original state ρ_W is stronger than that applied to ρ_W', because any local transformation (13) preserves the positivity of the partial transpose. For instance, if $\tanh r>\tanh^2(s)$, then the entangled states (5) for which

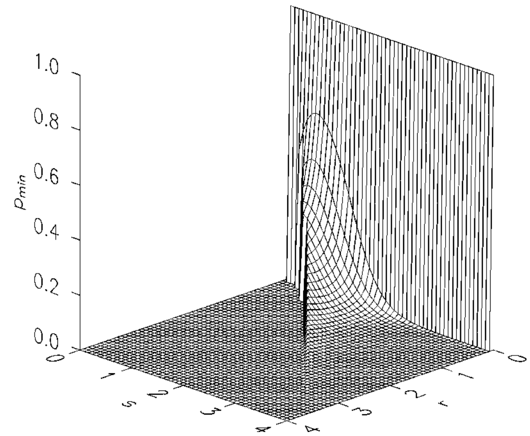

FIG. 1. Minimal probability $p_{\min}$ characterizing the entanglement of the CV Werner state as a function of the squeezing parameter r and the thermal noise parameter s. The state ρ_W is entangled if $p>p_{\min}$.

$$\frac{\tanh^2(2s)}{\tanh^2(2s)+2\tanh(2r)}\geqslant p>0 \tag{25}$$

are mapped into separable two-qubit states (16).

The region of Werner state inseparability is depicted in Fig. 1. We can see that the Werner state is entangled almost for every p if the squeezing is sufficiently large. We emphasize again that the negative partial transpose is only a sufficient condition for the entanglement and there may exist the entangled Werner states with positive partial transpose. One may even ask whether there exist any nontrivial separable CV Werner states (5). Although we do not have the sufficient separability condition for generic bipartite CV states at present, it is possible to find conditions under which the Werner state is separable, i.e. it can be written as a convex combination of product states.

The state (5) can be rewritten in the following form:

$$\rho_W=\sum_{m=0}^{\infty}P_m|mm\rangle\langle mm|+\sum_{m\neq n=0}^{\infty}\rho^{mn}, \tag{26}$$

where

$$P_m=p(1-\lambda_1^2)^2\lambda_1^{4m}+(1-p)(1-\lambda_2^2)^2(1-\lambda_2^4)\lambda_2^{8m}, \tag{27}$$

and ρ^{mn} are matrices in four-dimensional Hilbert subspaces spanned by the basis vectors $\{|mm\rangle,|mn\rangle,|nm\rangle,|nn\rangle\}$,

$$\rho^{mn}=\frac{1}{2}\begin{pmatrix}\alpha_{mn}&0&0&\beta_{mn}\\0&\gamma_{mn}&0&0\\0&0&\gamma_{mn}&0\\\beta_{mn}&0&0&\alpha_{mn}\end{pmatrix}, \tag{28}$$

where

$$\alpha_{mn}=p(1-\lambda_1^2)^2\lambda_1^{2(m+n)}+(1-p)(1-\lambda_2^2)^2(1-\lambda_2^4)\lambda_2^{4(m+n)},$$

$$\beta_{mn}=p(1-\lambda_1^2)\lambda_1^{m+n},$$

$$\gamma_{mn}=(1-p)(1-\lambda_2^2)^2\lambda_2^{2(m+n)}. \tag{29}$$

Obviously, if all ρ^{mn} are positive-semidefinite separable matrices, then ρ_W is separable. From the matrix form (28) of ρ^{mn} one easily obtains the positivity condition $\alpha_{mn}\geqslant\beta_{mn}$ and the separability condition $\gamma_{mn}\geqslant\beta_{mn}$. Consequently, the Werner state is separable if both these inequalities are satisfied for all $m\neq n$. The second condition is identical to the necessary condition for separability of the Werner state $p\leqslant p_{m+n}$, where p_{m+n} is given in the inequality (21). A further constraint on p follows from the positivity condition $\alpha_{mn}\geqslant\beta_{mn}$,

$$p\leqslant\frac{1}{1+\frac{(1-\lambda_1^2)}{(1-\lambda_2^2)^2(1-\lambda_2^4)}\left[\left(\frac{\lambda_1}{\lambda_2^4}\right)^{m+n}-(1-\lambda_1^2)\left(\frac{\lambda_1^2}{\lambda_2^4}\right)^{m+n}\right]}, \tag{30}$$

where $m\neq n=0,1,\ldots$. Similarly to the case of entanglement we have to distinguish three regions in dependence on the value of the ratio $\tilde{q}=\lambda_1/\lambda_2^4$:

(i) If $\tilde{q}>1$, then the RHS of the inequality (30) goes to zero in the limit $m+n\to\infty$. Hence, the state (5) is separable only for $p=0$.

(ii) If $\tilde{q}=1$, then the expression in square brackets in the RHS of the inequality (30) attains its maximum equal to unity in the limit $m+n\to\infty$ and the Werner state can be separable if

$$p\leqslant\frac{(1-\lambda_2^2)^2}{2(1-\lambda_2^2+\lambda_2^4)}. \tag{31}$$

In this case the condition $p\leqslant p_1$ is weaker than the inequality (31) that is thus a sufficient condition for separability of the Werner state.

(iii) If $\tilde{q}<1$, then the RHS of the inequality (30) attains its minimum for $m+n=1$ and the state (5) can be separable if

$$p\leqslant\frac{1}{1+\frac{(1-\lambda_1^2)}{(1-\lambda_2^2)^2(1-\lambda_2^4)}\left[\left(\frac{\lambda_1}{\lambda_2^4}\right)-(1-\lambda_1^2)\left(\frac{\lambda_1^2}{\lambda_2^4}\right)\right]}. \tag{32}$$

Since in this case the inequality (32) is stronger than the condition $p\leqslant p_1$, the inequality (32) is a sufficient condition for separability of the Werner state. The region of separable CV Werner states is depicted in Fig. 2.

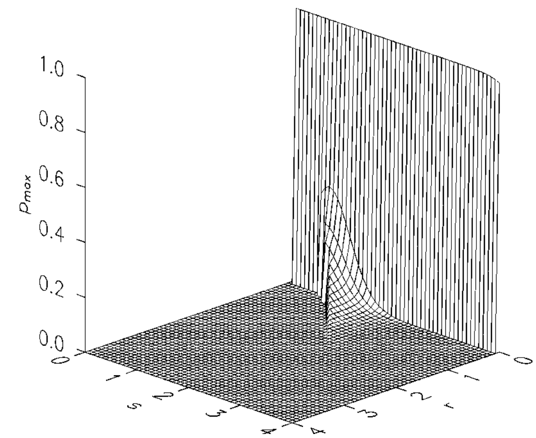

FIG. 2. Maximal probability $p_{\max}$ characterizing the separability of the CV Werner state as a function of the squeezing parameter r and the thermal noise parameter s. The state ρ_W is separable if $p\leqslant p_{\max}$.

V. NONLOCALITY

Due to the commutation rules (7) the nonlocality of the Werner state (5) can be investigated employing the standard two-qubit CHSH Bell inequalities in which the Pauli matrices are replaced with the single-mode operators (6),

$$2\geqslant|\langle(\mathbf{a}\cdot\mathbf{S}^A)(\mathbf{b}\cdot\mathbf{S}^B)\rangle+\langle(\mathbf{a}'\cdot\mathbf{S}^A)(\mathbf{b}\cdot\mathbf{S}^B)\rangle+\langle(\mathbf{a}\cdot\mathbf{S}^A)(\mathbf{b}'\cdot\mathbf{S}^B)\rangle-\langle(\mathbf{a}'\cdot\mathbf{S}^A)(\mathbf{b}'\cdot\mathbf{S}^B)\rangle|, \tag{33}$$

where $\mathbf{a},\mathbf{a}',\mathbf{b},\mathbf{b}'$ are real three-dimensional unit vectors and the angle brackets denote the averaging over the density matrix ρ_W. It is instructive to formulate this approach in terms of the mapping introduced in Sec. III. We map the Werner state ρ_W into the state of two qubits ρ_W' and then we analyze the nonlocality of the state ρ_W' characterized by the correlation tensor $\mathcal{T}$ whose elements are given by Eq. (15).

Now, according to the Horodecki criterion [10], if the sum of the two largest eigenvalues of the matrix $U=\mathcal{T}^T\mathcal{T}$ is greater than unity, then the state (5) violates the inequalities (33) for some choice of vectors $\mathbf{a},\mathbf{a}',\mathbf{b},\mathbf{b}'$. The matrix U has a twofold eigenvalue t_{11}^2 and single eigenvalue t_{33}^2. It can be shown that $t_{11}^2\leqslant t_{33}^2$ holds for any λ_1, λ_2, and p. Thus the maximal Bell factor is given by

$$B_{\max}=2\sqrt{t_{11}^2+t_{33}^2}. \tag{34}$$

Hence, the Bell inequality (33) is violated if $t_{11}^2+t_{33}^2>1$. Substituting here from the formulas (15) one obtains after some algebra that the state (5) violates the Bell inequalities (33) if

$$p>\frac{a(a-1)+\sqrt{a(a-ab^2+2b^2)}}{a^2+b^2}, \tag{35}$$

where $a=\tanh^2(2s)$ and $b=\tanh(2r)$. The region of nonlocality of the state (16) is depicted in Fig. 3. In the strong squeezing limit the direct analog of the Werner state (5) with

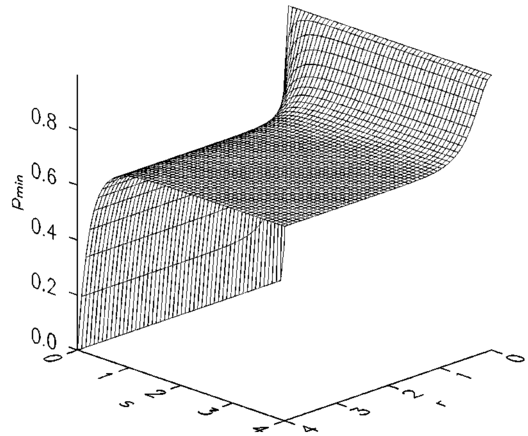

FIG. 3. Minimal probability $p_{\rm min}$ characterizing the nonlocality of the state ρ_W' as a function of the squeezing parameter r and the thermal noise parameter s. The state is nonlocal if $p > p_{\rm min}$.

$r=s$ is nonlocal if $p>(1/\sqrt{2})$ as in the case of two-qubit Werner state. However, it was found in the preceding section, that the original state ρ_W in infinite-dimensional Hilbert space may be entangled even if the two-qubit state ρ_W' is separable. We can conjecture that the nonlocality has a similar behavior and that the state ρ_W may violate some Bell inequalities although the state ρ_W' admits local realistic description.

VI. SQUEEZING

Apart from entanglement and nonlocality, the nonclassicality of the Werner state (5) can be judged by means of squeezing. For this purpose, it is useful to arrange the quadrature operators $x_\alpha, p_\beta([x_\alpha,p_\beta]=i\delta_{\alpha\beta}), \alpha,\beta=A,B$ into the vector $\xi=(x_A,p_A,x_B,p_B)$, and to define the 4×4 variance matrix $V_{\rm W}$ of the state (5), $(V_{\rm W})_{\alpha\beta}=(\langle\Delta\xi_\alpha\Delta\xi_\beta\rangle+\langle\Delta\xi_\beta\Delta\xi_\alpha\rangle)/2$, where $\Delta\xi_\alpha=\xi_\alpha-\langle\xi_\alpha\rangle$ and $\langle\xi_\alpha\rangle={\rm Tr}\ (\rho_W\xi_\alpha)$. After some calculations one arrives at

$$V_{\rm W}=\frac{1}{2}\begin{pmatrix} x & 0 & y & 0\\ 0 & x & 0 & -y\\ y & 0 & x & 0\\ 0 & -y & 0 & x\end{pmatrix}, \tag{36}$$

where

$$x=p\cosh(2r)+(1-p)(2\langle n\rangle_T+1),$$

$$y=p\sinh(2r). \tag{37}$$

A convenient measure of the squeezing is the generalized squeeze variance λ_G [25] that coincides with the lowest eigenvalue of the variance matrix $V_{\rm W}$. By definition, the state is squeezed if $\lambda_G<1/2$. Since $\lambda_G=x-y$ one obtains the following squeezing condition:

$$p>\frac{1}{1+\dfrac{1-e^{-2r}}{2\langle n\rangle_T}}, \tag{38}$$

where $\langle n\rangle_T=\sinh^2(s)$.

Interestingly, since the RHS of this inequality approaches unity in the strong squeezing limit if $r=s$, we arrive at the family of Werner states that are never squeezed. This is a counterintuitive example of the Werner states that are not squeezed; however, they can be entangled or even nonlocal.

It is worth noting that the inequality $\lambda_G=x-y<1/2$ is not only the squeezing condition, but also the sufficient condition for inseparability of the state (5) according to Duan's inseparability criterion [17] that is currently used in experimental tests of inseparability [26]. It is evident that the criterion becomes useless if $r=s$ and in the strong squeezing limit in contrast to the PH criterion employed in Sec. IV.

VII. TELEPORTATION

An interesting application of the two-qubit Werner state arises in quantum teleportation [1,7]. By analogy, the proposed Werner state (5) can be utilized in the BK scheme of the CV coherent-state teleportation [13]. In this scheme, the two-mode entangled state is shared between Alice and Bob. The Wigner function $W_{in}(x_1,p_1)$ of Alice's input state and the Wigner function $W_{out}(x_2,p_2)$ of Bob's output state are related by the convolution [27]

$$W_{out}(x_2,p_2)=\frac{1}{4}\int_{-\infty}^{\infty}K_{AB}(x_2-x_1,p_2-p_1)\times W_{in}(x_1,p_1)dx_1dp_1. \tag{39}$$

The kernel function $K_{AB}(x_-,p_+)$ reads

$$K_{AB}(x_-,p_+)=\int_{-\infty}^{\infty}\mathcal{W}_{AB}(-x_-,x_+,p_-,p_+)dx_+dp_-, \tag{40}$$

where $x_\pm=x_A\pm x_B$ and $p_\pm=p_A\pm p_B$, and $W_{AB}(x_A,p_A,x_B,p_B)=\mathcal{W}_{AB}(x_-,x_+,p_-,p_+)$ is the Wigner function of the state shared by Alice and Bob (quantum channel). The fidelity between Alice's and Bob's states can be calculated as follows:

$$F=\frac{\pi}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W_{in}(x_2,p_2)K_{AB}(x_2-x_1,p_2-p_1)\times W_{in}(x_1,p_1)dx_1dp_1dx_2dp_2. \tag{41}$$

The BK scheme is designed in such a way that the fidelity is invariant under displacement transformations. In particular, all coherent states are teleported with the same fidelity. If the quantum channel is in the NOPA state, then this fidelity reads

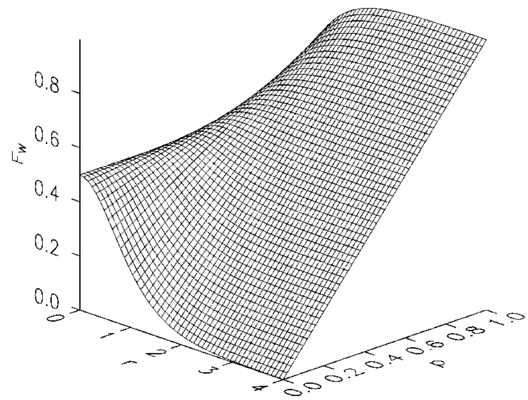

FIG. 4. The dependence of the fidelity F_W in the standard BK scheme employing the shared CV Werner state on the squeezing parameter $r=s$ and the probability p.

$$F_{\mathrm{NOPA}}=\frac{1}{1+e^{-2r}}. \tag{42}$$

This exceeds the maximal classical value $F=1/2$ for every $r>0$. Now we consider the Werner state (5) in symmetric form with $r=s$. One can find that the fidelity of teleportation in the standard BK scheme utilizing such a Werner state is changed as follows:

$$F_W=pF_{\mathrm{NOPA}}+(1-p)\frac{1}{d}, \tag{43}$$

where $d=2/(1-\lambda_1^2)=2\cosh^2 r$. The dependence of the fidelity F_W on the probability p and the squeezing parameter r is depicted in Fig. 4.

In the strong squeezing limit, the fidelity (43) approaches the value $F_W \approx p$. In order to teleport the coherent state with fidelity $F_W>1/2$, we need to employ the CV Werner state with $p>1/2$. Note that, for every $p\neq 0$, this Werner state is entangled (recall that we assume $r=s$ here). This should be contrasted with the results obtained for teleportation with the d-dimensional Werner state, where if the shared Werner state is entangled it is then useful for quantum teleportation [8]. In our case, some of the entangled Werner states in infinite-dimensional Hilbert space are not useful for the BK teleportation protocol. It is an open question whether the BK scheme can be modified in such a way that the coherent states would be teleported with a fidelity higher than 1/2 even when using entangled Werner states with $p<1/2$.

VIII. CONCLUSIONS

A natural extension of the Werner state into CV systems is presented, and separability, nonlocality, and squeezing of this state are analyzed. In a certain sense, the CV Werner state can be considered as a counterpart of the two-qubit Werner state. This relationship is established by the mapping (12). On the other hand, some features of the CV Werner state correspond to those of d-dimensional Werner states when $d\to\infty$. For instance, in the simplest case when $r=s$, the CV Werner state (5) is entangled for any $p>0$. Since the d-dimensional Werner state is entangled when $p>1/(1+d)$, the above behavior of the CV Werner state corresponds to the limit $d\to\infty$.

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